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Research Article

A p-Sum Representation of the 1-Jets of p-Velocities

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Abstract. The theory of jets provides a useful tool for various fields in mathematics, enabling the solution of higher-order differential equations and partial differential equations that model complex mechanical systems. This theory adopts a geometric approach to generalized mechanics and field theory. For instance, in Lagrangian particle mechanics, the formalism of higher-order jet bundles proves useful. Thus, the study of jets is not only beneficial to mathematics but also extends its applicability to other fields such as physics. In this study, we approach jet bundles from a differential geometry perspective. Specifically, we use structure of the bundle of all 1-jets of maps from $\mathbb{R}^p$ to M with source at 0. By employing normal coordinates on the manifold M, we demonstrate that this bundle is diffeomorphic to the p-Whitney sum of tangent bundles. Then, we prove that this bundle carries a vector bundle structure. Using its vector bundle structure, the paper establishes the existence of the isomorphism for tangent bundles of p^1 velocities, and extends the previous result by proving that the vector bundle of 1-jets of p-velocities is isometric to p-sum of tangent bundles, even in cases where the base manifold does not carry a Banach structure.

Keywords. First order jets, Tangent bundle of higher order, Tangent bundle of p^1 velocities, Induced metric

Mathematics Subject Classification (2020). 58A20, 55R25, 58A05

1. Introduction

The concept of jet bundles, and in particular the tangent bundle of p^k velocities, is a versatile mathematical tool with applications in various fields of mathematics. It has been used extensively for the solutions of partial differential equations (Baker and Doran [1], and Barco [2]), second order differential equations modeling certain types of dynamical systems (Sarelet *et al*. [13]), and singularity theory (Golubitsky and Guillemin [7]). Jet bundles also have applications in higher order mechanics (Deleón and Rodrigues [4], and Krupková [10]) and variational calculus (Musilová and Hronek [12]), offering a unique perspective on formulating fundamental physical theories like special relativity, classical electrodynamics, and quantum mechanics.

There are basically two different approach on the notion of jets in the literature. One approach is to define an equivalence relation among local sections of a given bundle (E,π,N). This type of jets is discussed in detail by Saunders [14]. The other approach is to use equivalence relation $\cong_k$ on a function between smooth functions N and M. Each equivalence class is called a k-jet. Also, k-jet bundle is defined as the set of all k-jets. In this study, we consider this approach.

A specific instance of k-jets is referred to as the tangent bundle of higher order, particularly when $N = \mathbb{R}^p$. This jet bundle exhibits a distinct geometric structure known as the almost tangent structure of higher order. In a broader context, if $N = \mathbb{R}^p$, the jet bundle is denoted as the tangent bundle of p^k velocities. This notion was aimed at advancing classical field theory in an autonomous manner (Deleón and Rodrigues [4], and Ehresmann [5]). The concept was also used to the construction of the geometry on frame bundles (Cordero *et al*.[3]).

A particular instance within the tangent bundle of p^k velocities arises when $p = 1$. This variant of the jet bundle is commonly referred to as the k-th order tangent bundle, owing to its intimate connection with the standard tangent bundle. According to this definition, a jet is simply an equivalence class of curves on an arbitrary manifold M such that two curves are equivalent if and only if their derivatives up to the k-th order at the origin are equal. The set of these type of jets carries a bundle structure (Morimoto [11]). Their geometry on Banach manifolds has also been studied by Suri [15, 16].

The third definition of jet bundles is constructed by using an equivalence relation on the functions from $\mathbb{R}^p$ to M, such that all of their partial derivatives up to order 1 at the origin are equal. That is, the 1-jets with the source at the origin of $\mathbb{R}^p$, directed towards M offer a sound approach for the generalization of tangent bundles. This type of approach is used in working with double tangent bundles (Fischer and Laquer [6], and Kadioğlu [9]), frame bundles (Cordero *et al*. [3]). In this paper, we focus on this type of jets.

In this paper, we demonstrate that J^1_pM can be expressed as the Whitney sum of p tangent bundles. It was proven by Suri [16] that the kth order tangent bundles (or the tangent bundle of 1^k velocities) are isometric to the k-Sum of tangent bundles if the base manifold carries a Banach structure. Here we prove that the manifold structure of 1-jets of p-velocities is equivalent to p-sum of tangent bundles even if the base manifold does not carry a Banach structure. We prove that each fiber of the jet bundle is indeed a vector space and then prove that the jet bundle J^1_pM is indeed a vector bundle. We also present local coordinates of the tangent vectors in TJ^1_pM. By using these local coordinates, we prove that the p-sum representation is actually a Whitney sum representation by showing that J^1_pM is bundle isomorphic to the p-Whitney sum.

2. Preliminaries

In this section, we provide a concise overview of essential preliminary materials.

2.1 First Order Jets: $J^1_p M$

Definition 2.1 ([3]). Suppose that $C^\infty(\mathbb{R}^p)$ denotes the algebra of smooth functions on the Euclidean space $\mathbb{R}^p$ with natural coordinates $(u_1, u_2, \dots, u_p)$, and $f, g \in C^\infty(\mathbb{R}^p)$. Then, there is an equivalence relation on $C^\infty(\mathbb{R}^p)$: $f \equiv g$ if and only if $f(0) = g(0)$ and $\frac{\partial}{\partial u_\alpha}\big|_0(f) = \frac{\partial}{\partial u_\alpha}\big|_0(g)$ for every $i = 1, 2, \dots, p$.

Now, let M be an m dimensional manifold, and $\phi, \xi : \mathbb{R}^p \to M$, two smooth maps. Then, we say that $\phi \equiv \xi$ if $f \circ \phi \equiv f \circ \xi$ for every $f \in C^\infty(M)$. This relation an equivalence relation, and we denote by $j^1(\phi)$ the equivalence class of $\phi \in C^\infty(\mathbb{R}^p; M)$ referred to as a 1-jet in M at $\phi(0)$.

We denote $J^1_p M$ the set of all equivalence classes in $C^\infty(\mathbb{R}^p; M)$, which is a $mp + m$ dimensional smooth manifold with local charts defined as follows:

If $(U, x_1, \dots x_m)$ is a local chart in M, then $(J^1_p U, x_1, \dots, x_m, x^1_\alpha, \dots, x^m_\alpha)$ is the local chart for $J^1_p M$, with $\alpha = 1, 2, \dots, p$ by

$$x^i(j^1(\phi)) = x^i(\phi(0)),$$

$$x^i_\alpha(j^1(\phi)) = \frac{\partial(x^i \circ \phi)}{\partial u_\alpha}\Big|_0. \tag{2.1}$$

Consider an arbitrary point $j^1\phi$, and suppose $\phi_\alpha : \mathbb{R} \to M$ represents the differentiable curve defined as $\phi_\alpha(u) = \phi(0, \dots, u, \dots 0)$, where u is at the αth position. Then, associated with $j^1\phi$, there exists a unique $(p+1)$-tuple $[x; X_1, \dots, X_p]$, determined as follows:

$$x = \phi(0), \quad X_\alpha = (\phi_\alpha)_* \left(\frac{d}{du}\Big|_0\right), \tag{2.2}$$

where $\frac{d}{du}$ is the canonical vector field tangent to $\mathbb{R}$. Henceforth, we will express $[x; X_1, \dots, X_p]$ more succinctly as $[x; X_\alpha]$. Furthermore, we will denote the equivalence between $j^1\phi$ and $[x; X_\alpha]$ if there is no confusion.

Remark 2.1. If $j^1\phi \in J^1_p M$ is represented by $[x; X_\alpha]$, then $X_\alpha = \frac{\partial \phi}{\partial u_\alpha}\big|_0$.

2.2 The Whitney Sum $\oplus_p(TM)$

Consider M as an m-dimensional manifold, with (TM, π, M) representing its tangent bundle, where $\Psi : \pi^{-1}U \to U \times \mathbb{R}^m$ is the local bundle trivialization, and $(x^i, \dot{x}^i)$ represents its corresponding local coordinate chart. Then, the Whitney sum $TM \oplus_M TM = \{(V_1, V_2) \in TM \times TM : \pi(V_1) = \pi(V_2)\}$ is a vector bundle on M, with bundle projection $\pi_{1,2} : TM \oplus_M TM \to M \to M$, $\pi_{1,2}(V_1, V_2) = \pi(V_1) = \pi(V_2)$, local trivializations $\Psi_{1,2} : (TM \oplus_M TM)_U \to U \times (\mathbb{R}^m \times \mathbb{R}^m)$ defined by

$$\Psi_{1,2}(V_1, V_2) = (\pi_{1,2}(V_1, V_2), (pr_2 \circ \Psi)(V_1), (pr_2 \circ \Psi)(V_2)).$$

If $(x^i, \dot{x}^i)$ being a local chart on $\pi^{-1}(U)$, then the local coordinate chart induced by the local trivializations on $\pi_{1,2}(U)$ is given by $(\bar{x}^i, \dot{\bar{y}}^i_1, \dot{\bar{y}}^i_2)$, where

$$\bar{x}^i = x^i \circ \pi_{1,2}, \quad y^i_1 = \dot{x}^i \circ Pr_1, \quad y^i_2 = \dot{x}^i \circ Pr_2.$$

Here, Pr_1 and Pr_2 are first and second projections on $TM \times_M TM$, respectively. The following remark is needed when we work on tangent vectors on J^1_pM.

Remark 2.2 ([3]). If $\dot{\phi}$ is a tangent vector to $\phi : \mathbb{R} \to J^1_pM$; then, there exist

$$\psi : \mathbb{R} \times \mathbb{R}^p \to M \tag{2.3}$$

and $\delta > 0$ such that $\phi(t) = j^1(\psi_t)$ for $|t| < \delta$, where

$$\psi_t(u) = \psi^u(t) = \psi(t,u), \tag{2.4}$$

for $t \in \mathbb{R}$ and $u \in \mathbb{R}^p$.

Now, let's delve into some preliminary information concerning the generalized Whitney sum of tangent bundles.

Definition 2.2. The pth Whitney sum $\oplus_p(TM) = \{(V_1, V_2, \ldots, V_p) \in TM \times TM \times \ldots \times TM : \pi(V_1) = \pi(V_2) = \ldots = \pi(V_p)\}$ is a vector bundle, with bundle projection

$$\pi_{1-p} : \oplus_p(TM) \to M$$
$$(V_1, V_2, \ldots, V_p) \to \pi_{1-p}(V_1, V_2, \ldots, V_p) = \pi(V_1) = \pi(V_2) = \ldots = \pi(V_p)$$

with local coordinate chart $(\bar{x}^i, y^i_1, y^i_2, \ldots, y^i_p)$ such that

$$\bar{x}^i = x^i \circ \pi_{1-p}, \quad y^i_1 = \dot{x}^i \circ pr_1,\ y^i_2 = \dot{x}^i \circ pr_2,\ \ldots,\ y^i_p = \dot{x}^i \circ pr_p, \tag{2.5}$$

where $pr_1, pr_2, \ldots, pr_p$ are usual projections $\oplus_p(TM) \to TM$. Here, $pr_\alpha(V_1, V_2, \ldots, V_p) = V_\alpha$. This definition offers a straightforward extension of the concept of $TM \oplus_M TM$ to the pth Whitney sum $\oplus_p(TM)$, extending the understanding of tangent bundles in a systematic manner.

3. J^1_pM as a Whitney Sum

Definition 3.1. Let $\oplus_p(TM)$ represent the generalized Whitney sum $TM \oplus TM \oplus TM \oplus \ldots \oplus TM$ on the manifold M. Now, we define $\Omega : J^1_pM \to \oplus_p(TM)$ as

$$\Omega(j^1\phi) = \left(\phi_{*0}\left(\frac{\partial}{\partial u_1}\right), \phi_{*0}\left(\frac{\partial}{\partial u_2}\right), \ldots, \phi_{*0}\left(\frac{\partial}{\partial u_p}\right)\right)\Big|_{\phi(0)}, \tag{3.1}$$

where $j^1\phi \in J^1_pM$ is a 1-jet in M at $\phi(0)$ and $(u_1, u_2, \ldots u_p)$ is the standard coordinate system in $\mathbb{R}^p$.

The mapping Ω defined in equation (3.1) establishes a connection between the 1-jets in M at $\phi(0)$ and the generalized Whitney sum $\oplus_p(TM)$, providing a valuable tool for further analysis.

Lemma 3.1. *Ω is well defined.*

Proof. Consider $j^1\phi$ and $j^1\phi'$ in J^1_pM with $j^1\phi = j^1\phi'$. By the definition of 1-jets,

$$\phi(0) = \phi'(0) \quad \text{and} \quad \frac{\partial(x_i \circ \phi)}{\partial u_\alpha}\Big|_0 = \frac{\partial(x_i \circ \phi')}{\partial u_\alpha}\Big|_0,$$

for all $\alpha \in \{1, 2, \ldots, p\}$. Then, for all $\alpha \in \{1, 2, \ldots, p\}$, $\phi_{*0}(\frac{\partial}{\partial u_\alpha}) = \phi'_{*0}(\frac{\partial}{\partial u_\alpha})$ and $\phi_{*0}(\frac{\partial}{\partial u_1})$ in the same tangent space at $\phi(0) = \phi'(0)$. Therefore $\Omega(j^1\phi) = \Omega(j^1\phi')$, thus Ω is well defined. □

Lemma 3.2. Ω *is an injection.*

Proof. Consider $\Omega(j^1\phi)=\Omega(j^1\phi')$ for $j^1\phi, j^1\phi' \in J^1_pM$. Then, we have

$$\left(\phi_{*0}\left(\frac{\partial}{\partial u_1}\right),\phi_{*0}\left(\frac{\partial}{\partial u_2}\right),\dots,\phi_{*0}\left(\frac{\partial}{\partial u_p}\right)\right)\Big|_{\phi(0)}=\left(\phi'_{*0}\left(\frac{\partial}{\partial u_1}\right),\phi'_{*0}\left(\frac{\partial}{\partial u_2}\right),\dots,\phi'_{*0}\left(\frac{\partial}{\partial u_p}\right)\right)\Big|_{\phi'(0)}.$$

Then, $\phi(0)=\phi'(0)$ and $\phi_{*0}(\frac{\partial}{\partial u_\alpha})=\phi'_{*0}(\frac{\partial}{\partial u_\alpha})$. Thus $\frac{\partial(x_i\circ\phi)}{\partial u_\alpha}|_0=\frac{\partial(x_i\circ\phi')}{\partial u_\alpha}|_0$, for all $\alpha\in\{1,2,\dots,p\}$ and $\phi(0)=\phi'(0)$, implying that $j^1\phi=j^1\phi'$. Therefore, Ω is an injection. □

Theorem 3.1. J^1_pM *is diffeomorphic to the sum* $\oplus_p(TM)$*.*

Proof. Since M is paracompact, there exists a normal neighborhood U at $x\in M$, and $\widetilde{U}\subset T_xM$ a neighborhood of 0 so that the exponential map $\mathrm{Exp}:\widetilde{U}\to U$ is a diffeomorphism. We now consider U as a normal neighborhood of x such that it is diffeomorphic to $\widetilde{U}$, and let $V_1,V_2,\dots,V_p\in T_xM$ be such that $\sum_{\alpha=1}^p u_\alpha V_\alpha\in\widetilde{U}$, $\forall(u_1,u_2,\dots,u_p)$ in a small neighborhood of $\in\mathbb{R}^p$. Then, we define a function

$$\phi_V:\mathbb{R}^p\to M$$

$$(u_1,u_2,\dots,u_p)\to\phi_V(u_1,u_2,\dots,u_p)=\mathrm{Exp}\left(\sum_{\alpha=1}^p u_\alpha V_\alpha\right)$$

The function ϕ_V is actually a combination of Exp and a linear function

$$\eta_V:\mathbb{R}^p\to T_xM$$

defined by $\eta_V(u_1,\dots,u_p)=\sum_{\alpha=1}^p u_\alpha V_\alpha$. Thus ϕ_V is clearly a smooth function.
On the other hand, from the definition of η and Exp, we have $\phi_V(0)=(\mathrm{Exp}\circ\eta)(0)=\mathrm{Exp}(0_x)=x$. Also we have

$$(\phi_V)_{*0}\left(\frac{\partial}{\partial u_\alpha}\Big|_0\right)=\mathrm{Exp}_{*0_x}\left(\eta_{*0}\left(\frac{\partial}{\partial u_\alpha}\Big|_0\right)\right)=\mathrm{Exp}_{*0_x}\left(\frac{\partial\eta}{\partial u_\alpha}\Big|_0\right)=\mathrm{Exp}_{*0_x}(V_\alpha)=V_\alpha.$$

Therefore,

$$\Omega(j^1(\phi_V))=\left((\phi_V)_{*0}\left(\frac{\partial}{\partial u_1}\Big|_0\right),(\phi_V)_{*0}\left(\frac{\partial}{\partial u_2}\Big|_0\right),\dots,(\phi_V)_{*0}\left(\frac{\partial}{\partial u_p}\Big|_0\right)\right)=(V_1,V_2,\dots,V_p)_x.$$

Then, Ω is a bijection.

Now, we continue with the local form of the function Ω. For any $j^1\phi\in J^1_pM$ the local form is $j^1\phi\in J^1_pM=[x,X_\alpha]$, where $X_\alpha=\phi^\alpha_*(\frac{d}{du}|_0)$. Therefore,

$$\Omega(j^1\phi)=\Omega([x,X_\alpha])=(X_1,X_2,\dots,X_p)|_x.$$

Thus, the local representation of Ω is the identity map of $\mathbb{R}^{mp+m}$, concluding the proof. □

Remark 3.1. The function Ω defined in Definition 3.1 is a natural generalization of the diffeomorphism in [8, Proposition 4].

So far, we have showed that the total space of the two bundles are equivalent. In the next section we focus on the bundle structure of the jet bundle J^1_pM, and define a vector bundle structure on it.

4. The Vector Bundle J^1_pM and Its Tangent Bundle

In this section, we prove that the bundle carries a vector bundle structure, and also, this chapter provides local expressions of each tangent vector on J^1_pM. First, we introduce the vector bundle structure of J^1_pM.

4.1 J^1_pM as a Vector Bundle

It is well known that the fiber space of the bundle J^1_pM is the vector space $L(\mathbb{R}^p,\mathbb{R}^m)$. In order for a bundle to carry a vector bundle structure, one needs to show that each fiber carries a vector space structure and is isomorphic to its fiber space. In the following theorem, we define this structure:

Theorem 4.1. *Let $j^1f, j^1g \in \pi^{-1}\{x\}$ be two jets in the same fiber. Therefore, we conclude that $f,g:\mathbb{R}^p \to M$ smooth functions where $f(0)=g(0)=x$. Suppose that $[x,X_\alpha]$, $[x,Y_\alpha]$ are the local coordinates (as in equation (2.2)) of j^1f, and j^1g respectively, and $c \in \mathbb{R}$ a scalar. We define*

$$+(f,g)(u)=\exp\left(\sum_{\alpha=1}^{p}(u_\alpha(X_\alpha+Y_\alpha))\right), \tag{4.1}$$

$$\bullet(c,f)=\exp\left(\sum_{\alpha=1}^{p}(u_\alpha(cX_\alpha))\right). \tag{4.2}$$

Then, $+(f,g)$ and $\bullet(c,f)$ are differentiable functions.

Proof. Suppose that $j^1f, j^1g \in \pi^{-1}\{x\}$ with the local forms $[x,X_\alpha]$, $[x,Y_\alpha]$, respectively. From equation (2.2), we know that $x=f(0)=g(0)$ and $X_\alpha=(f_\alpha)_*\left(\frac{d}{du}\big|_0\right)$ and $Y_\alpha=(g_\alpha)_*\left(\frac{d}{du}\big|_0\right)$. By Remark 2.1, we have $X_\alpha=f_{*0}\left(\frac{\partial}{\partial u_\alpha}\big|_0\right)$ and $Y_\alpha=g_{*0}\left(\frac{\partial}{\partial u_\alpha}\big|_0\right)$. Thus $X_\alpha, Y_\alpha \in T_xM$ are tangent vectors at the point x, then $X_\alpha+Y_\alpha \in T_xM$ and $cX_\alpha \in T_xM$ as well.

Since M is paracompact, a linear connection ∇ on M, and a normal neighbourhood N of $x \in M$, exists such that there is $\widetilde{U} \subset T_xM$ a neighborhood such that $\exp:\widetilde{U} \to N$ is a diffeomorphism. Now, consider such neighborhood in T_xM such that $\sum_{\alpha=1}^{p} u_\alpha(X_\alpha+Y_\alpha) \in \widetilde{U}$, for all $(u_1,u_2,\dots,u_p) \in U$, where U be a small neighborhood of $0 \in \mathbb{R}^p$. We now define functions $\eta_{(X+Y)}:\mathbb{R}^p \to T_xM$, and $\eta_{cX}:\mathbb{R}^p \to T_xM$ defined by

$$\eta_{(X+Y)}(u_1,\dots,u_p)=\sum_{\alpha=1}^{p} u_\alpha(X_\alpha+Y_\alpha),$$

$$\eta_{(cX)}(u_1,\dots,u_p)=\sum_{\alpha=1}^{p} u_\alpha(cX_\alpha),$$

where $X,Y \in T_xM$ tangent vectors, and $c \in \mathbb{R}$ a scalar. From the definition of the functions above, both $\eta_{(X+Y)}$ and η_{cX} are linear functions. Since the exponential function exp is differentiable, then the functions $+(f,g)$ and $\bullet(c,f)$ are differentiable. □

Now, we consider the jets of $j^1(+(f,g))$ and $j^1(\bullet(c,f))$.

Lemma 4.1. *If $j^1f, j^1g \in \pi^{-1}\{x\}$, then $j^1(+(f,g)) \in \pi^{-1}\{x\}$, and $j^1(\bullet(c,f)) \in \pi^{-1}\{x\}$.*

Proof. Let $j^1f, j^1g \in \pi^{-1}\{x\}$, then $\pi(j^1f) = \pi(j^1g) = x$. On the other hand, since $X_\alpha + Y_\alpha \in T_xM$, then $+(f,g)(0) = \exp\left(\sum_{\alpha=1}^{p}(u_\alpha(cX_\alpha))\right) = \exp(0_x) = x$. Here 0_x represents the zero vector of T_xM. Therefore $\pi(j^1 + (f,g)) = +(f,g)(0) = x$, i.e. $j^1(+(f,g)) \in \pi^{-1}\{x\}$. Using the same method, it can be proven that $j^1(\bullet(c,f)) \in \pi^{-1}\{x\}$. □

In the following definition, we define the addition and multiplication of two jets explicitly:

Definition 4.1. Let $j^1f, j^1g \in \pi^{-1}\{x\}$ and $c \in \mathbb{R}$. We define

$$\begin{aligned} &\boxplus : \pi^{-1}\{x\} \times \pi^{-1}\{x\} \to \pi^{-1}\{x\} \\ &\qquad (j^1f, j^1g) \to j^1f \boxplus j^1g = j^1(+(f,g)), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\boxdot : \mathbb{R} \times \pi^{-1}\{x\} \times \pi^{-1}\{x\} \to \pi^{-1}\{x\} \\ &\qquad (c, j^1f) \to c \boxdot j^1f = j^1(\bullet(c,f)) \end{aligned} \tag{4.4}$$

Remark 4.1. The local expressions of the addition $j^1f + j^1g = j^1\phi^+_{f,g}$ and the scalar multiplication $cj^1f = j^1\phi^c_f$ are as follows:

$$\begin{aligned} j^1\phi^+_{f,g} &\equiv [x, X_\alpha + Y_\alpha], \\ j^1\phi^c_f &\equiv [x, cX_\alpha]. \end{aligned} \tag{4.5}$$

The proof directly stems from the definitions of the operations $\boxplus$ and $\boxdot$ outlined in Definition 4.1. Therefore we skip the proof.

Theorem 4.2. *$\boxplus$ and $\boxdot$ are well defined.*

Proof. Let $j^1f, j^1g, j^1\bar{f}, j^1\bar{g} \in \pi^{-1}\{x\}$ and their local forms are given by $[x, X_\alpha]$, $[x, Y_\alpha]$, $[x, \bar{X}_\alpha]$, $[x, \bar{Y}_\alpha]$, respectively.
Suppose that$(j^1f, j^1g) = (j^1f_1, j^1g_1)$. Then, $\frac{\partial(x^i \circ f)}{\partial u_\alpha}\Big|_{u=0} = \frac{\partial(x^i \circ f_1)}{\partial u_\alpha}\Big|_{u=0}$, and $\frac{\partial(x^i \circ g)}{\partial u_\alpha}\Big|_{u=0} = \frac{\partial(x^i \circ g_1)}{\partial u_\alpha}\Big|_{u=0}$. Then, $X_\alpha = \bar{X}_\alpha$, and $Y_\alpha = \bar{Y}_\alpha$. From equation (4.5), the local form of $j^1f \boxplus j^1g = j^1(+(f,g)) = [x, X_\alpha + Y_\alpha]$ and the local form of $j^1\bar{f} \boxplus j^1\bar{g} = j^1(+(\bar{f},\bar{g})) = [x, \bar{X}_\alpha + \bar{Y}_\alpha]$. Then, the local forms of $j^1f \boxplus j^1g$ and $j^1\bar{f} \boxplus j^1\bar{g}$ are equal. Therefore, $\boxplus$ is well defined. Using the same way, one can prove that $\boxdot$ is well defined too. □

By Theorem 4.2, we defined addition and scalar multiplication on J^1_pM.

Corollary 4.1. *With the addition and the scalar multiplication defined by equations* (4.1) *and* (4.2)*, each fiber $\pi^{-1}\{x\}$ is a vector space. $+(f,g)(0) = \exp(\eta_{(X+Y)}(0)) = \exp(0_x) = x$, where 0_x represents the zero vector of T_xM.*

We skip the proof as it can be proven with using the definitions of $j^1\phi^+_{f,g}$ and $j^1\phi^c_f$. In the next theorem, we show that J^1_pM carries a vector bundle structure.

Theorem 4.3. *With the bundle structures given by equation* (2.1)*, and the vector space structure on each fiber given by equation* (4.5)*, J^1_pM is a vector bundle on M with fiber space $L(\mathbb{R}^p, \mathbb{R}^m)$*

and bundle trivializations

$$\varphi : J^1_p U \to U \times L(\mathbb{R}^p, \mathbb{R}^m),$$

where u_α *is the usual coordinate functions of* $\mathbb{R}^p$, $1 \le i \le m$, $1 \le \alpha \le p$, *and* U *is an open subset of* M.

Proof. The coordinate charts of $J^1_p M$ is given by equation (2.1). Using these given structures, $J^1_p M$ is a smooth bundle. Now we present the vector bundle structure. Each fiber is defined as $\pi^{-1}\{x\} = \{j^1 f : \phi(0) = x\}$. Here recall that $f : \mathbb{R}^p \to M$ smooth functions. The addition and scalar multiplication on each fiber as follows:

$$j^1 f + j^1 g = [x, X_\alpha + X'_\alpha],$$
$$\lambda j^1 f = [x, \lambda X_\alpha],$$

where $j^1 f = [x, X_\alpha]$, and $j^1 g = [x, X'_\alpha]$. Using the operations defined in Definition 4.1, fibers of $J^1_p M$ carry vector space structure. The fiber map

$$\varphi_x : \pi^{-1}\{x\} \to L(\mathbb{R}^p, \mathbb{R}^m)$$
$$j^1 f \to \left[\frac{\partial (x^i \circ f)}{\partial u^\alpha} \Big|_0 \right] \tag{4.6}$$

for all $x \in M$. Therefore, we have

$$\begin{aligned} \varphi_x(j^1 f + \lambda j^1 g) &= \varphi_x([x, X_\alpha] + [x, \lambda X'_\alpha]) \\ &= \varphi_x([x, X_\alpha + \lambda X'_\alpha)] \\ &= \left[\frac{\partial (x^i \circ f)}{\partial u^\alpha} \Big|_0 + \lambda \frac{\partial (x^i \circ g)}{\partial u^\alpha} \Big|_0 \right] \\ &= \varphi_x(j^1 f) + \lambda \psi_x(j^1 g). \end{aligned}$$

Now we prove that it is a bijection:

Let $\varphi_x(j^1 f) = [0] \in L(\mathbb{R}^p, \mathbb{R}^m)$. Then $\left[\frac{\partial (x^i \circ f)}{\partial u^\alpha}\big|_0\right] = [0]$ for all $1 \le i \le m$, and for all α, $1 \le \alpha \le p$. Then $j^1 f = [x, 0]$, which proves that φ_x is one to one. From the rank-nullity theorem, φ_x is a bijection. Thus φ_x is a linear isomorphism for all $x \in M$. □

4.2 The Tangent Bundle $TJ^1_p M$

In this section, we compute the tangent vectors of $J^1_p M$. For this purpose, we define the local coordinates of a curve on $J^1_p M$.

Remark 2.2 points out that for each curve ϕ corresponding to a tangent vector $\dot{\phi}$, there exists functions ψ, ψ_t, ψ^u as in equation (2.3) and (2.4). Using the correspondence we define local coordinates of $\phi(t)$ if ϕ is a smooth curve on the bundle $J^1_p M$ as following:

Definition 4.2. If ϕ is a curve on the bundle $J^1_p M$, then the local coordinates of $\phi(t)$ is

$$\phi(t) = \left[(x^i \circ \psi_t)(0); \frac{\partial (x^i \circ \psi_t)}{\partial u_\alpha} \Big|_{u=0} \right],$$

where $\psi : \mathbb{R} \times \mathbb{R}^p \to M$ is a smooth function such that $\phi(t) = j^1(\psi_t)$ for $\psi_t(u) = \psi(t, u) = \psi^u(t)$, for all $(t, u) \in \mathbb{R} \times \mathbb{R}^p$.

Lemma 4.2. *Let $\dot{\phi} \in T(J^1_p M)$ and ψ, ψ_t, ψ^u are the smooth functions defined by equations* (2.3) *and* (2.4)*. Then the local form of $\dot{\boldsymbol{\phi}}$ is given by:*

$$\dot{\phi} \in T(J^1_p M) \equiv \left(\left[(x^i \circ \psi)(0,0); \frac{\partial (x^i \circ \psi_0)}{\partial u_\alpha}\Big|_{u=0} \right], \left[(\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{0}}); \frac{\partial (\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{u}})}{\partial u_\alpha}\Big|_0 \right] \right). \tag{4.7}$$

Proof. Let $\phi : \mathbb{R} \to J^1_p M$ be a smooth curve such that, for all $t \in \mathbb{R}$, local form of $\phi(t)$, as defined in (4.2), is $\phi(t) = \left[(x^i \circ \psi_t)(0); \frac{\partial (x^i \circ \psi_t)}{\partial u_\alpha}\big|_{u=0} \right]$. The derivative of ϕ is given by:

$$\frac{d}{dt}\Big|_{t=0} (j^1(\psi_t)) = \frac{d}{dt}\Big|_{t=0} (x^i \circ \psi_t(0)) \frac{\partial}{\partial x^i}\Big|_{j^1\psi_0} + \frac{d}{dt}\Big|_{t=0} (X^i_\alpha \circ \psi_t) \frac{\partial}{\partial X^i_\alpha}\Big|_{j^1\psi_0}. \tag{4.8}$$

As $\dot{\boldsymbol{\phi}} \in T_{j^1\psi_0}(J^1_p M)$, the first two components of local form are $(x^i \circ \psi_0)(0)$, and $\frac{\partial (x^i \circ \psi_0)}{\partial u_\alpha}\big|_{u=0}$.

The computation of the terms in equation (4.8) is as follows:

$$\frac{d}{dt}\Big|_{t=0} (x^i \circ \psi_t)(0)) = \frac{d}{dt}\Big|_{t=0} (x^i \circ \psi^0)(t)) = (x^i \circ \psi^0)'(0).$$

Additionally,

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \left(\frac{\partial (x^i \circ \psi_t)}{\partial u_\alpha}\Big|_{u=0} \right) &= \frac{d}{dt}\Big|_{t=0} \left(\left(\frac{\partial (x^i \circ \psi)(t,u)}{\partial u_\alpha} \right)\Big|_{u=0} \right) \\
&= \frac{\partial^2 (x^i \circ \psi)(t,u)}{dt \partial u_\alpha}\Big|_{(0,0)}.
\end{aligned}$$

Since ψ is a smooth function, we have

$$\begin{aligned}
\frac{\partial^2 (x^i \circ \psi)(t,u)}{dt \partial u_\alpha}\Big|_{(0,0)} &= \frac{\partial^2 (x^i \circ \psi)(t,u)}{\partial u_\alpha dt}\Big|_{(0,0)} \\
&= \frac{\partial}{\partial u_\alpha}\Big|_{u=0} \left(\frac{d (x^i \circ \psi)(t,u)}{dt}\Big|_{t=0} \right) \\
&= \frac{\partial}{\partial u_\alpha}\Big|_{u=0} \left(\frac{d (x^i \circ \psi^u)}{dt}\Big|_{t=0} \right) \\
&= \frac{\partial (\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{u}})}{\partial u_\alpha}\Big|_{u=0}.
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\Big|_{t=0} \left(\frac{\partial (x^i \circ \psi_t)}{\partial u_\alpha}\Big|_{u=0} \right) = \frac{\partial (\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{u}})}{\partial u_\alpha}\Big|_{u=0}.$$

Using this equation in equation (4.8), we have

$$\dot{\boldsymbol{\phi}} = (\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{0}}) \frac{\partial}{\partial x^i}\Big|_{j^1\psi_0} + \frac{\partial (\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{u}})}{\partial u_\alpha}\Big|_{u=0} \frac{\partial}{\partial X^i_\alpha}\Big|_{j^1\psi_0}.$$

Therefore, the local form of $\dot{\boldsymbol{\phi}}$ is $\left(\left[(x^i \circ \psi)(0,0); \frac{\partial (x^i \circ \psi_0)}{\partial u_\alpha}\big|_{u=0} \right], \left[(\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{0}}); \frac{\partial (\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{u}})}{\partial u_\alpha}\big|_0 \right] \right)$. □

In the following lemma, we demonstrate that the local coordinates of the tangent vector remain independent of the choice of ψ.

Lemma 4.3. *The local coordinates of the tangent vector $\dot{\phi}$ do not depend the choice of ψ.*

Proof. Suppose that ψ and $\bar{\psi}$ are the smooth functions corresponding to the tangent vector $\dot{\phi}$. According to Definition 4.2, the local coordinates of $\phi(t)$ for all $t \in \mathbb{R}$ are given by

$$\phi(t) = \left[(x^i \circ \psi_t)(0); \frac{\partial(x^i \circ \psi_t)}{\partial u_\alpha}\Big|_{u=0}\right] = \left[(x^i \circ \bar{\psi}_t)(0); \frac{\partial(x^i \circ \bar{\psi}_t)}{\partial u_\alpha}\Big|_{u=0}\right].$$

The following equalities hold for all $t \in \mathbb{R}$:

$$(x^i \circ \psi)(t,0) = (x^i \circ \bar{\psi})(t,0), \tag{4.9}$$

$$\frac{\partial(x^i \circ \psi_t)}{\partial u_\alpha}\Big|_{u=0} = \frac{\partial(x^i \circ \bar{\psi}_t)}{\partial u_\alpha}\Big|_{u=0}. \tag{4.10}$$

Now, by setting $t = 0$ in both equations, we obtain

$$(x^i \circ \psi)(0,0) = (x^i \circ \bar{\psi})(0,0), \quad \frac{\partial(x^i \circ \psi_0)}{\partial u_\alpha}\Big|_{u=0} = \frac{\partial(x^i \circ \bar{\psi}_0)}{\partial u_\alpha}\Big|_{u=0}.$$

Rewriting equations (4.9) and (4.10), we obtain

$$(x^i \circ \psi^0)(t) = (x^i \circ \bar{\psi}^0)(t) \tag{4.11}$$

and

$$\frac{\partial(x^i \circ \psi)(t,u)}{\partial u_\alpha}\Big|_{u=0} = \frac{\partial(x^i \circ \bar{\psi})(t,u))}{\partial u_\alpha}\Big|_{u=0}. \tag{4.12}$$

Therefore, third component of the local coordinates remains the same. Now, let's compute last components. Taking the derivative of both sides of equation (4.11) at the point $t = 0$, we get

$$(x^i \circ \psi^0)'(0) = (x^i \circ \bar{\psi}^0)'(0).$$

Similarly, taking the derivative of both sides of equation (4.12), we have

$$\frac{d}{dt}\Big|_{t=0}\left(\frac{\partial(x^i \circ \psi)(t,u)}{\partial u_\alpha}\Big|_{u=0}\right) = \frac{d}{dt}\Big|_{t=0}\left(\frac{\partial(x^i \circ \bar{\psi})(t,u)}{\partial u_\alpha}\Big|_{u=0}\right),$$

$$\Rightarrow \quad \frac{\partial^2(x^i \circ \psi)(t,u)}{dt\partial u_\alpha}\Big|_{(0,0)} = \frac{\partial^2(x^i \circ \bar{\psi})(t,u)}{\partial u_\alpha dt}\Big|_{(0,0)}.$$

Since both ψ and $\bar{\psi}$ are smooth functions, this implies

$$\frac{\partial}{\partial u_\alpha}\Big|_{u=0}\left(\frac{d(x^i \circ \psi^u)}{dt}\Big|_{t=0}\right) = \frac{\partial}{\partial u_\alpha}\Big|_{u=0}\left(\frac{d(x^i \circ \bar{\psi}^u)}{dt}\Big|_{t=0}\right)$$

$$\Rightarrow \quad \frac{\partial(x^i \circ \psi^u)'(0)}{\partial u_\alpha}\Big|_{u=0} = \frac{\partial(x^i \circ \bar{\psi}^u)'(0)}{\partial u_\alpha}\Big|_{u=0}.$$

Therefore, the last components of the local coordinates are the same, concluding the proof. □

Thus far, we have demonstrated the equivalence of the total spaces of the two bundles. To establish that the bundles are isomorphic, we now aim to show that the differential Ω_* is injective. To achieve this, we need to express the local form of Ω_*.

Lemma 4.4. *For a tangent vector $\dot{\boldsymbol{\phi}} \in T(J^1_p M)$, the local coordinates of $\Omega_*(\dot{\boldsymbol{\phi}})$ are given by*

$$\Omega_*(\dot{\boldsymbol{\phi}}) \equiv \left((x^i \circ \psi^0)(0), \frac{(x^i \circ \psi_0)}{\partial u_\alpha}\Big|_{u=0}; (\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{0}}), \frac{\partial(\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{u}})}{\partial u_\alpha}\Big|_{u=0}\right).$$

Proof. Let $\dot{\boldsymbol{\phi}} \in T(J^1_p M)$ be a tangent vector, and let ψ be the function corresponding to ϕ as in equations (2.3) and (2.4). Then

$$\left((\bar{x}^i; y^i_1, \dots, y^i_p)(\Omega\circ\phi)(0); \frac{d(\bar{x}^i\circ\Omega\circ\phi)}{dt}\Big|_{t=0}, \frac{d(y^i_1\circ\Omega\circ\phi)}{dt}\Big|_{t=0}, \frac{d(y^i_2\circ\Omega\circ\phi)}{dt}\Big|_{t=0}, \dots, \frac{d(y^i_p\circ\Omega\circ\phi)}{dt}\Big|_{t=0}\right),$$

where

$$(\Omega\circ\phi)(t) = \Omega(j^1\psi_t) = \left(\frac{\partial(x^i\circ\psi_t)}{\partial u_1}\Big|_{u=0}\frac{\partial}{\partial x^i}\Big|_{\psi_t(0)}, \frac{\partial(x^i\circ\psi_t)}{\partial u_2}\Big|_{u=0}\frac{\partial}{\partial x^i}\Big|_{\psi_t(0)}, \dots, \frac{\partial(x^i\circ\psi_t)}{\partial u_p}\Big|_{u=0}\frac{\partial}{\partial x^i}\Big|_{\psi_t(0)}\right).$$

We denote αth component of $(\Omega\circ\phi)(t)$ as $V_\alpha(t)$. Then

$$(\Omega\circ\Phi)(t) = (\psi_t(0); V_1(t), V_2(t), \dots, V_p(t)),$$

where $V_\alpha(t) = \frac{\partial(x^i\circ\psi_t)}{\partial u_\alpha}\big|_{u=0}\frac{\partial}{\partial x^i}\big|_{\psi_t(0)} \in T_{\psi_t(0)}M$. Finally,

$$\Omega_*(\dot{\boldsymbol{\phi}}) = (\Omega\circ\phi)'(0) \in T_{\Omega(j^1\psi_0)}(\oplus_p(T_xM)),$$

where $x = \psi_0(0)$.

From Equation 2.5, the local coordinate chart for $\oplus_p(TM)$ be $(\bar{x}^i; y^i_1, y^i_2, \dots, y^i_p)$ is given by $(\bar{x}^i; y^i_1, y^i_2, \dots, y^i_p)$, where

$$\bar{x}^i = x^i\circ\pi_{1-p}, \quad y^i_1 = \dot{x}^i\circ pr_1,\ y^i_2 = \dot{x}^i\circ pr_2,\ \dots\ y^i_p = \dot{x}^i\circ pr_p.$$

Now we compute each of the local coordinates of $(\Omega\circ\phi)(t)$:

$$\begin{aligned}\bar{x}^i((\Omega\circ\phi)(t)) &= (x^i\circ\pi_{1,2})((\Omega\circ\phi)(t))\\ &= (x^i\circ\pi_{1,2})(\Omega(j^i\psi_t))\\ &= \left((x^i\circ\pi_{1,2})(\psi_t)(0); \frac{\partial(x^i\circ\psi_t)}{\partial u_\alpha}\Big|_{u=0}\right)\\ &= (x^i\circ\psi^0)(t)\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}y^i_\alpha((\Omega\circ\phi)(t)) &= y^i_\alpha((\Omega\circ\phi)(t))\\ &= y^i_\alpha(\Omega(j^i\psi_t))\\ &= y^i_\alpha\left((\psi_t)(0); \frac{\partial(x^i\circ\psi_t)}{\partial u_\alpha}\Big|_{u=0}\right)\\ &= \frac{\partial(x^i\circ\psi_t)}{\partial u_\alpha}\Big|_{u=0}\end{aligned} \tag{4.14}$$

for all $\alpha\in\{1,2,\dots,p\}$. Therefore,

$$\begin{aligned}\Omega_*(\dot{\boldsymbol{\phi}}) = {}&\frac{d(\bar{x}^i\circ\Omega\circ\phi)}{dt}\Big|_{t=0}\frac{\partial}{\partial\bar{x}^i}\Big|_{\Omega(j^1\psi_0)} + \frac{d(y^i_1\circ\Omega\circ\phi)}{dt}\Big|_{t=0}\frac{\partial}{\partial y^i_1}\Big|_{\Omega(j^1\psi_0)} + \dots\\ &+ \frac{d(y^i_2\circ\Omega\circ\phi)}{dt}\Big|_{t=0}\frac{\partial}{\partial y^i_2}\Big|_{\Omega(j^1\psi_0)} + \frac{d(y^i_p\circ\Omega\circ\phi)}{dt}\Big|_{t=0}\frac{\partial}{\partial y^i_p}\Big|_{\Omega(j^1\psi_0)}.\end{aligned}$$

Conversely, using the local coordinates of $(\Omega\circ\phi)(t)$, we have

$$\Omega_*(\dot{\boldsymbol{\phi}}) = \left(\frac{d(\bar{x}^i\circ\Omega\circ\phi)}{dt}\Big|_{t=0}, \frac{d(y^i_1\circ\Omega\circ\phi)}{dt}\Big|_{t=0}, \frac{d(y^i_2\circ\Omega\circ\phi)}{dt}\Big|_{t=0}, \dots, \frac{d(y^i_p\circ\Omega\circ\phi)}{dt}\Big|_{t=0}\right)_{(\Omega\circ\phi)(0)}.$$

From equation (4.13), the derivative is $\frac{d(\bar{x}^i \circ \Omega \circ \phi)}{dt}\Big|_{t=0} = (\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\mathbf{0}})$.

On the other hand, referring to equation (4.14), we observe

$$\begin{aligned}
\frac{d(y^i_\alpha \circ \Omega \circ \phi)}{dt}\Big|_{t=0} &= \frac{d}{dt}\Big|_{t=0}\left(\frac{\partial(x^i \circ \psi_t)}{\partial u_\alpha}\Big|_{u=0}\right) \\
&= \frac{\partial^2(x^i \circ \psi(t,u))}{dt \partial u_\alpha}\Big|(0,0) \\
&= \frac{\partial^2(x^i \circ \psi(t,u))}{\partial u_\alpha dt}\Big|(0,0) \\
&= \frac{\partial}{\partial u_\alpha}\Big|_{u=0}\left(\frac{d}{\partial(x^i \circ \psi^u)(t)}\Big|_{t=0}\right) \\
&= \frac{\partial(\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{u}})}{\partial u_\alpha}\Big|_{u=0},
\end{aligned}$$

for all $\alpha \in \{1,2,\dots,p\}$. Then, the local coordinates of $\Omega_*(\dot{\boldsymbol{\phi}})$ are

$$\Omega_*(\dot{\boldsymbol{\phi}}) \equiv \left((x^i \circ \psi^0)(0), \frac{(x^i \circ \psi_0)}{\partial u_\alpha}\Big|_{u=0}; (\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\mathbf{0}}), \frac{\partial(\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{u}})}{\partial u_\alpha}\Big|_{u=0}\right). \qquad \square$$

Theorem 4.4. Ω *is both an immersion and submersion.*

Proof. Let the local function of Ω_* be $\widetilde{\Omega}_*$, and $\dot{\boldsymbol{\phi}} \in T(J^1_pM)$ be a tangent vector. From Lemma 4.2, the local form of $\dot{\boldsymbol{\phi}}$ is

$$\left(\left[(x^i \circ \psi)(0,0); \frac{\partial(x^i \circ \psi_0)}{\partial u_\alpha}\Big|_{u=0}\right], \left[(\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\mathbf{0}}); \frac{\partial}{\partial u_\alpha}\Big|_0 (\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{u}})\right]\right).$$

On the other hand, from Lemma 4.4, the local coordinate chart of $\Omega_*(\dot{\boldsymbol{\phi}})$ is

$$\Omega_*(\dot{\boldsymbol{\phi}}) \equiv \left((x^i \circ \psi^0)(0), \frac{(x^i \circ \psi_0)}{\partial u_\alpha}\Big|_{u=0}; (\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\mathbf{0}}), \frac{\partial(\boldsymbol{x^i} \circ \dot{\boldsymbol{\psi}}^{\boldsymbol{u}})}{\partial u_\alpha}\Big|_{u=0}\right).$$

Therefore, the local function $\widetilde{\Omega}_* : \mathbb{R}^{mp+p} \times \mathbb{R}^{mp+p} \to \mathbb{R}^{mp+p} \times \mathbb{R}^{mp+p}$ is an identity map. Hence, Ω is both an immersion and submersion. $\square$

Corollary 4.2. *With the vector bundle structure given in Theorem* 4.3, J^1_pM *is bundle isomorphic to the Whitney sum* $\oplus_p(TM)$.

Proof. Theorem 3.1 indicates that the total spaces of the bundles J^1_pM and $\oplus_p(TM)$ are diffeomorphic. It can easily be seen that Figure 1 commutes.

$$\begin{array}{ccc}
J^1_pM & \xrightarrow{\Omega} & \oplus_p(TM) \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi_M} \\
M & \xrightarrow{id_M} & M
\end{array}$$

Figure 1

Therefore, Ω is a fiber-preserving map. The local form of Ω is the identity map. Therefore, Ω maps the fiber $J^1_pM_x$ to the fiber $\oplus_p(TM)_x$.

Now lets consider the restriction of Ω to the fiber $J_p^1 M_x$. $\Omega_{J_p^1 M_x} : J_p^1 M_x \to \oplus_p(T_x M)$. Let $j^1\sigma, j^1\tau \in J_p^1 M_x$. Then, $\sigma(0) = \tau(0) = x$. Thus, Ω is a vector bundle isomorphism. □

5. Conclusion

We demonstrate that the jet bundle $J_p^1 M$ can be expressed as the Whitney sum of p tangent bundles. For future research, the vector bundle isomorphism between the p-jet bundle and the p-Whitney sum can be used to lift a Riemannian metric on the manifold M to $J_p^1 M$. This can be achieved by extending the product metric $\oplus_p g$ on $\oplus_p(TM)$ to $J_p^1 M$ and using the pull back metric along the isomorphism Ω. This approach opens the door to studying the behavior and properties of Riemannian metrics in the context of jet bundles, which could lead to new insights in Riemannian geometry and its applications in related fields.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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