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Research Article

Observations on Haüy Rhombic Dodecahedral Numbers with Some Special Numbers

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Abstract. In this paper, the basic definition of Haüy rhombic dodecahedral numbers and nasty numbers are given. The Haüy rhombic dodecahedral number is taken into study and we prove this number is a combination of some special numbers. Also, we investigate about the relationships between the Haüy rhombic dodecahedral, stella octangula, hex number, centered tetrahedral and gnomonic numbers. For the clear understanding of the properties of special numbers we proved fifteen theorems. Sixteen special numbers are taken into consideration for this observation. The key points are given in the proof of the theorems.

Keywords. Haüy rhombic dodecahedral number, Stella octangula number, Centered tetrahedral number, Dodecahedral numbers, Gnomonic numbers

Mathematics Subject Classification (2020). 11B39, 11Y55

1. Introduction

There are an endless number of unanswered puzzles presented by Haüy rhombic dodecahedral sums. Significant absorption in both homogeneous and non-homogeneous Haüy rhombic dodecahedral sums is revealed by the mathematicians (Özkam *et al*. [10], Reka [12], Shanmuganandham and Deepika [13], Keskin and Karaath [6]). Anyone can make reference to our sources in this situation for a variety of Haüy rhombic dodecahedral sums with unknowns difficulties (Gnanam and Anitha [5], Miller [8], Rajpoot [11], Niven *et al*. [9], Dance and Denze [3], Conway and Guy [2]). Here, consideration is given to the sums' whole number solutions. This increases the sum's interest a little bit. Additionally, there are a few connections made between the numbers and the special integers (Bezdek [1], Meyyappan [7], Sloane [14], Friedberg [4]).

2. Preliminaries

Definition 2.1. The general form of Haüy rhombic dodecahedral number is given by $k^3 + b \sum_{i=13,\dots,k-2} i^2$ for k an odd number, when $k = 2n-1$ gives

$$(\text{HauyRhoDod})_n = (2n-1)(8n^2-14n+7).$$

Example 2.2. When $n = 1,2,3,\dots$, respectively, Definition 2.1 yields $1,33,185,553,1233,\dots$.

Definition 2.3. A natural number N is said to be '*nasty number*' if N can be written as the product of two numbers in to different ways such that the sum of one pair is equal to the difference of other pair.

Example 2.4. Let $N = 24$,

$$24 = 4\times 6 = 12\times 2,$$
$$4+6 = 10 = 12-2.$$

Therefore, 24 is a nasty number.

3. Method of Analysis

Theorem 3.1. $(\text{HRdo})_n = 8(\text{Soct})_n - 12(\text{Hex})_n + 36(\text{Gno})_n + 41$.

Proof.

$$\begin{aligned}(\text{HRdo})_n &= 16n^3-36n^2+28n-7\\ &= 16n^3-(36n^2+36n+12)+64n+5\\ &= (16n^3+64n+5)-12(\text{Hex})_n\\ &= 8(\text{Soct})_n-12(\text{Hex})_n+(72n-36)+41\\ &= 8(\text{Soct})_n-12(\text{Hex})_n+36(\text{Gno})_n+41.\end{aligned}$$

Hence

$$(\text{HRdo})_n - 8(\text{Soct})_n + 12(\text{Hex})_n - 36(\text{Gno})_n \equiv 0 \pmod{41}.$$ □

Theorem 3.2. $24(\text{Ctet})_n - 6C_{20,n} + 16(\text{Gno})_n - (\text{HRdo})_n$ *is a perfect square number.*

Proof.

$$\begin{aligned}(\text{HRdo})_n &= 16n^3-36n^2+28n-7\\ &= (16n^3+24n^2+56n+24)-60n^2-28n-31\\ &= 24(\text{Ctet})_n-(60n^2+60n+6)+32n-25\\ &= 24(\text{Ctet})_n-6C_{20,n}+(32n-16)-9\\ &= 24(\text{Ctet})_n-6C_{20,n}+16(\text{Gno})_n-9.\end{aligned}$$

Hence

$24(\text{Ctet})_n - 6C_{20,n} + 16(\text{Gno})_n - (\text{HRdo})_n$ is a perfect square number. □

Theorem 3.3. $144(\text{Ctet})_n - 36C_{20,n} + 96(\text{Gno})_n - 6(\text{HRdo})_n$ *is a nasty number.*

Proof. From Theorem 3.2, we get

$$144(\text{Ctet})_n - 36C_{20,n} + 96(\text{Gno})_n - 6(\text{HRdo})_n = 9 \text{ is a nasty number.}$$ □

Theorem 3.4. $24(\text{oc})_n - 6P_{14,n} - 5(\text{Gno})_n - (\text{HRdo})_n \equiv 0 \pmod{2,3,4,6,12}$.

Proof.

$$\begin{aligned}(\text{HRdo})_n &= 16n^3 - 36n^2 + 28n - 7\\ &= (16n^3 + 8n) - 36n^2 + 20n - 7\\ &= 24(\text{oc})_n - (36n^2 - 30n) - 10n - 7\\ &= 24(\text{oc})_n - 6P_{14,n} - (10n - 5) - 12\\ &= 24(\text{oc})_n - 6P_{14,n} - 5(\text{Gno})_n - 12\end{aligned}$$

$$24(\text{oc})_n - 6P_{14,n} - 5(\text{Gno})_n - (\text{HRdo})_n = 12.$$

Hence

$$24(\text{oc})_n - 6P_{14,n} - 5(\text{Gno})_n - (\text{HRdo})_n \equiv 0 \pmod{2,3,4,6,12}.$$ □

Corollary 3.1. $48(\text{oc})_n - 12P_{14,n} - 10(\text{Gno})_n - 2(\text{HRdo})_n$ *forms nasty number.*

Theorem 3.5. $(\text{HRdo})_n = 12(\text{Coc})_n - 4C_{30,n} + 28(\text{Gno})_n + 13$.

Proof.

$$\begin{aligned}(\text{HRdo})_n &= 16n^3 - 36n^2 + 28n - 7\\ &= (16n^3 + 24n^2 + 32n + 12) - 60n^2 - 4n - 19\\ &= 12(\text{Coc})_n - (60n + 60n^2 + 4) + 56n - 15\\ &= 12(\text{Coc})_n - 4C_{30,n} + (56n - 28) + 13(\text{HRdo})_n\\ &= 12(\text{Coc})_n - 4C_{30,n} + 28(\text{Gno})_n + 13.\end{aligned}$$ □

Corollary 3.2. $(\text{HRdo})_n - 12(\text{Coc})_n + 4C_{30,n} - 28(\text{Gno})_n \equiv 0 \pmod{13}$.

Theorem 3.6. $(\text{HRdo})_n - 4(\text{Ode})_n + (\text{Soct})_n + 12C_{13,n} - 20(\text{Gno})_n - n$ *is a square.*

Proof.

$$\begin{aligned}(\text{HRdo})_n &= 16n^3 - 36n^2 + 28n - 7\\ &= (18n^3 - 18n^2 + 4n) - 2n^3 - 18n^2 + 24n - 7\\ &= 4(\text{Ode})_n - (2n^3 - n) - 18n^2 + 23n - 7\\ &= 4(\text{Ode})_n - (\text{Soct})_n - (18n^2 + 18n + 12)\\ &= 4(\text{Ode})_n - (\text{Soct})_n - 12C_{3,n} + 41n + 5\\ &= 4(\text{Ode})_n - (\text{Soct})_n - 12C_{3,n} + 20(\text{Gno})_n + n + 25\end{aligned}$$

$$(\text{HRdo})_n - 4(\text{Ode})_n + (\text{Soct})_n + 12C_{13,n} - 20(\text{Gno})_n - 25 = 0.$$

Hence

$$(\text{HRdo})_n - 4(\text{Ode})_n + (\text{Soct})_n + 12C_{13,n} - 20(\text{Gno})_n - n \text{ is a square.}$$ □

Corollary 3.3. $6[(\text{HRdo})_n - 4(\text{Dde})_n + (\text{Soct})_n + 12C_{13,n} - 20(\text{Gno})_n - n]$ *forms a nasty number.*

Theorem 3.7. $(\text{HRdo})_n = 6(\text{Ico})_n + 6(\text{Tet})_n - 24(\text{Pro})_n + 22(\text{Gno})_n + 15$.

Proof.

$$\begin{aligned}(\mathrm{HRdo})_n &= 16n^3-36n^2+28n-7\\ &= (15n^3-15n^2+6n)+n^3-21n^2+22n-7\\ &= 6(\mathrm{Ico})_n+(n^3+3n^2+2n)-24n^2+20n-7\\ &= 6(\mathrm{Ico})_n+6(\mathrm{Tet})_n-(24n+24n)+44n-7\\ &= 6(\mathrm{Ico})_n+6(\mathrm{Tet})_n-24(\mathrm{Pro})_n+(44n-22)+15.\end{aligned}$$

Hence

$$(\mathrm{HRdo})_n = 6(\mathrm{Ico})_n+6(\mathrm{Tet})_n-24(\mathrm{Pro})_n+22(\mathrm{Gno})_n+15.$$ □

Theorem 3.8. $(\mathrm{HRdo})_n-(\mathrm{Pr}\,i)_{16}^n+4C_{10,n}-(46n-3)=0$.

Proof.

$$\begin{aligned}(\mathrm{HRdo})_n &= 16n^3-36n^2+28n-7\\ &= (16n^3-16n^2+n)-20n^2+26n-7\\ &= (\mathrm{Pr}\,i)_{16}^n-(20n^2+20n+4)+46n-3\\ &= (\mathrm{Pr}\,i)_{16}^n+4C_{10,n}-(46n-3).\end{aligned}$$

Hence

$$(\mathrm{HRdo})_n-(\mathrm{Pr}\,i)_{16}^n+4C_{10,n}-(46n-3)=0.$$ □

Theorem 3.9. $(\mathrm{HRdo})_n=4(\mathrm{Rdod})_n-10P_{14,n}-19(\mathrm{Gno})_n-30$.

Proof.

$$\begin{aligned}(\mathrm{HRdo})_n &= 16n^3-36n^2+28n-7\\ &= (16n^3+24n^2+16n+4)-60n^2+12n-11\\ &= 4(\mathrm{Rdod})_n-(60n^2-50n)-38n-11\\ &= 4(\mathrm{Rdod})_n-10P_{14,n}-(38n-19)-30\\ &= 4(\mathrm{Rdod})_n-10P_{14,n}-19(\mathrm{Gno})_n-30.\end{aligned}$$

Hence

$$(\mathrm{HRdo})_n=4(\mathrm{Rdod})_n-10P_{14,n}-19(\mathrm{Gno})_n-30.$$ □

Theorem 3.10. $(\mathrm{HRdo})_n-6(\mathrm{Ico})_n-6P_{9,n}-7(n-1)$ *is a cubic integer.*

Proof.

$$\begin{aligned}(\mathrm{HRdo})_n &= 16n^3-36n^2+28n-7\\ &= (15n^3-15n^2+6n)+n^3-21n^2+12n-7\\ &= 6(\mathrm{Ico})_n+n^3-(21n^2-15n)+7n-7\\ &= 6(\mathrm{Ico})_n-6P_{9,n}+7(n-1)+n^3.\end{aligned}$$

Hence

$(\mathrm{HRdo})_n-6(\mathrm{Ico})_n-6P_{9,n}-7(n-1)$ is a cubic integer. □

Theorem 3.11.

$$8(\mathrm{CCG})_n-8P_{17,n}-24(\mathrm{Gno})_n-(\mathrm{HRdo})_n\equiv 0 \pmod 3.$$

Proof.

$$
\begin{aligned}
(\mathrm{HRdo})_n &= 16n^3 - 36n^2 + 28n - 7 \\
&= (16n^3 + 24n^2 + 24n + 8) - 60n^2 + 4n - 15 \\
&= 8(\mathrm{CCG})_n - (60n - 52n) - 48n - 15 \\
&= 8(\mathrm{CCG})_n - 8P_{17,n} - (48n - 24) - 39 \\
&= 8(\mathrm{CCG})_n - 8P_{17,n} - 24(\mathrm{Gno})_n - 39 \\
&= 8(\mathrm{CCG})_n - 8P_{17,n} - 24(\mathrm{Gno})_n - (\mathrm{HRdo})_n = 39.
\end{aligned}
$$

Hence

$$8(\mathrm{CCG})_n - 8P_{17,n} - 24(\mathrm{Gno})_n - (\mathrm{HRdo})_n \equiv 0 \pmod 3$$

Theorem 3.12.

$$(\mathrm{HRdo})_n = 6(\mathrm{Cpy})_{11}^n + 2[(\mathrm{Ico})_n + C_{11,n} + C_{20,n}] + 31(\mathrm{Gno})_n + 20.$$

Proof.

$$
\begin{aligned}
(\mathrm{HRdo})_n &= 16n^3 - 36n^2 + 28n - 7 \\
&= (16n^3 - 5n^2 - 3n) - 31n^2 + 31n - 7 \\
&= 6(\mathrm{Cpy})_{11}^n + 2(\mathrm{Ico})_n - (31n^2 + 31n + 4) + 62n - 11 \\
&= 6(\mathrm{Cpy})_{11}^n + 2(\mathrm{Ico})_n - 2C_{11,n} - 2C_{20,n} + (62n - 31) + 20 \\
&= 6(\mathrm{Cpy})_{11}^n + 2[(\mathrm{Ico})_n + C_{11,n} - C_{20,n}] + 31(\mathrm{Gno})_n + 20.
\end{aligned}
$$

Hence

$$(\mathrm{HRdo})_n = 6(\mathrm{Cpy})_{11}^n + 2[(\mathrm{Ico})_n + C_{11,n} + C_{20,n}] + 31(\mathrm{Gno})_n + 20.$$ □

Theorem 3.13. $(\mathrm{HRdo})_n = 6[(\mathrm{TruTet})_n - (\mathrm{Py})_n^9] - 2P_{8,n} + 5(\mathrm{Gno})_n - 2.$

Proof.

$$
\begin{aligned}
(\mathrm{HRdo})_n &= 16n^3 - 36n^2 + 28n - 7 \\
&= (16n^3 - 30n^2 - 14n) - 6n^2 + 14n - 7 \\
&= 6[(\mathrm{TruTet})_n - (\mathrm{Py})_n^9] - (6n^2 - 4n) + 10n - 7 \\
&= 6[(\mathrm{TruTet})_n - (\mathrm{Py})_n^9] - 2P_{8,n} + 5(\mathrm{Gno})_n - 2.
\end{aligned}
$$

Hence

$$(\mathrm{HRdo})_n = 6[(\mathrm{TruTet})_n - (\mathrm{Py})_n^9] - 2P_{8,n} + 5(\mathrm{Gno})_n - 2.$$ □

Theorem 3.14. $n(\mathrm{HRdo})_n + 8[(\mathrm{Ode})_n - (\mathrm{Gno})_n] + 4C_{4,n} - 5 = (2n)^4$ *is a even bi-quadratic number.*

Proof.

$$
\begin{aligned}
(\mathrm{HRdo})_n &= 16n^3 - 36n^2 + 28n - 7 \\
n(\mathrm{HRdo})_n &= 16n^4 - (36n^3 - 36n^2 + 8n) - 8n^2 + 8n - 7 \\
&= 16n^4 - 8(\mathrm{Ode})_n - (8n^2 + 8n + 4) + 16n - 3 \\
&= 16n^4 - 8(\mathrm{Ode})_n - 4C_{4,n} + 8(\mathrm{Gno})_n + 5
\end{aligned}
$$

$$
\begin{aligned}
n(\mathrm{HRdo}) + 8[(\mathrm{Ode})_n - (\mathrm{Gno})_n] + 4C_{4,n} - 5 &= 16n^4 \\
n(\mathrm{HRdo}) + 8[(\mathrm{Ode})_n - (\mathrm{Gno})_n] + 4C_{4,n} - 5 &= (2n)^4
\end{aligned}
$$

Hence

$$n(\text{HRdo})_n + 8[(\text{Ode})_n - (\text{Gno})_n] + 4C_{4,n} - 5 = (2n)^4 \text{ is a even bi-quadratic number.}$$ □

Theorem 3.15. $3(\text{Nex})n + 24(\text{Pen})n - 13(\text{Rdod})n + 10C17, n - 22(\text{Gno})n - n(\text{HRdo})n \equiv 0 \pmod{17}$.

Proof. From Theorem 3.14,

$$\begin{aligned} n(\text{HRdo})_n &= 16n^4 - 36n^3 + 28n^2 - 7 \\ &= (16n^4 + 16n^3 + 21n^2 + 11n + 1) - 52n^3 + 7n^2 - 11n - 8 \\ &= 3(\text{Nex})_n + 24(\text{Pen})_n - (52n^3 + 78n^2 + 52n + 13) + 85n^2 + 41n + 5 \\ &= 3(\text{Nex})_n + 24(\text{Pen})_n - 13(\text{Rdod})_n + (85n^2 + 85n + 10) - 44n + 5 \\ &= 3(\text{Nex})_n + 24(\text{Pen})_n - 13(\text{Rdod})_n + 10C_{17,n} - (44n - 22) - 17 \end{aligned}$$

Hence

$$3(\text{Nex})_n + 24(\text{Pen})_n - 13(\text{Rdod})_n + 10C_{17,n} - 22(\text{Gno})_n - n(\text{HRdo})_n \equiv 0 \pmod{17}.$$ □

4. Conclusion

The numbers mentioned above are really fascinating and exist all around the planet. We took into account a sufficient number of interesting results involving Haüy rhombic dodecahedral, stella octangula, hex number, centered tetrahedral, pyramidal, prism, pronic, and gnomonic numbers. We sincerely believe that this effort will serve as a welcome impetus for further investigation into these data.

Notations:

$(\text{HRdo})_n$	:	Hauy Rhombic dodecahedral number of rank n
$(\text{Soct})_n$	:	Stella octangula number of rank n
$(\text{Hex})_n$	:	Hex number with rank n
$(\text{Gno})_n$	:	Gnomonic number
$(\text{Ctet})_n$	:	Centered tetrahedral number of rank n
$(\text{oc})_n$	:	Octahedral number of rank n
$(\text{Ico})_n$	:	Icosahedral number of rank n
$(\text{Rdod})_n$	:	Rhombic dodecahedral number of rank n
$(\text{Tet})_n$	:	Tetrahedral number of rank n
$(\text{Nex})_n$	:	Nexus number of rank n
$(\text{Pen})_n$	:	Pentatope number of rank n
$(\text{Dde})_n$	:	Dodecahedral number of rank n
$(\text{Coc})_n$	:	Centered octahedral number of rank n
$(\text{TruTet})_n$	:	Truncated Tetrahedral number of rank n
$(\text{Py})_t^s$	:	Pyramidal number of rank s it side t
$(\text{Cpy})_t^s$	:	Centered pyramidal number of rank s it side t

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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