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Research Article

Certain Inequalities Involving Generalized Hyperbolic Sine and Cosine Functions

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Abstract. In this article, we established generalizations of some inequalities involving generalized hyperbolic functions to establish several lower and upper bounds of known inequalities. The implanted results give the sharpness of inequalities.

Keywords. Hyperbolic function, Generalized hyperbolic functions

Mathematics Subject Classification (2020). 26D05, 26D15, 26D20

1. Introduction

In 1995, Lindqvist [5] defined the sine function in the p-generalization form which is symbolized by $\sin_p(\zeta)$ and it is the inverse of the function,

$$\arcsin_p(\zeta) = \int_0^\zeta (1-t^p)^{\frac{-1}{p}}\,dt,$$

where $1 < p < \infty$ and $0 \le \zeta \le 1$. For the value $p = 2$, the function $\sin_p(\zeta)$ is equivalent to general sine function and it will be extended to $(-\infty,\infty)$ (Nantomah and Prempeh [7]). Similarly, cosine and tangent functions are also given in the p-generalization form using the sine function.

With the addition of this, the hyperbolic sine function defined in the p-generalization form which is symbolized by $\sinh_p(\zeta)$ and it is the inverse of the function,

$$\operatorname{arcsinh}_p(\zeta)=\begin{cases}\int_0^{\zeta}(1+t^p)^{\frac{-1}{p}}dt, & \zeta\in(0,\infty),\\ -\operatorname{arcsinh}_p(-\zeta), & \zeta\in(-\infty,0).\end{cases}$$

In 2012, Takeuchi [9] given a further generalization of the sine function in the form of pq-generalization, which is symbolized by $\sin_{p,q}(\zeta)$ and is inverse of the function,

$$\arcsin_{p,q}(\zeta)=\int_0^{\zeta}(1-t^q)^{\frac{-1}{p}}dt,$$

where $0\le\zeta\le1$ and $p,q\in(1,\infty)$. Now the function $\sin_{p,q}(\zeta)$ match $\sin_p(\zeta)$ for $p=q$ and it will be extended upto $(-\infty,\infty)$. Similarly, p,q-generalization of tangent and cosine functions is also defined as a generalization of the sine function.

Now, the hyperbolic sine function is also defined in the form p,q-generalization, which is symbolized by $\sinh_{p,q}(\zeta)$ and is inverse of the function,

$$\operatorname{arcsinh}_{p,q}(\zeta)=\int_0^{\zeta}(1+t^q)^{\frac{-1}{p}}dt,\quad \zeta\in(0,\infty).$$

Similarly, using the hyperbolic sine function we can also define the hyperbolic cosine and hyperbolic tangent functions in the form of p,q-generalization.

In this paper, we present a generalization of inequalities that involve the generalized hyperbolic functions. Our primary goal is to refine and sharpen certain inequalities related to the generalized hyperbolic cosine and hyperbolic sine functions. The main result we establish is a broader generalization of inequalities involving the sine and cosine hyperbolic functions.

2. Preliminaries

In this section, we define the generalized hyperbolic sine and cosine functions and present some key results related to these functions.

Definition 2.1 ([7]). The hyperbolic sine, hyperbolic cosine and hyperbolic tangent functions can be defined in the generalized form respectively as (Darkunde and Ghodechor [3], Nantomah and Prempeh [7], and Nantomah *et al*. [8]),

$$\sinh_a(\zeta)=\frac{a^{\zeta}-a^{-\zeta}}{2},\quad \cosh_a(\zeta)=\frac{a^{\zeta}+a^{-\zeta}}{2},$$
$$\tanh_a(\zeta)=\frac{a^{\zeta}-a^{-\zeta}}{a^{\zeta}+a^{-\zeta}},\quad \coth_a(\zeta)=\frac{a^{\zeta}+a^{-\zeta}}{a^{\zeta}-a^{-\zeta}},$$

where $a>1$ and $\zeta\in(-\infty,\infty)$.

All these generalized results satisfies the identities as follows:

$$\cosh_a(\zeta)+\sinh_a(\zeta)=a^{\zeta},\qquad \cosh_a(\zeta)-\sinh_a(\zeta)=a^{-\zeta},$$
$$\frac{d}{d\zeta}[\sinh_a(\zeta)]=\ln a\cosh_a(\zeta),\qquad \frac{d}{d\zeta}[\cosh_a(\zeta)]=\ln a\sinh_a(\zeta),$$
$$\frac{d}{d\zeta}[\tanh_a(\zeta)]=\ln a\,\mathrm{sech}_a^2(\zeta),\qquad \frac{d}{d\zeta}[\coth_a(\zeta)]=\ln a\,\mathrm{csc}h_a^2(\zeta),$$
$$\frac{d^2}{d\zeta^2}[\cosh_a(\zeta)]+\frac{d^2}{d\zeta^2}[\sinh_a(\zeta)]=(\ln a)^2a^{\zeta},\qquad \frac{d^2}{d\zeta^2}[\cosh_a(\zeta)]-\frac{d^2}{d\zeta^2}[\sinh_a(\zeta)]=(\ln a)^2a^{-\zeta},$$

$$\cosh_a^2(\zeta)+\sinh_a^2(\zeta)=\cosh_a(2\zeta), \qquad \cosh_a^2(\zeta)-\sinh_a^2(\zeta)=1,$$
$$2\cosh_a(\zeta)\cdot\sinh_a(\zeta)=\sinh_a(2\zeta), \qquad \cosh_a^2(\zeta)=\frac{\cosh_a(2\zeta)+1}{2},$$
$$\sinh_a^2(\zeta)=\frac{\cosh_a(2\zeta)-1}{2}.$$

In particular, Nantomah [8] for the Euler's number $a = e = 2.71828$, all these defined identities and results coincide with their basic definitions.

Lemma 2.1 (Hermite-Hadamard Inequality, [6])**.** *If $f : I \subseteq (-\infty,\infty) \to (-\infty,\infty)$ be the convex function then*
$$f\left(\frac{u_1+v_1}{2}\right) \le \frac{1}{v_1-u_1}\int_{u_1}^{v_1} f(t)dt \le \frac{f(u_1)+f(v_1)}{2},$$
where $u_1, v_1 \in I$ and $u_1 < v_1$.

Lemma 2.2 ([2])**.** *Let the function $f, g : [a,b] \to \mathbb{R}$ are continuous and be differentiable on (a,b) and $g'(x) \neq 0$ on (a,b). Let $A_1(x) = \frac{f(x)-f(a)}{g(x)-g(a)}$, $A_2(x) = \frac{f(x)-f(b)}{g(x)-g(b)}$, $x \in (a,b)$,*

(i) *$A_1(\cdot)$ and $A_2(\cdot)$ are increasing (strictly increasing) on (a,b) if $\frac{f'(\cdot)}{g'(\cdot)}$ is increasing (strictly increasing) on (a,b),*

(ii) *$A_1(\cdot)$ and $A_2(\cdot)$ are decreasing (strictly decreasing) on (a,b) if $\frac{f'(\cdot)}{g'(\cdot)}$ is decreasing (strictly decreasing) on (a,b).*

Lemma 2.3 ([1])**.** *If, $M(x) = \sum_{k=0}^{\infty} m_k(x)x^k$ and $N(x) = \sum_{k=0}^{\infty} n_k(x)x^k$ are convergent for $|x| < R$, where m_k and n_k are the real numbers for $k = 0,1,2,3,\dots$ such that $n_k > 0$. Now if the sequence $\frac{m_k}{n_k}$ is strictly increasing (or decreasing) on $(0,R)$, then the function $\frac{M(x)}{N(x)}$ is also strictly increasing (or decreasing) on $(0,R)$.*

The power series of generalized hyperbolic sine and hyperbolic cosine functions is defined as (Gradshteyn *et al.* [4]):
$$\sinh_a(\zeta) = \sum_{n=0}^{\infty} (\ln a)^{2n+1}\frac{\zeta^{2n+1}}{(2n+1)!},$$
$$\cosh_a(\zeta) = \sum_{n=0}^{\infty} (\ln a)^{2n}\frac{\zeta^{2n}}{(2n)!}.$$
We will use these power series for the proof of our main result.

3. Main Results

In this section, we present the inequalities with sharpened bounds for the generalized hyperbolic functions. These refined inequalities provide more accurate bounds for the generalized hyperbolic functions.

Theorem 3.1. *Let* $m \geq \frac{(\ln a)^3}{15}$ *then the function* $f(\zeta) = \frac{\left(\ln \frac{\sinh_a(\zeta)}{\zeta}\right)}{\ln(1+m\zeta^2)}$ *is strictly increasing on* $(0,r)$ *where* $r \in (0,\infty)$. *In specific, for this fixed value of m the best suitable constants* α *and* β *are* $\frac{(\ln a)^3}{6m}$ *and* $\frac{\left(\ln \frac{\sinh_a(r)}{r}\right)}{\ln(1+\frac{r^2}{15})}$ *respectively such that,*

$$(1+m\zeta^2)^\alpha \leq \frac{\sinh_a(\zeta)}{\zeta} \leq (1+m\zeta^2)^\beta.$$

Proof. Let

$$f(\zeta) = \frac{\ln \frac{\sinh_a(\zeta)}{\zeta}}{\ln(1+m\zeta^2)} = \frac{f_1(\zeta)}{f_2(\zeta)},$$

where $f_1(\zeta) = \ln \frac{\sinh_a(\zeta)}{\zeta}$ and $f_2(\zeta) = \ln(1+m\zeta^2)$ with $f_1(0)=0$ and $f_2(0)=0$.
Differentiating $f_1(\zeta)$ and $f_2(\zeta)$, we get

$$\begin{aligned} f_1'(\zeta) &= \frac{\zeta}{\sinh_a(\zeta)}\left[\frac{\zeta \ln a \cosh_a(\zeta) - \sinh_a(\zeta)}{\zeta^2}\right] \\ &= \frac{\zeta \ln a \cdot \cosh_a(\zeta) - \sinh_a(\zeta)}{\zeta \sinh_a(\zeta)} \end{aligned}$$

and $f_2'(\zeta) = \frac{2m\zeta}{1+m\zeta^2}$, therefore

$$\begin{aligned} \frac{f_1'(\zeta)}{f_2'(\zeta)} &= \frac{\zeta \ln a \cdot \cosh_a(\zeta) - \sinh_a(\zeta)}{\zeta \sinh_a(\zeta)} \cdot \frac{(1+m\zeta^2)}{2m\zeta} \\ &= \frac{(1+m\zeta^2)}{2m}\left(\frac{\zeta \ln a \cdot \cosh_a(\zeta) - \sinh_a(\zeta)}{\zeta^2 \sinh_a(\zeta)}\right) \\ &= \frac{1}{2m}\left[\frac{\zeta \ln a \cdot \cosh_a(\zeta) - \sinh_a(\zeta)}{\zeta^2 \sinh_a(\zeta)} + \frac{m\zeta^3 \ln a \cdot \cosh_a(\zeta) - m\zeta^2 \sinh_a(\zeta)}{\zeta^2 \sinh_a(\zeta)}\right] \\ &= \frac{1}{2m}\left[\ln a \frac{\coth_a(\zeta)}{\zeta} - \frac{1}{\zeta^2} + \ln a . m\zeta \coth_a(\zeta) - m\right] \\ &= \frac{1}{2m} f_3(\zeta), \end{aligned}$$

where $f_3(\zeta) = \ln a \frac{\coth_a(\zeta)}{\zeta} - \frac{1}{\zeta^2} + \ln a \cdot m\zeta \coth_a(\zeta) - m$.
Now $f_3(\zeta)$ is increasing iff $f_3'(\zeta) > 0$ therefore by using Lemma 2.2, we see that $f(\zeta)$ is increasing if $f_3'(\zeta) > 0$.

$$f_3'(\zeta) = (\ln a)\left[\frac{-(\ln a)\operatorname{cosech}_a^2(\zeta)\zeta - \coth_a(\zeta)}{\zeta^2}\right] + \frac{2}{\zeta^3} + m \ln a \left[-\zeta(\ln a)\operatorname{cosech}_a^2(\zeta) + \coth_a(\zeta)\right] > 0$$

which is equivalent to

$$\frac{2 - (\ln a)^2\left(\frac{\zeta}{\sinh_a(\zeta)}\right)^2 - (\ln a)\zeta \coth_a(\zeta)}{\zeta^2(\ln a)\left[(\ln a)\left(\frac{\zeta}{\sinh_a(\zeta)}\right)^2 - \zeta \coth_a(\zeta)\right]} > m.$$

Due to well known following result (Alzer and Qiu [1]),

$$\left(\frac{\zeta}{\sinh_a(\zeta)}\right)^2 < 1 < \frac{\zeta}{\tanh_a(\zeta)}.$$

We see that, $F_4(\zeta) > m$, where

$$F_4(\zeta) = \frac{2-(\ln a)^2\left(\frac{\zeta}{\sinh_a(\zeta)}\right)^2-(\ln a)\zeta\coth_a(\zeta)}{\zeta^2(\ln a)\left[(\ln a)\left(\frac{\zeta}{\sinh_a(\zeta)}\right)^2-\zeta\coth_a(\zeta)\right]}$$

$$= \frac{2-(\ln a)^2\left[\left(\frac{\zeta}{\sinh_a(\zeta)}\right)^2-(\ln a)\zeta\frac{\cosh_a(\zeta)}{\sinh_a(\zeta)}\right]}{\zeta^2(\ln a)\left[(\ln a)\left(\frac{\zeta}{\sinh_a(\zeta)}\right)^2-\zeta\frac{\cosh_a(\zeta)}{\sinh_a(\zeta)}\right]}$$

$$= \frac{2-(\ln a)^2\left[\frac{\zeta^2-(\ln a)\zeta\sinh_a(\zeta)\cosh_a(\zeta)}{\sinh^2(\zeta)}\right]}{\zeta^2(\ln a)\left[\frac{(\ln a)\zeta^2-\zeta\sinh_a(\zeta)\cosh_a(\zeta)}{\sinh^2(\zeta)}\right]}$$

$$= \frac{2\sinh_a^2(\zeta)-(\ln a)^2\zeta^2-(\ln a)^3\zeta\sinh_a(\zeta)\cosh_a(\zeta)}{(\ln a)^2\zeta^4-(\ln a)\zeta^3\sinh_a(\zeta)\cosh_a(\zeta)}$$

$$= \frac{\cosh_a(\zeta)-1-(\ln a)^2\zeta^2-\frac{(\ln a)\zeta\sinh_a(2\zeta)}{2}}{(\ln a)^2\zeta^4-\frac{(\ln a)\zeta^3\sinh_a(2\zeta)}{2}}$$

$$= \frac{2\cosh_a(\zeta)-2-2(\ln a)^2\zeta^2-(\ln a)\zeta\sinh_a(2\zeta)}{2(\ln a)^2\zeta^4-(\ln a)\zeta^3\sinh_a(2\zeta)}.$$

Using power series expansion of $\sinh_a(\zeta)$ and $\cosh_a(\zeta)$ above equation can be written as,

$$F_4(\zeta) = \frac{2\sum\limits_{n=0}^{\infty}2^{2n}(\ln a)^{2n}\frac{\zeta^{2n}}{(2n)!}-2-2(\ln a)^2\zeta^2-(\ln a)\zeta\sum\limits_{n=0}^{\infty}\frac{(\ln a)^{2n+1}2^{2n+1}\zeta^{2n+1}}{(2n+1)!}}{2(\ln a)^2\zeta^4-(\ln a)\zeta^3\sum\limits_{n=0}^{\infty}\frac{(\ln a)^{2n+1}2^{2n+1}\zeta^{2n+1}}{(2n+1)!}}$$

$$= \frac{2\sum\limits_{n=0}^{\infty}2^{2n}(\ln a)^{2n}\frac{\zeta^{2n}}{(2n)!}-2-2(\ln a)^2\zeta^2-\sum\limits_{n=0}^{\infty}\frac{(\ln a)^{2n+2}2^{2n+1}\zeta^{2n+2}}{(2n+1)!}}{2(\ln a)^2\zeta^4-\sum\limits_{n=0}^{\infty}\frac{(\ln a)^{2n+2}2^{2n+1}\zeta^{2n+4}}{(2n+1)!}}$$

$$= \frac{\sum\limits_{n=2}^{\infty}2^{2n+1}(\ln a)^{2n}\frac{\zeta^{2n}}{(2n)!}-\sum\limits_{n=0}^{\infty}\frac{(\ln a)^{2n+2}2^{2n+1}\zeta^{2n+2}}{(2n+1)!}}{2(\ln a)^2\zeta^4-\sum\limits_{n=0}^{\infty}\frac{(\ln a)^{2n+2}2^{2n+1}\zeta^{2n+4}}{(2n+1)!}}$$

$$= \frac{\sum\limits_{n=2}^{\infty}2^{2n+1}(\ln a)^{2n}\frac{\zeta^{2n}}{(2n)!}-\sum\limits_{n=2}^{\infty}\frac{(\ln a)^{2n}2^{2n-1}\zeta^{2n}}{(2n-1)!}}{-\sum\limits_{n=3}^{\infty}\frac{(\ln a)^{2n-3}2^{2n-3}\zeta^{2n}}{(2n-3)!}}$$

$$= \frac{\sum\limits_{n=2}^{\infty}\left[\frac{(\ln a)^{2n}2^{2n+1}}{(2n)!}-\frac{(\ln a)^{2n}-2^{2n-1}}{(2n-1)!}\right]\zeta^{2n}}{-\sum\limits_{n=3}^{\infty}\frac{(\ln a)^{2n-3}2^{2n-3}\zeta^{2n}}{(2n-3)!}}$$

$$= \frac{\sum\limits_{n=3}^{\infty}\left[\frac{(\ln a)^{2n}-2^{2n-1}}{(2n-1)!}-\frac{(\ln a)^{2n}2^{2n+1}}{(2n)!}-\right]\zeta^{2n}}{\sum\limits_{n=3}^{\infty}\frac{(\ln a)^{2n-3}2^{2n-3}\zeta^{2n}}{(2n-3)!}}$$

$$= \frac{\sum_{n=3}^{\infty} a_n \zeta^{2n}}{\sum_{n=3}^{\infty} b_n \zeta^{2n}},$$

where

$$\frac{a_n}{b_n} = \frac{(\ln a)^{2n}\frac{2^{2n-1}}{(2n-1)!} - (\ln a)^{2n}\frac{2^{2n+1}}{(2n)!}}{(\ln a)^{2n-3}\frac{2^{2n-3}}{(2n-3)!}}$$
$$= \frac{(\ln a)^{2n}(2n)(2)^{2n-1} - (\ln a)^{2n}(2)^{2n+1}(2n-3)!}{(\ln a)^{2n-3}(2)^{2n-3}(2n)!}$$
$$= \frac{(\ln a)^{2n}(2n)(2)^{2n-1} - (\ln a)^{2n}(2)^{2n+1}}{(\ln a)^{2n-3}(2)^{2n-3}(2n)(2n-1)(2n-2)}$$
$$= \frac{(\ln a)^3 8(n-2)}{(2n)(2n-1)(2n-2)}$$
$$= \frac{4(\ln a)^3(n-2)}{(n)(2n-1)(2n-2)}$$
$$= c_n.$$

Suppose, $c_n \le c_{n+1}$,

$$\implies \quad 2(n-2)(n+1)(2n+1) < (n-1)(2n-1)(2n-2)$$
$$\implies \quad 2\left[(n^2-n-2)(2n+1)\right] < \left[(2n^2-3n+1)(2n-2)\right]$$
$$\implies \quad 2n^2-12n-2<0$$
$$\implies \quad n \ge 3$$

Therefore the sequence is strictly increasing and $b_n > 0$, for all $n \ge 3$.
By Lemma 2.3, $F_4(\zeta)$ is strictly increasing on $(0,r)$, which implies that $F(\zeta)$ is also increasing. So, $\sup\{F_4(\zeta):\zeta>0\} \ge m$ and $m \ge \frac{(\ln a)^3}{15}$ since,

$$\lim_{\zeta\to 0+} F_4(\zeta) = \frac{(\ln a)^3}{15}.$$

We have,

$$\lim_{\zeta\to 0+} F(\zeta) \le F(\zeta) \le \lim_{\zeta\to r-} F(\zeta).$$

Hence theorem is proved. □

Theorem 3.2. *Let $m \ge \frac{(\ln a)}{3}$, then the function $G(\zeta) = \frac{\log\cosh_a(\zeta)}{\log 1+m\zeta^2}$ is strictly increasing in $(0,r)$, where $r \in (0,\infty)$.*
In particular, the best possible value of α_1 and β_1 are $\frac{(\ln a)}{2m}$ and $\frac{\ln(\cosh_a(r))}{\ln(1+mr^2)}$, such that

$$(1+m\zeta^2)^{\alpha_1} < \cosh_a(\zeta) < (1+m\zeta^2)^{\beta_1}.$$

Proof. Let

$$G(\zeta) = \frac{\ln\cosh_a(\zeta)}{\ln(1+m\zeta^2)} = \frac{G_1(\zeta)}{G_2(\zeta)},$$

where $G_1(\zeta) = \ln\cosh_a(\zeta)$ and $G_2(\zeta) = \ln(1+m\zeta^2)$ with $G_1(0)=G_2(0)=0$.

Differentiating $G_1(\zeta)$ and $G_2(\zeta)$, we get

$$\frac{G_1'(\zeta)}{G_2'(\zeta)} = \frac{(\ln a)(1+m\zeta^2)\sinh_a(\zeta)}{(2m\zeta)\cosh_a(\zeta)} = (\ln a)G_3(\zeta),$$

where

$$G_3(\zeta) = \frac{(1+m\zeta^2)\sinh_a(z)}{(2m\zeta)\cosh_a(\zeta)},$$

$\frac{G_1'(\zeta)}{G_2'(\zeta)}$ is increasing iff $G_3'(\zeta) > 0$.

By Lemma 2.2, $G(\zeta)$ will be increasing if $G_3'(\zeta) > 0$, where

$$G_3(\zeta) = \frac{\tanh_a(\zeta)}{\zeta} + mz\tanh_a(\zeta),$$

$$G_3'(\zeta) = \frac{\zeta(\ln a)\operatorname{sech}_a^2(\zeta) - \tanh_a(\zeta)}{\zeta^2} + (m\tanh_a(\zeta) + mz(\ln a)\operatorname{sech}_a^2(\zeta))$$

$$= \frac{\zeta(\ln a)(1-\sinh_a(\zeta)\cosh_a(\zeta))}{\zeta^2} + m\frac{\zeta(\ln a) + \sinh_a(\zeta)\cosh_a(\zeta)}{\cosh_a^2(\zeta)}$$

$$(\ln a)\left[(\zeta)\cos_a^2(\zeta) - (\zeta)\sinh_a^2(\zeta)\right] - \sinh_a(\zeta)\cosh_a(\zeta)$$

$$> m\zeta^2\left[(\ln a)(-\zeta)\cosh_a^2(\zeta) + (\zeta)\sinh_a^2(\zeta)) - \sinh_a(\zeta)\cosh_a(\zeta)\right].$$

$$\implies \quad 2(\ln a)(\zeta) - \sinh_a(2\zeta) > m\zeta^2[-2\zeta(\ln a) - \sinh_a(2\zeta)]$$

$$\implies \quad 2(\ln a)(\zeta) - \sinh_a(2\zeta) > -2(\ln a)m\zeta^3 - m\zeta^2\sinh_a(2\zeta)$$

$$\implies \quad \frac{2(\ln a)(\zeta) - \sinh_a(2\zeta)}{-2(\ln a)\zeta^3 - \zeta^2\sinh_a(2\zeta)} > m.$$

It is equivalent to

$$G_4(\zeta) = \frac{2(\ln a)(\zeta) - \sinh_a(2\zeta)}{2(\ln a)\zeta^3 + \zeta^2\sinh_a(2\zeta)} < m.$$

It implies $G_4(\zeta) < m$ and $G_4(\zeta) = \frac{1}{G_3(\zeta)}$. It shows $G_3(\zeta)$ is strictly increasing in $(0,r)$, which implies $G(\zeta)$ is increasing for specified values of $m = \frac{(\ln a)}{3}$.

So, $\sup\{G_4(\zeta) : \zeta \in (0,r)\} \le m$ and

$$\lim_{\zeta\to 0+} G_4(\zeta) = \frac{(\ln a)}{3}.$$

Now we have that,

$$\lim_{\zeta\to 0+} G(\zeta) \le G(\zeta) \le \lim_{\zeta\to r-} G(\zeta).$$

Hence theorem is proved. □

4. Scope for Further Research

These generalized results can be used in further research to refine and sharpen inequalities involving hyperbolic functions. They provide a broader framework for exploring new inequalities and enhancing the precision of existing ones in various mathematical and applied contexts.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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