Communications in Mathematics and Applications
Vol. 15, No. 2, pp. 619–634, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i2.2625
http://www.rgnpublications.com

Research Article

Monophonic Cover Pebbling Number of Standard and Algebraic Graphs

A. Lourdusamy[1], S. Kither iammal*[2] and I. Dhivviyanandam[3]

[1] *Department of Mathematics, St. Xavier's College (Autonomous) (affiliated to Manonmaniam Sundaranar University), Palayamkottai 627002, Tamil Nadu, India*
[2] *Department of Mathematics, Jayaraj Annapackiam College for Women (affiliated to Mother Teresa Women's University), Periyakulam 625601, Tamil Nadu, India*
[3] *Department of Mathematics, North Bengal St. Xavier's College (affiliated to the University of North Bengal), Rajganj, West Bengal, India*
***Corresponding author:** cathsat86@gmail.com

Received: March 5, 2024 **Accepted:** April 21, 2024

Abstract. Given a connected graph G and a configuration D of pebbles on the vertices of G, a pebbling transformation takes place by removing two pebbles from one vertex and placing one pebble on its adjacent vertex. A monophonic path is considered to be a longest chordless path between two vertices u and v which are not adjacent. A monophonic cover pebbling number, $\gamma_\mu(G)$, is a minimum number of pebbles required to cover all the vertices of G with at least one pebble each on them after the transferring of pebbles by using monophonic paths. In this paper, we determine the monophonic cover pebbling number of cycles, square of cycles, shadow graph of cycles, complete graphs, Jahangir graphs, fan graphs, zero divisor graphs and unit graphs.

Keywords. Cover pebbling, Monophonic pebbling, Monophonic cover pebbling, Zero divisor, Unit graph

Mathematics Subject Classification (2020). 05E15, 05C12, 05C25, 05C38, 05C76

1. Introduction

Beeler *et al*. [2] stated that Lagarias and Saks suggested the concept of graph pebbling to solve a number theoretic conjecture. Then, Chung [3] gave further developmental ideas using graph pebbling concepts to solve the number theory problems. A pebbling move is defined as extracting two pebbles from one vertex and keeping one pebble on the adjacent vertex and eliminating the other pebble. Crull *et al*. [4], defined the cover pebbling number $\gamma(G)$, as follows: It is the minimum number of pebbles needed to cover all the vertices with at least one pebble however we place pebbles in the initial configuration. Lourdusamy *et al*. [6, 7] defined detour pebbling number, and monophonic pebbling number. A monophonic cover pebbling number, $\gamma_\mu(G)$, is a minimum number of pebbles require to cover all the vertices of G with at least one pebble each on them after shifting of pebbles by using monophonic paths which is chordless and the longest. The application of this concept plays a vital role in the supply of goods and transportation problems. This is also applied in the network transmission of the information from one node to the other. The application of monophonic cover pebbling number decides the equal distribution of goods on every customers by using the monophonic path. In this paper, we determine the monophonic cover pebbling number of some graphs. To prove the worst condition, we use the stacking theorem (Crull *et al*. [4]). It is stated as: Let D be the initial configuration of pebbles. When the initial configuration D is placed on a single vertex v such that the dist(v) is a maximum, such a way $s(v) = \sum_{u \in V(G)} 2^{\text{dis}(u,v)}$, and do this for every vertex $v \in V(G)$. Then, $\gamma(G)$ is the largest $s(v)$.

Note 1.1. The notation $D_2(G)$ stands for shadow graph which is taken from Jayagopal and Raju [5]. The notation $\Gamma(Z)$ stands for zero-divisor graph of a ring R which is taken from Anderson and Livingston [1]. The notation $U(R)$ stands for the unit graph which is taken from Maimani *et al*. [8].

Theorem 1.1. *For the path P_n, $\gamma_\mu(P_n)$ is $2^n - 1$.*

Theorem 1.2. *For $K_{1,n}$, $\gamma_\mu(K_{1,n}) = 4n - 1$.*

Result 1.1 ([6])**.** *Let G be a connected graph. The monophonic distance between u and v is 0 if and only if $u = v$ and 1 if and only if $u - v$ is an edge of G.*

Definition 1.1 ([9])**.** Let $v \in V(G)$. Then, v is called a key or source vertex if dis(v) is maximum.

Notation

Throughout this article, we denote

- β as the source vertex,
- M_i is the monophonic path and $M_i^{\sim}$ contains the vertices which are not on M_i,
- We use *MCPN* for monophonic cover pebbling number,
- $N(v_0)$ is the neighborhood of v_0.

2. Monophonic Cover Pebbling Number of Some Standard Graphs

Theorem 2.1. *For $C_n, \gamma_\mu(C_n)$ is*
$$\begin{cases} 2\sum_{k=\frac{n}{2}+1}^{n-2} 2^k + 2^{\frac{n}{2}} + 5, & \text{if } n \text{ is even},\\ 2\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2} +5, & \text{if } n \text{ is odd}. \end{cases}$$

Proof. Let $V(C_n)=\{u_1,u_2,\ldots,u_n\}$ and $E(C_n)=\{u_iu_{i+1},u_nu_1\}$, where $1\le i\le n-1$.

Case 1: When n is even.

Let $p(u_1) = 2\sum_{k=\frac{n}{2}+1}^{n-2} 2^k + 2^{\frac{n}{2}} + 4$. Now to cover the vertices u_2,u_n, we use 4 pebbles; to cover the vertex $u_{\frac{n}{2}}$, we use $2^{\frac{n}{2}}$ pebbles; subsequently, to place one pebble each on $u_3,u_4,\cdots,u_{\frac{n}{2}-1},u_{\frac{n}{2}+1},u_{\frac{n}{2}+2},\cdots,u_{n-2},u_{n-1}$, we have the following pebble distributions: $2(2^{n-2},2^{n-3},2^{n-4},\cdots,2^{\frac{n}{2}+2},2^{\frac{n}{2}+1})$ and so the total number of pebbles is $2\left(\sum_{k=\frac{n}{2}+1}^{n-2} 2^k\right)+2^{\frac{n}{2}}+4$.

Now there is no pebble to cover u_1. Thus, $\gamma_\mu(C_n)\ge 2\sum_{k=\frac{n}{2}+1}^{n-2} 2^k+2^{\frac{n}{2}}+5$.

To prove $\gamma_\mu(C_n)\le 2\sum_{k=\frac{n}{2}+1}^{n-2} 2^k+2^{\frac{n}{2}}+5$, let us consider any configuration of $2\sum_{k=\frac{n}{2}+1}^{n-2} 2^k+2^{\frac{n}{2}}+5$ pebbles on $V(C_n)$. Let $\beta=u_1$. To cover the vertices of $N(u_1)$, we require 4 pebbles; to cover the vertices $u_3,u_4,\cdots,u_{\frac{n}{2}-1},u_{\frac{n}{2}+1},u_{\frac{n}{2}+2},\cdots,u_{n-2},u_{n-1}$, we require $2(2^{n-2},2^{n-3},2^{n-4},\cdots,2^{\frac{n}{2}+2},2^{\frac{n}{2}+1})$ pebbles; to cover the vertex $u_{\frac{n}{2}}$, we require $2^{\frac{n}{2}}$ pebbles; to cover u_1, we require 1 pebbles. Thus, to cover the vertices in C_n we require $2\left(\sum_{k=\frac{n}{2}+1}^{n-2} 2^k\right)+2^{\frac{n}{2}}+5$. By symmetry the proof follows for any source vertex u_i where $2\le i\le n$.

Case 2: When n is odd.

Let $p(u_1) = 2\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2} +4$. Now to cover the vertices u_2,u_n, we use 4 pebbles; subsequently, to place one pebble each on $u_3,u_4,\cdots,u_{\lceil\frac{n}{2}\rceil},u_{\lceil\frac{n}{2}\rceil+1},\cdots,u_{n-2},u_{n-1}$, we have the following pebble distributions: $2(2^{n-2},2^{n-3},2^{n-4},\cdots,2^{\lceil\frac{n}{2}\rceil+1},2^{\lceil\frac{n}{2}\rceil})$ and so the total number of pebbles is $2\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2} 2^k\right)+4$. Now there is no pebble to cover u_1. Thus, $\gamma_\mu(C_n)\ge 2\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2} 2^k\right)+5$.

To prove $\gamma_\mu(C_n)\le 2\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2} 2^k\right)+5$, let us consider any configuration of $2\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2} 2^k\right)+5$ pebbles on $V(C_n)$. Let $\beta=u_1$. To cover the vertices of $N(u_1)$, we require 4 pebbles; to cover the vertices $u_3,u_4,\cdots,u_{n-2},u_{n-1}$ we need $2(2^{n-2},2^{n-3},2^{n-4},\cdots,2^{\lceil\frac{n}{2}\rceil+1},2^{\lceil\frac{n}{2}\rceil})$ pebbles; to cover u_1, we need 1 pebble. Thus, to cover the vertices in C_n, we need $2\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2} 2^k\right)+5$ pebbles. By symmetry the proof follows for any source vertex u_i, where $2\le i\le n$.

Theorem 2.2. *For $D_2(C_n),\gamma_\mu(D_2(C_n))$ is*
$$\begin{cases} 4\left(\sum_{k=\frac{n}{2}+1}^{n-2} 2^k\right)+2^{\frac{n}{2}}+13, & \textit{if } n \textit{ is even},\\ 4\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2}\right)+13, & \textit{if } n \textit{ is odd}.\end{cases}$$

Proof. Let $V(D_2(C_n))=\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n\}$ and $E(D_2(C_n))=\{u_ju_{j+1},u_nu_1,v_jv_{j+1},v_nv_1,u_jv_{j+1},u_nv_1,v_ju_{j+1},v_nu_1\}$, where $j=1,2,\cdots,n-1$.

Case 1: When n is even.
Let $p(u_1) = 4\left(\sum_{k=\frac{n}{2}+1}^{n-2} 2^k\right) + 2^{\frac{n}{2}} + 12$. To cover the vertices v_2, v_n, u_2, u_n, we need 8 pebbles; to cover v_1, we need 4 pebbles; to cover the vertices $u_3, u_4, \cdots, u_{\frac{n}{2}-1}, u_{\frac{n}{2}+1}, \cdots, u_{n-1}, v_3, v_4, \cdots, v_{\frac{n}{2}-1}, v_{\frac{n}{2}+1}, \cdots, v_{n-1}$ we need $4(2^{\frac{n}{2}+1}, 2^{\frac{n}{2}+2}, \cdots, n-3, n-2)$ pebbles; to cover $u_{\frac{n}{2}}$ we need $2_{\frac{n}{2}}$ pebbles. Now there is no pebble to cover u_1. Thus, $p(D_2(C_n)) \geq 4\left(\sum_{k=\frac{n}{2}+1}^{n-2} 2^k\right) + 2^{\frac{n}{2}} + 13$.

To prove $p(D_2(C_n)) \leq 4\left(\sum_{k=\frac{n}{2}+1}^{n-2} 2^k\right) + 2^{\frac{n}{2}} + 13$, let us consider any configuration of $4\left(\sum_{k=\frac{n}{2}+1}^{n-2} 2^k\right) + 2^{\frac{n}{2}} + 13$ pebbles on $V(D_2(C_n))$. Let $\beta = u_1$. To cover the vertices of $N(u_1)$, we need 4(2) pebbles; to cover v_1, which is at the monophonic distance 2, we need 4 pebbles; to cover the vertices $u_3, u_4, \cdots, u_{\frac{n}{2}-1}, u_{\frac{n}{2}+1}, \cdots, u_{n-1}, v_3, v_4, \cdots, v_{\frac{n}{2}-1}, v_{\frac{n}{2}+1}, \cdots, v_{n-1}$ we need $4(2^{\frac{n}{2}+1}, 2^{\frac{n}{2}+2}, \cdots, n-3, n-2)$ pebbles; to cover $u_{\frac{n}{2}}$ we need $2_{\frac{n}{2}}$ pebbles; to cover u_1 we need 1 pebble. Thus, the total number of pebbles used is $4\left(\sum_{k=\frac{n}{2}+1}^{n-2} 2^k\right) + 2^{\frac{n}{2}} + 13$. By symmetry the proof follows for any source vertex u_i where $2 \leq i \leq n$, and v_k where $1 \leq k \leq n$.

Case 2: When n is odd.
Let $p(u_1) = 4\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2} 2^k\right) + 12$. To cover the vertices v_2, v_n, u_2, u_n, we need 8 pebbles; to cover v_1, we need 4 pebbles; to cover the vertices $u_3, u_4, \cdots, u_{n-1}, v_3, v_4, \cdots, v_{n-1}$ we need $4(2^{\lceil\frac{n}{2}\rceil}, 2^{\lceil\frac{n}{2}\rceil+1}, \cdots, n-3, n-2)$ pebbles. Now there is no pebble to cover u_1. Thus, $p(D_2(C_n)) \geq 4\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2} 2^k\right) + 13$.

To prove $p(D_2(C_n)) \leq 4\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2} 2^k\right) + 13$, let us consider any configuration of $4\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2} 2^k\right) + 13$ pebbles on $V(D_2(C_n))$. Let $\beta = u_1$. To cover the vertices of $N(u_1)$, we need 4(2) pebbles; to cover v_1, which is at the monophonic distance 2, we need 4 pebbles; to cover the vertices $u_3, u_4, \cdots, u_{n-1}, v_3, v_4, \cdots, v_{n-1}$ we need $4(2^{\lceil\frac{n}{2}\rceil}, 2^{\lceil\frac{n}{2}\rceil+1}, \cdots, n-3, n-2)$ pebbles; to cover u_1 we need 1 pebble. Thus, the total number of pebbles used is $4\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n-2} 2^k\right) + 13$. By symmetry the proof follows for any source vertex u_i where $2 \leq i \leq n$, and v_k where $1 \leq k \leq n$. □

Theorem 2.3. *For the graph $F_n, \gamma_\mu(F_n) = 2^{n-1} + 1$.*

Proof. Let $V(F_n) = \{v_0, v_1, \cdots, v_{n-1}\}$ and $E(F_n) = \{v_i v_{i+1}, v_0 v_j\}$ where $i = 0, 1, \cdots, n-2$ and $j = 1, 2, \cdots, n-1$. Let $p(v_1) = 2^{n-1}$. By Theorem 1.1 to cover $n-1$ vertices of the fan graph from v_1 to v_{n-1} we require $2^{n-1} - 1$ pebbles. We are left with 2 pebbles on v_1 which can be used to cover v_1 or v_0. So there will be a vertex which is not covered. Thus, $\gamma_\mu(F_n) \geq 2^{n-1} + 1$.

Now we prove $\gamma_\mu(F_n) \leq 2^{n-1} + 1$.

Case 1: Let the key vertex be v_k, where $k = 1$ or $n-1$.
Let $k = 1$ and $p(v_1) = 2^{n-1} + 1$. To cover v_{n-1} we require 2^{n-2} pebbles and to cover v_{n-2} we require 2^{n-3} pebbles. Following this process to cover the remaining vertices by using the monophonic path we need $2^{n-2} + 2^{n-3} + \cdots + 2^1 + 2^0$ pebbles. Thus, we need $2^{n-1} - 1$ pebbles to cover v_1 to

v_{n-1}. In order to cover v_0, we require 2 pebbles. Thus, using $2^{n-1}+1$ pebbles we are able to cover $V(F_n)$.

Case 2: Let the key vertex be v_0.
The monophonic distance from v_0 to any vertex is 1 and degree of v_0 is $n-1$. Hence, using $2(n-1)$ pebbles we can cover $n-1$ vertices and to cover v_0, we need an additional pebble. Thus, to cover all the vertices we need $2(n-1)+1 = 2n-1 < 2^{n-1}-1$.

Case 3: Let the key vertex be v_l where $1 < l < n-1$.
The monophonic distance from v_l to v_{n-1} is $n-l$ and the monophonic distance from v_l to v_1 is $l-1$. Thus, to cover v_l to v_{n-l} we require $2^{n-l}-1$ pebbles and to cover v_l to v_1 we require $2^{l-1}-2$ pebbles. To cover v_0 we require 2 pebbles. Thus, the total number of pebbles to cover F_n is $2^{n-l}-1+2^{l-1}-2+2 = 2^{n-l}+2^{l-1}+-1 < 2^{n-1}-1$. Hence, $\gamma_\mu(F_n) = 2^{n-1}+1$. □

Theorem 2.4. *For the complete graph $K_n, \gamma_\mu(K_n)$ is $2n-1$.*

Proof. Let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$, where every pair of distinct vertices are connected. Let $p(v_1) = 2n-2$. All the vertices are adjacent to each other. Therefore, to cover $n-1$ adjacent vertices of v_1, we require $2n-2$ pebbles. But v_1 is not covered. Therefore, $\gamma_\mu(K_n) \geq 2n-1$. Let us prove $\gamma_\mu(K_n) \leq 2n-1$. Let $p(v_n) = 0$. Then, there is i such that $p(v_i) \geq 2$, where $1 \leq i \leq n-1$. Using the pigeonhole principle we can shift a pebble from v_i to v_n. Then, using $2n-3$ pebbles we can cover the remaining $n-1$ vertices. □

Theorem 2.5. *For $J_{m,n}, \gamma_\mu(J_{m,n})$ is*
$$\begin{cases} 2\left(\sum\limits_{k=\lceil \frac{nm}{2}\rceil}^{nm-2} 2^k\right)+2^n+5, & \textit{if } nm \textit{ is odd},\\ 2\left(\sum\limits_{k=\frac{nm}{2}+1}^{nm-2}\right)+2^{\frac{nm}{2}}+2^n+5, & \textit{if } nm \textit{ is even}.\end{cases}$$

Proof. Let $V(J_{m,n}) = \{v_0, v_1, \cdots, v_{mn-1}, v_{mn}\}$ and $E(J_{m,n}) = \{v_iv_{i+1}, v_{nm}v_1, v_0v_1, v_0v_{n+1}, v_0v_{2n+1}, v_0v_{3n+1}, \cdots, v_0v_{(m-1)n+1}\}$, where $1 \leq i \leq nm-1$.

Case 1: When nm is odd.
Let $p(v_2) = 2\left(\sum\limits_{k=\lceil \frac{nm}{2}\rceil}^{nm-2} 2^k\right)+2^n+4$. To cover the vertex v_0, we need 2^n pebbles; to cover v_1, v_3, we need 4 pebbles; to cover the vertices $v_4, v_5, \cdots, v_{nm-1}, v_{nm}$ we need $2(2^{\lceil \frac{nm}{2}\rceil}+2^{\lceil \frac{nm}{2}\rceil+1}+\cdots+2^{nm-3}+2^{nm-2})$ pebbles, i.e., $2\left(\sum\limits_{k=\lceil \frac{nm}{2}\rceil}^{nm-2} 2^k\right)$. Now there is no pebble to cover v_2. Thus, $V(J_{m,n}) \geq 2\left(\sum\limits_{k=\lceil \frac{nm}{2}\rceil}^{nm-2} 2^k\right)+2^n+5$.

Now we prove $V(J_{m,n}) \leq 2\left(\sum\limits_{k=\lceil \frac{nm}{2}\rceil}^{nm-2} 2^k\right)+2^n+5$.

Subcase 1.1: Let $\beta = N(v_0)$.
Without loss of generality, let $\beta = v_1$. From Table 1, to cover the vertices from v_3 to v_{nm-1} we need $2(2^{\lceil \frac{nm}{2}\rceil}, 2^{\lceil \frac{nm}{2}\rceil+1}, \cdots, 2^{nm-3}, 2^{nm-2})$ pebbles, i.e., $2\left(\sum\limits_{k=\lceil \frac{nm}{2}\rceil}^{nm-2} 2^k\right)$. To cover $N(v_0)$ we need 6

pebbles; to cover v_1 we need 1 pebble. Thus, in this we require $2\left(\sum_{k=\lceil \frac{nm}{2}\rceil}^{nm-2} 2^k\right)+7$ pebbles.

Table 1. Monophonic distance from v_1 to $V(J_{n,m})$

	v_0	v_1	v_2	v_3	v_4	$\cdots$	$v_{\lceil \frac{nm}{2}\rceil}$	$v_{\lceil \frac{nm}{2}\rceil+1}$	$\cdots$	v_{nm-2}	v_{nm-1}	v_{nm}
v_1	1	0	1	$nm-2$	$nm-3$	$\cdots$	$v_{\lceil \frac{nm}{2}\rceil+1}$	$v_{\lceil \frac{nm}{2}\rceil+1}$	$\cdots$	$nm-3$	$nm-2$	1

Subcase 1.2: Let $\beta = v_k$ where v_k is an adjacent vertex of a vertex in $N(v_0)$.
Without loss of generality, let $\beta = v_{nm}$. From Table 2, to cover the vertices $v_2, v_3, \cdots, v_{nm-2}$, we need $2\left(\sum_{k=\lceil \frac{nm}{2}\rceil}^{nm-2} 2^k\right)$ pebbles; to cover the vertices v_1, v_{nm-1}, we need 4 pebbles; to cover v_0, we require 2^n pebbles; to cover v_{nm}, we need 1 pebble. Thus, the number of pebbles to cover $V(J_{n,m})$ is $2\left(\sum_{k=\lceil \frac{nm}{2}\rceil}^{nm-2} 2^k\right)+2^n+5$.

Table 2. Monophonic distance from v_{nm} to $V(J_{n,m})$

	v_0	v_1	v_2	v_3	v_4	$\cdots$	$v_{\lceil \frac{nm}{2}\rceil}$	$v_{\lceil \frac{nm}{2}\rceil+1}$	$\cdots$	v_{nm-2}	v_{nm-1}	v_{nm}
v_{nm}	n	1	$nm-2$	$nm-3$	$nm-4$	$\cdots$	$v_{\lceil \frac{nm}{2}\rceil}$	$v_{\lceil \frac{nm}{2}\rceil+1}$	$\cdots$	$nm-2$	1	0

Subcase 1.3: Let $\beta = v_s$ where $v_s \notin N(v_0)$ and $v_s \notin N(N(v_0))$.
Covering the vertices $v_{s+2}, v_{s+3}, \cdots, v_{nm}, v_1, v_2, \cdots, v_{s-2}$, we require $2(2^{\lceil \frac{nm}{2}\rceil}+2^{\lceil \frac{nm}{2}\rceil+1}+\cdots+2^{nm-3}+2^{nm-2})$ pebbles; to cover the vertices v_{s-1}, v_{s+1}, we need 4 pebbles; to cover v_s, we require 1 pebble. Now covering v_0 which is of the monophonic distance $< n$, it will cost $< 2^n$ pebbles. Thus, using fewer $2\left(\sum_{k=\lceil \frac{nm}{2}\rceil}^{nm-2} 2^k\right)+2^n+5$, pebbles we cover all the vertices of the graph.

Subcase 1.4: Let $\beta = v_0$.
We have $m-1$ paths of having the same length n from v_0. To cover the vertices of $N(v_0)$, we need $2m$ pebbles; to cover v_0, we need 1 pebble; to cover the remaining vertices we need $2m\left(\sum_{\lceil \frac{n}{2}\rceil+1}^{n} 2^k\right)$ pebbles. Thus, using $2m\left(\sum_{\lceil \frac{n}{2}\rceil+1}^{n} 2^k\right)+2m+1$ pebbles we cover $V(J_{n,m})$.

Case 2: When nm is even.
Let $p(v_2) = 2\left(\sum_{k=\frac{nm}{2}+1}^{nm-2}\right)+2^{\frac{nm}{2}}+2^n+4$. To cover the vertex v_0, we need 2^n pebbles; to cover v_1, v_3, we need 4 pebbles; to cover the vertices $v_4, v_5, \cdots, v_{\frac{nm}{2}-1}, v_{\frac{nm}{2}+1}, \cdots, v_{nm-1}, v_{nm}$ we need $2(2^{\frac{nm}{2}+1}+2^{\frac{nm}{2}+2}+\cdots+2^{nm-3}+2^{nm-2})$ pebbles, i.e., $2\left(\sum_{k=\frac{nm}{2}+1}^{nm-2}\right)$ pebbles. To cover the vertex $v_{\frac{nm}{2}}$, we need $2^{\frac{nm}{2}}$ pebbles. Now there is no pebble to cover v_2. Thus, $V(J_{m,n}) \geq 2\left(\sum_{k=\frac{nm}{2}+1}^{nm-2}\right)+2^{\frac{nm}{2}}+2^n+5$.

Now we prove $V(J_{m,n}) \leq 2\left(\sum_{k=\frac{nm}{2}+1}^{nm-2}\right) + 2^{\frac{nm}{2}} + 2^n + 5$.

Subcase 2.1: Let $\beta = N(v_0)$.

Without loss of generality, let $\beta = v_1$. From Table 3, to cover the vertices from v_3 to v_{nm-1} we need $2(2^{\frac{nm}{2}+1} + 2^{\frac{nm}{2}+2} + \cdots + 2^{nm-3} + 2^{nm-2})$ pebbles, i.e., $2\left(\sum_{k=\frac{nm}{2}+1}^{nm-2} 2^k\right)$. To cover $N(v_0)$ we need 6 pebbles; to cover v_1 we need 1 pebble. Thus, in this we require $2\left(\sum_{k=\frac{nm}{2}+1}^{nm-2} 2^k\right) + 7$ pebbles.

Table 3. Monophonic distance from v_1 to $V(J_{n,m})$

	v_0	v_1	v_2	v_3	v_4	$\cdots$	$v_{\frac{nm}{2}-1}$	$v_{\frac{nm}{2}}$	$v_{\frac{nm}{2}+1}$	$\cdots$	v_{nm-2}	v_{nm-1}	v_{nm}
v_1	1	0	1	$nm-2$	$nm-3$	$\cdots$	$v_{\frac{nm}{2}+1}$	$v_{\frac{nm}{2}}$	$v_{\frac{nm}{2}+1}$	$\cdots$	$nm-3$	$nm-2$	1

Subcase 2.2: Let $\beta = v_k$ where v_k is an adjacent vertex of a vertex in $N(v_0)$.

Without loss of generality, let $\beta = v_{nm}$. From Table 4, to cover the vertices $v_2, v_3, \cdots, v_{\frac{nm}{2}-2}, v_{\frac{nm}{2}}, v_{\frac{nm}{2}+1}, \cdots, v_{nm-2}$, we need $2\left(\sum_{k=\frac{nm}{2}+1}^{nm-2} 2^k\right)$ pebbles; to cover the vertices v_1, v_{nm-1}, we need 4 pebbles; to cover v_0, we require 2^n pebbles; to cover v_{nm}, we need 1 pebble; to cover $v_{\frac{nm}{2}-1}$, we need $2^{\frac{nm}{2}}$ pebbles. Thus, the number of pebbles to cover $V(J_{n,m})$ is $2\left(\sum_{k=\frac{nm}{2}+1}^{nm-2} 2^k\right) + 2^{\frac{nm}{2}} + 2^n + 5$.

Table 4. Monophonic distance from v_{nm} to $V(J_{n,m})$

	v_0	v_1	v_2	$\cdots$	$v_{\frac{nm}{2}-2}$	$v_{\frac{nm}{2}-1}$	$v_{\frac{nm}{2}}$	$\cdots$	v_{nm-3}	v_{nm-2}	v_{nm-1}	v_{nm}
v_{nm}	n	1	$nm-2$	$\cdots$	$v_{\frac{nm}{2}+1}$	$v_{\frac{nm}{2}}$	$v_{\frac{nm}{2}+1}$	$\cdots$	$nm-3$	$nm-2$	1	0

Subcase 2.3: Let $\beta = v_s$ where $v_s \notin N(v_0)$ and $v_s \notin N(N(v_0))$.

Covering the vertices $v_{s+2}, v_{s+3}, \cdots, v_{nm}, v_1, v_2, \cdots, v_{s-2}$, we require $2(2^{\frac{nm}{2}+1} + 2^{\frac{nm}{2}+2} + \cdots + 2^{nm-3} + 2^{nm-2})$ and $2^{\frac{nm}{2}}$ pebbles; to cover the vertices v_{s-1}, v_{s+1}, we need 4 pebbles; to cover v_s, we require 1 pebble. Now covering v_0 which is of the monophonic distance $< n$, it will cost $< 2^n$ pebbles. Thus, using fewer $2\left(\sum_{k=\frac{nm}{2}+1}^{nm-2} 2^k\right) + 2^{\frac{nm}{2}} + 2^n + 5$, pebbles we cover all the vertices of the graph.

Subcase 2.4: Let $\beta = v_0$.

We have $m-1$ paths of having the same length n from v_0. If n is even then to cover the vertices of $N(v_0)$, we need $2m$ pebbles; to cover v_0, we need 1 pebble; to cover the remaining vertices we need $2m\left(\sum_{k=\frac{n}{2}+2}^{n} 2^k\right)$ and $2^{\frac{n}{2}+1}$ pebbles. Thus, using $2m\left(\sum_{k=\frac{n}{2}+2}^{n} 2^k\right) + 2^{\frac{n}{2}+1} + 2m + 1$ pebbles we cover $V(J_{n,m})$. If n is odd then to cover the vertices of $N(v_0)$, we need $2m$ pebbles; to cover v_0, we need 1 pebble; to cover the remaining vertices we need $2m\left(\sum_{k=\lceil\frac{n}{2}\rceil+1}^{n} 2^k\right)$ pebbles. Thus, using $2m\left(\sum_{k=\lceil\frac{n}{2}\rceil+1}^{n} 2^k\right)+2m+1$ pebbles we cover $V(J_{n,m})$. □

3. The Monophonic Cover Pebbling Number of Some Zero Divisor Graphs

Theorem 3.1. *For* $\Gamma(Z_6)$*,* $\gamma_\mu(\Gamma(Z_6))=7$.

Proof. Let $V(\Gamma(Z_6))$ be $\{v_2,v_3,v_4\}$. Then, $E(\Gamma(Z_6))$ be $\{(v_2,v_3),(v_3,v_4)\}$. Since $\Gamma(Z_6)\cong P_3$, the proof follows by Theorem 1.1. □

Theorem 3.2. *For* $\Gamma(Z_8)$*,* $\gamma_\mu(\Gamma(Z_8))=7$.

Proof. Let $V(\Gamma(Z_8))=\{v_2,v_4,v_6\}$. Then, $E(\Gamma(Z_8))=\{(v_2,v_4),(v_4,v_6)\}$. Since $\Gamma(Z_8)\cong P_3$, we are done by Theorem 1.1. □

Theorem 3.3. *For* $\Gamma(Z_9)$*,* $\gamma_\mu(\Gamma(Z_9))=3$.

Proof. Let $V(\Gamma(Z_9))$ be $\{v_3,v_6\}$. Then, $E(\Gamma(Z_9))$ be $\{(v_3,v_6)\}$. We note that $\Gamma(Z_9)\cong P_2$. Hence, we are done by Theorem 1.1. □

Theorem 3.4. *For* $\Gamma(Z_{10})$*,* $\gamma_\mu(\Gamma(Z_{10}))=15$.

Proof. Let $V(\Gamma(Z_{10}))$ be $\{v_2,v_4,v_5,v_6,v_8\}$ and $E(\Gamma(Z_{10}))$ be $\{(v_2,v_5),(v_4,v_5),(v_6,v_5),(v_8,v_5)\}$. Since $\Gamma(Z_{10})\cong K_{1,4}$, by Theorem 1.2, $\mu(\Gamma(Z_{10}))=15$. □

Theorem 3.5. *For* $\Gamma(Z_{12})$*,* $\gamma_\mu(\Gamma(Z_{12}))=31$.

Table 5. Monophonic distances of all the pairs of vertices in $\Gamma(Z_{12})$

	v_2	v_3	v_4	v_6	v_8	v_9	v_{10}	$d_\mu(v_i,v_j)$
v_2	0	3	2	1	2	3	2	3
v_3	3	0	1	2	1	2	3	3
v_4	2	1	0	1	2	1	2	2
v_6	1	2	1	0	1	2	1	2
v_8	2	1	2	1	0	1	2	2
v_9	3	2	1	2	1	0	3	3
v_{10}	2	3	2	1	2	3	0	3

Proof. Let $V(\Gamma(Z_{12}))=\{v_2,v_3,v_4,v_6,v_8,v_9,v_{10}\}$. Then, $E(\Gamma(Z_{12}))=\{(v_2,v_6),(v_6,v_8),(v_6,v_4),(v_6,v_{10}),(v_8,v_9),(v_4,v_9),(v_4,v_3),(v_8,v_3)\}$. Here $n=12$. Let the monophonic path M_1 be $\{v_2,v_6,v_8,v_3\}$. From Table 5, consider the monophonic distances of all the pairs of vertices in $\Gamma(Z_n)$. If we place $2^{\frac{n}{6}}(2^{\frac{n}{4}})-2$ pebbles on the vertex v_2 we cannot cover all the vertices in $\Gamma(Z_n)$. Thus, $\gamma_\mu(\Gamma(Z_n)) \geq 2^{\frac{n}{6}}(2^{\frac{n}{4}})-1$. Now let us prove the sufficient condition.

Case 1: Let v_2 be the source vertex.
Consider the monophonic path $M_3: v_2, v_6, v_8, v_9$. To cover the vertices of M_3, by Theorem 1.1, we need 2^4-1 pebbles; covering v_9 which is at distance 3 it will cost 8 pebbles, and covering v_4, v_{10} which is at distance 2 it will cost 8 pebbles. So with 31 pebbles we can put a pebble on all vertices simultaneously. By symmetry the proof follows for v_{10}.

Case 2: Let v_9 be the source vertex.
$N(v_9)$ consists of v_4 and v_8. Let $M_2: v_9, v_8, v_6, v_2$ be the monophonic path. We note that v_8 is on the monophonic path M_2. To cover the vertices of M_2 we require $2^{\frac{n}{3}}-1$ pebbles and to cover v_4 we require 2 pebbles. Now we are left with v_3 and v_{10} which are at the monophonic distance of 2 and 3 respectively. Thus, we require $2^{\frac{n}{6}}+2^{\frac{n}{4}}$ pebbles. The number of pebbles needed to cover all the vertices is $2^{\frac{n}{3}}+1+2^{\frac{n}{6}}+2^{\frac{n}{4}}$ which are fever than $2^{\frac{n}{6}}(2^{\frac{n}{4}})-1$. By symmetry the proof follows for the source vertex v_3.

Case 3: Let v_4 be the source vertex.
The vertices in $N(v_4)$ is $\{v_3, v_9, v_6\}$. To place a pebble on the vertices of $N(v_4)$ we require 6 pebbles. The remaining 3 vertices are at distance 2. To cover these vertices we require 12 pebbles and 1 pebble for source vertex. Thus, we are done using a fewer than 31. By symmetry the proof follows for the source vertex v_8.

Case 4: Let v_6 be the source vertex.
The vertices in $N(v_6)$ are v_{10}, v_8, v_4, v_2. To cover the vertices in $N(v_6)$ and v_6 we need 9 pebbles; to cover v_3, v_9 which are at distance 2, we need 8 pebbles. Thus, with fewer than 31 pebbles we put a pebble on all the vertices simultaneously.
Thus, $\gamma_\mu(\Gamma(Z_{12})) = 31$. □

Theorem 3.6. *For* $\Gamma(Z_{14})$*,* $\gamma_\mu(\Gamma(Z_{14})) = 23$.

Proof. Let $V(\Gamma(Z_{14})) = \{v_2, v_4, v_6, v_7, v_8, v_{10}, v_{12}\}$. Then $E(\Gamma(Z_{14}))$ is $\{(v_2, v_7), (v_4, v_7), (v_6, v_7), (v_8, v_7), (v_{10}, v_7), (v_{12}, v_7)\}$. Since $\Gamma(Z_{14}) \cong K_{1,6}$, $\gamma_m u(\Gamma(Z_{14})) = 8$ by Theorem 1.2.

Theorem 3.7. *For* $\Gamma(Z_{15})$*,* $\gamma_\mu(\Gamma(Z_{15})) = 17$.

Proof. Let $V(\Gamma(Z_{15}))$ be $\{v_3, v_5, v_6, v_9, v_{10}, v_{12}\}$. Then $E(\Gamma(Z_{14}))$ be $\{(v_3, v_5), (v_9, v_5), (v_{12}, v_5), (v_{10}, v_3), (v_{10}, v_9), (v_{10}, v_{12}), (v_6, v_5), (v_6, v_{10})\}$. The graph we obtain for $\Gamma(Z_{15})$ is a complete bipartite graph with bipartite sets of sizes 2 and 4. Let $p(v_3) = 16$. To cover v_6, v_9, v_{12} which are at distance 2, we need 12 pebbles and to cover v_5, v_{10} which are in $N(v_3)$ we need 4 pebbles and there are zero pebble to cover v_3. Thus, $\gamma_\mu(\Gamma(Z_{15})) \geq 17$. Now we prove $\gamma_\mu(\Gamma(Z_{15})) \leq 17$.

Case 1: Let v_5 be the source vertex.
$N(v_5)$ is $\{v_3, v_6, v_9, v_{12}\}$. To cover the vertices in $N(v_5)$, it will cost 8 pebbles; to cover v_5 it will cost one pebble and to cover v_{10} which is at the monophonic distance 2 it will cost 4 pebbles. Thus, with fewer than 17 pebbles we could cover all the vertices. By symmetry the proof follows for the source vertex v_{10}.

Case 2: Let the source vertex be v_6.
Then $N(V_6)$ is $\{v_5, v_{10}\}$. To cover the vertices in $N(v_6)$, it will cost 4 pebbles; to cover v_6 it will cost one pebble and to cover v_3, v_9, v_{12} which are at the monophonic distance 2 it will cost 12 pebbles. By symmetry the proof follows for the source vertices v_3 and v_{12}.
Thus, $\gamma_\mu(\Gamma(Z_{15})) = 17$. □

Theorem 3.8. *For* $\Gamma(Z_{16})$*,* $\gamma_\mu(\Gamma(Z_{16})) = 23$.

Proof. Let $V(\Gamma(Z_{16})) = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}\}$. Then, $E(\Gamma(Z_{16})) = \{(v_8, v_{12}), (v_8, v_4), (v_8, v_6), (v_8, v_{10}), (v_8, v_{12}), (v_8, v_{14}), (v_4, v_{12})\}$. Here $n = 16$. Consider the monophonic path $M_1 : v_2, v_6, v_{14}$. Place $6(2^{\frac{n}{8}}) - 2$ pebbles on v_2. To cover $v_4, v_6, v_{10}, v_{12}, v_{14}$ which are at the monophonic distance 2, it will cost 20 pebbles and to cover the vertices in $N(v_2)$ it will cost 2 pebbles and so there are zero pebbles to cover v_2. Thus, $\gamma_\mu(\Gamma(Z_n)) \geq 6(2^{\frac{n}{8}}) - 1$. Now we show $\gamma_\mu(\Gamma(Z_n)) \leq 6(2^{\frac{n}{8}}) - 1$.

Case 1: Let the source vertex be v_{14}.
To cover the vertices $v_2, v_4, v_6, v_{10}, v_{12}$ which is at the distance 2 it will cost 20 pebbles; to cover v_{14} it will cost 1 pebble and to cover v_8 it will cost 2 pebbles. Thus, with a configuration of $6(2^{\frac{n}{8}}) - 1$ pebbles we can cover all the vertices. By symmetry the proof follows for the source vertices v_2, v_6, v_{10}.

Case 2: Let v_{12} be the source vertex.
The vertices in $N(v_{12})$ are v_4, v_8. There are four vertices that are at the distance 2 and so to cover these vertices it will cost 16 pebbles; to cover v_4, v_8 it will cost 4 pebbles and to cover v_{12} will cost 1 pebble. Thus, with a configuration of 21 pebbles we are able to cover all the vertices. By symmetry the proof follows for the source vertex v_4.

Case 3: Let v_8 be the source vertex.
We note that six vertices are adjacent to the source vertex. So to cover these six vertices we need 12 pebbles and to cover v_8 we need 1 pebble. Hence, with a configuration of 13 pebbles we can cover all the vertices.
Thus, $\gamma_\mu(\Gamma(Z_{16})) = 23$. □

Theorem 3.9. *For* $\Gamma(Z_{18})$*,* $\gamma_\mu(\Gamma(Z_{18})) = 61$.

Proof. Let $V(\Gamma(Z_{18})) = \{v_2, v_{13}, v_4, v_6, v_8, v_9, v_{10}, v_{12}, v_{14}, v_{15}, v_{16}\}$.
Then, $E(\Gamma(Z_{18})) = \{v_9 v_i, v_6 v_j, v_{12} v_{15}, v_{12} v_{13}\}$ where $i = 6, 12, 2, 4, 8, 10, 14, 16$ and $j = 12, 13, 15$. Here $n = 18$. Let us place $7(2^{\frac{n}{6}}) + 4$ pebbles on v_{13}. There are six vertices at the monophonic distance of $\frac{n}{6}$, which will cost $6(2^{\frac{n}{6}})$ pebbles to cover; there are 2 vertices at the monophonic distance of $\frac{n}{9}$ which will cost $2(2^{\frac{n}{9}})$ pebble to cover; there are 2 vertices at the monophonic distance 1 which will cost 4 pebbles to cover. So obviously the source vertex is not covered. Hence, $\gamma_\mu(\Gamma(Z_n)) \geq 7(2^{\frac{n}{6}}) + 5$.
Now we prove $\gamma_\mu(\Gamma(Z_n)) \leq 7(2^{\frac{n}{6}}) + 5$.

Case 1: Let v_{13} be the source vertex.
Now consider the monophonic distance from v_{13} to any other vertex in $\Gamma(Z_n)$. There will be 6 vertices at the monophonic distance $\frac{n}{6}$ which needs $6(2^{\frac{n}{6}})$ pebbles to cover; 2 vertices at the

Table 6. Monophonic distance between all pairs of vertices in $\Gamma(Z_{18})$

	v_2	v_{13}	v_4	v_6	v_8	v_9	v_{10}	v_{12}	v_{14}	v_{15}	v_{16}	$d_\mu(v_i,v_j)$
v_2	0	3	2	2	2	1	2	2	2	3	2	3
v_{13}	3	0	3	1	3	2	3	1	3	2	3	3
v_4	2	3	0	2	2	1	2	2	2	3	2	3
v_6	2	1	2	0	2	1	2	1	2	1	2	2
v_8	2	3	2	2	0	1	2	2	2	3	2	3
v_9	1	2	1	1	1	0	1	1	1	2	1	2
v_{10}	2	3	2	2	2	1	0	2	2	3	2	3
v_{12}	2	1	2	1	2	1	2	0	2	1	1	2
v_{14}	2	3	2	2	2	1	2	2	0	3	2	3
v_{15}	3	2	3	1	3	2	3	1	3	0	3	3
v_{16}	2	3	2	2	2	1	2	2	2	3	0	3

monophonic distance of $\frac{n}{9}$ which will cost $2(2^{\frac{n}{9}})$ pebbles to cover and 2 vertices which are at monophonic distance 1 which will cost 4 pebbles to cover and the remaining pebble will cover the source vertex. Thus, with a configuration $6(2^{\frac{n}{6}})+2(2^{\frac{n}{9}})+4+1=7(2^{\frac{n}{6}})+5$ pebbles we cover all the vertices. By symmetry the proof follows for the source vertex v_{15}.

Case 2: Let v_{16} be the source vertex.
Table 2 gives the monophonic distances between all pairs of vertices in $\Gamma(Z_n)$. There are 7 vertices v_{13} at the monophonic distance 2 which will cost $7(2^{\frac{n}{9}})$ pebbles to cover; 2 vertices at the monophonic distance 3 which will cost $2(2^{\frac{n}{6}})$ pebbles to cover and with the remaining pebble we cover v_{16}. So with a configuration of less than $7(2^{\frac{n}{6}})+5$ pebbles we cover all the vertices. By symmetry the proof follows for the source vertices $v_{14},v_{10},v_8,v_4,v_2$.

Case 3: Let v_9 be the source vertex.
By considering the monophonic distances from Table 6, we have 8 vertices at distance 1 which will cost $8(2^{\frac{n}{18}})$ pebbles to cover them and 2 vertices at distance 2 which will cost $2(2^{\frac{n}{9}})$ pebbles to cover them. With one pebble we can cover v_9. Thus, to cover all the vertices using monophonic path it will cost $8(2^{\frac{n}{18}})+2(2^{\frac{n}{9}})+1<7(2^{\frac{n}{6}})+5$ pebbles. □

Theorem 3.10. *For* $\Gamma(Z_{2p})$, $\gamma_\mu(\Gamma(Z_{2p}))=4p-1$, *where* p *is any prime number.*

Proof. Let $V(\Gamma(Z_{2p}))=\{v_2,v_4,\cdots,v_{2p-2},v_p\}$. Then $E(\Gamma(Z_{2p}))=\{v_iv_p,\ 2\le i\le 2p-2\}$. Since $\Gamma(Z_{2p})\cong K_{1,p-1}$, by Theorem 1.2 the result follows. □

4. Monophonic Cover Pebbling Number for Unit Graphs of $\mathbb{Z}_n$

In this section, we compute the monophonic cover pebbling number of unit graphs of $\mathbb{Z}_n$ where $2 \le n \le 10$.

Theorem 4.1. *For* $\mathbb{Z}_2, \gamma_\mu(U(\mathbb{Z}_2))$ *is* 3.

Proof. Let $V(U(\mathbb{Z}_2))$ be $\{v_0, v_1\}$. Then, $E(U(\mathbb{Z}_2)) = v_0v_1$. The resulting graph is of a path of length 1. By Theorem 1.1, $\gamma_\mu(U(\mathbb{Z}_2)) = 3$. □

Theorem 4.2. *For* $\mathbb{Z}_3, \gamma_\mu(U(\mathbb{Z}_3))$ *is* 7.

Proof. Let $V(U(\mathbb{Z}_3))$ be $\{v_0, v_1, v_2\}$. Then, $E(U(\mathbb{Z}_3)) = \{v_ov_1, v_0v_2\}$. The resulting graph is a path of length 2. By Theorem 1.1, $\gamma_\mu(U(\mathbb{Z}_3)$ is 7. □

Theorem 4.3. *For* $U(\mathbb{Z}_4), \gamma_\mu(U(\mathbb{Z}_4))$ *is* 9.

Proof. Let $V(U(\mathbb{Z}_4)) = \{v_0, v_1, v_2, v_3\}$. Then, $E(U(\mathbb{Z}_4)) = \{v_0v_1, v_0v_3, v_1v_2, v_2v_3\}$. Now $U(\mathbb{Z}_4) \cong C_4$. By Theorem 2.1, $\gamma_\mu(U(\mathbb{Z}_4))$ is 9. □

Theorem 4.4. *For* $U(\mathbb{Z}_5), \gamma_\mu(U(\mathbb{Z}_5))$ *is* 11.

Proof. Let $V(U(\mathbb{Z}_5))$ be $\{v_0, v_1, v_2, v_3, v_4\}$. Then $E(U(\mathbb{Z}_5)) = \{v_0v_1, v_0v_2, v_0v_3, v_0v_4, v_1v_2, v_1v_3, v_2v_4, v_3v_4\}$. In the resulting graph $\deg(v_0) = 4$ and $\deg(v_i) = 3$ where $1 \le i \le 4$. Let the source vertex be v_0. If we place 10 pebbles on v_1 then to cover the vertices in $N(v_1)$ it will cost 6 pebbles; to cover v_4 which is at the monophonic distance 2 it will cost 4 pebbles and there are no pebbles to cover v_1. Thus, $\gamma_\mu(U(\mathbb{Z}_5)) \ge 11$. Now let us prove $\gamma_\mu(U(\mathbb{Z}_5)) \le 11$.

Case 1: Let v_0 be the source vertex.
The vertex v_0 is adjacent to every vertex. To cover $N(v_0)$ it will cost 8 pebbles and to cover the source vertex one pebble is used. So with 8 pebbles we cover all the vertices of the graph.

Case 2: Let v_2 be the source vertex.
The vertices in $N(v_2)$ are v_0, v_1, v_4. To cover the vertices in $N(v_2)$ it will cost 6 pebbles; to cover the vertex v_3 which is at the monophonic distance 2 it will cost 4 pebbles and one pebble is used to cover the source vertex. Thus, to cover all the vertices it will cost $6 + 4 + 1 = 11$ pebbles. By symmetry the proof follows for the source vertices v_1, v_3, v_4.
Hence, $\gamma_\mu(U(\mathbb{Z}_5)) = 11$. □

Theorem 4.5. *For* $U(\mathbb{Z}_6), \gamma_\mu(U(\mathbb{Z}_6))$ *is* 33.

Proof. Let $V(U(\mathbb{Z}_6)) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$. Then, $E(U(\mathbb{Z}_5)) = \{v_0v_1, v_0v_5, v_2v_3, v_2v_5, v_3v_4, v_4v_1\}$. Since $U(\mathbb{Z}_6) \cong C_6$, by Theorem 3.1, $\gamma_\mu(U(\mathbb{Z}_6))$ is 33. □

Theorem 4.6. *For* $U(\mathbb{Z}_7), \gamma_\mu(U(\mathbb{Z}_7))$ *is 15.*

Proof. Let $V(U(\mathbb{Z}_7)) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$. Then, $E(U(\mathbb{Z}_7)) = \{v_0v_1, v_0v_2, v_0v_3, v_0v_4, v_0v_5, v_0v_6, v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_2v_4, v_2v_6, v_3v_5, v_3v_6, v_4v_5, v_4v_6, v_5v_6\}$. In resulting graph $\deg(v_0) = 6$ and $\deg(v_i) = 5$ where $1 \le i \le 6$. If we place 14 pebbles on v_1 then to cover the vertices in $N(v_1)$

it will cost 10 pebbles; to cover v_6 which is at the monophonic distance 2 it will cost 4 pebbles and there are zero pebbles to cover v_1. Thus, $\gamma_\mu(U(\mathbb{Z}_7)) \geq 15$.
Now let us prove $\gamma_\mu(U(\mathbb{Z}_7)) \leq 15$.

Case 1: Let v_0 be the source vertex.
The vertex v_0 is adjacent to every vertex. To cover $N(v_0)$ it will cost 12 pebbles and one pebble is used for the source vertex. Thus, we need 9 pebbles to cover all the vertices of the graph.

Case 2: Let v_2 be the source vertex.
$N(v_2) = \{v_0, v_1, v_3, v_4, v_6\}$. To cover the vertices in $N(v_2)$ it will cost 10 pebbles; to cover the vertex v_5 which is at the monophonic distance 2 it will cost 4 pebbles and one pebble is used to cover the source vertex. Thus, to cover all the vertices it will cost $10+4+1=15$ pebbles. By symmetry the proof follows for the source vertices v_1, v_3, v_4, v_5, v_6.
Hence, $\gamma_\mu(U(\mathbb{Z}_5)) = 11$. □

Theorem 4.7. *For* $U(\mathbb{Z}_8), \gamma_\mu(U(\mathbb{Z}_8))$ *is* 21.

Proof. Let $V(U(\mathbb{Z}_8)) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. Then, $E(U(\mathbb{Z}_8)) = \{v_0v_1, v_0v_3, v_0v_5, v_0v_7, v_1v_2, v_2v_3, v_2v_5, v_2v_7, v_1v_4, v_3v_4, v_4v_5, v_4v_7, v_1v_6, v_3v_6, v_5v_6, v_6v_7\}$. In the resulting graph $\deg(v_j) = 4$ where $0 \leq j \leq 7$ and $U(\mathbb{Z}_8)$ is a complete bipartite graph with partite sets of sizes 4 and 4. Let $p(v_0) = 20$. To cover v_2, v_4, v_6 which are at distance 2, we need 12 pebbles and to cover v_1, v_3, v_5, v_7 which are in $N(v_0)$ we need 8 pebbles and there are zero pebbles to cover v_0. Thus, $\gamma_\mu(U(\mathbb{Z}_8)) \geq 21$.
Now we prove $\gamma_\mu(U(\mathbb{Z}_8)) \leq 21$.

Case 1: Let v_1 be the source vertex.
$N(v_1)$ is $\{v_0, v_2, v_4, v_6\}$. To cover the vertices in $N(v_1)$ it will cost 8 pebbles; to cover v_1 it will cost one pebble and to cover v_3, v_5, v_7 which are at the monophonic distance 2 it will cost 12 pebbles. Thus, with 21 pebbles we could cover all the vertices. By symmetry the proof follows for the source vertices $v_0, v_2, v_3, v_4, v_5, v_6, v_7$. □

Theorem 4.8. *For* $U(\mathbb{Z}_9), \gamma_\mu(U(\mathbb{Z}_9))$ *is* 23.

Proof. Let $V(U(\mathbb{Z}_9)) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. Then, $E(U(\mathbb{Z}_9)) = \{v_0v_1, v_0v_2, v_0v_4, v_0v_5, v_0v_7, v_0v_8, v_1v_3, v_1v_4, v_1v_6, v_1v_7, v_2v_3, v_2v_5, v_2v_6, v_2v_8, v_3v_4, v_3v_5, v_3v_7, v_3v_8, v_4v_6, v_4v_7, v_5v_6, v_5v_8, v_6v_7, v_6v_8\}$. In the resulting graph $\deg(v_j) = 6$, where $j = 0, 3, 6$ and $\deg(v_k) = 5$, where $k = 1, 2, 4, 5, 7, 8$. Let $p(v_1) = 22$. Then, there will be 5 vertices at the monophonic distance 1 and 3 vertices at the monophonic distance 2. Thus, to cover these vertices it will cost $5 \times 2 + 3 \times 4 = 22$ pebbles and there are zero pebbles to cover v_1. Thus, $\gamma_\mu(U(\mathbb{Z}_9)) \geq 23$.
Now we prove $\gamma_\mu(U(\mathbb{Z}_9)) \leq 23$.

Case 1: Let v_0 be the source vertex.
$N(v_0)$ is $\{v_1, v_2, v_4, v_5, v_7, v_8\}$. To cover the vertices in $N(v_0)$ it will cost 12 pebbles; to cover v_0 it will cost 1 pebble and to cover the vertices v_3, v_6 which are at the monophonic distance 2 it will cost 8 pebbles. Thus, to cover all the vertices in the graph it will cost 21 pebbles. By symmetry the proof follows for the source vertices v_3, v_6.

Case 2: Let v_2 be the source vertex.
$N(v_2)$ is $\{v_0,v_3,v_5,v_6,v_8\}$. To cover the vertices in $N(v_2)$ it will cost 10 pebbles; to cover v_2 it will cost 1 pebble and to cover the vertices v_1,v_4,v_7 which are at the monophonic distance 2 it will cost 12 pebbles. Thus, to cover all the vertices in the graph it will cost 23 pebbles. By symmetry the proof follows for the source vertices v_1,v_4,v_5,v_7,v_8.
Hence, $\gamma_\mu(U(\mathbb{Z}_9)) = 23$. □

Theorem 4.9. *For* $U(\mathbb{Z}_{10}), \gamma_\mu(U(\mathbb{Z}_{10}))$ *is* 81.

Table 7. Monophonic distance of all the vertices $\mathbb{Z}_{10}$

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	$d_\mu(v_i,v_j)$
v_0	0	1	4	1	4	3	4	1	4	1	4
v_1	1	0	1	4	3	4	1	4	1	4	4
v_2	4	1	0	3	4	1	4	1	4	1	4
v_3	1	4	3	0	1	4	1	4	1	4	4
v_4	4	3	4	1	0	1	4	1	4	1	4
v_5	3	4	1	4	1	0	1	4	1	4	4
v_6	4	1	4	1	4	1	0	1	4	3	4
v_7	1	4	1	4	1	4	1	0	3	4	4
v_8	4	1	4	1	4	1	4	3	0	1	4
v_9	1	4	1	4	1	4	3	4	1	0	4

Proof. Let $V(U(\mathbb{Z}_{10})) = \{v_0,v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9\}$. Then $E(U(\mathbb{Z}_{10})) = \{v_0v_1, v_0v_3, v_0v_7, v_0v_9, v_1v_2, v_1v_6, v_1v_8, v_2v_5, v_2v_7, v_2v_9, v_3v_4, v_3v_6, v_3v_8, v_4v_5, v_4v_7, v_4v_9, v_5v_6, v_5v_8, v_6v_7, v_8v_9\}$. In the resulting graph $\deg(v_j) = 4$ where $0 \le i \le 9$. Let $p(v_0) = 80$ and $N(v_0)$ is $\{v_1,v_3,v_7,v_9\}$. To cover the vertices in $N(v_0)$ it will cost 8 pebbles; to cover v_5 which is at the monophonic distance 3 it will cost 8 pebbles; to cover v_2,v_4,v_6,v_8 which are at the monophonic distance 4 it will cost 64 pebbles and there are zero pebbles to cover v_0. Hence, $\gamma_\mu(U(\mathbb{Z}_{10})) \ge 81$.
Now we prove $\gamma_\mu(U(\mathbb{Z}_{10})) \le 81$.

Case 1: Let v_1 be the source vertex.
$N(v_1)$ is $\{v_0,v_2,v_6,v_8\}$. To cover the vertices in $N(v_1)$ it will cost 8 pebbles; to cover the vertex v_4 which is at the monophonic distance 3 it will cost 8 pebbles; from Table 7, to cover the vertices v_3,v_5,v_7,v_9 which are at the monophonic distance 4 it will cost 64 pebbles and one pebble is used to cover v_1. Thus, to cover all the vertices in the graph it will cost 81 pebbles. By symmetry the proof follows for the source vertices $v_0,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9$. □

Theorem 4.10. *If p is a prime number where $p \ge 3$, then $\gamma_\mu(U(\mathbb{Z}_p))$ is* $2p+1$.

Proof. Let $V(U(Z_p)) = \{x_0, x_1, x_2, \ldots, x_{p-1}\}$. The unit graph of $\mathbb{Z}_p$ forms a connected graph with $E(U(Z_p)) = \left\{E(K_p) - \{x_i x_{p-i}\} \mid i = 1, 2, \cdots, \frac{p-1}{2}\right\}$. Moreover, $\deg(x_0) = p-1$ and $\deg(x_j) = p-2$ where $j = 1, 2, \cdots, p-1$. In the resulting graph $\deg(v_0) = p-1$ and $\deg(v_i) = p-2$, where $1 \le i \le p-1$. Let $p(v_1) = 2p$. Now to cover the vertices in $N(v_1)$ it will cost $2p-4$ pebbles and to cover the vertex v_{p-1} which is at the monophonic distance 2 it will cost 4 pebbles and there are zero pebbles to cover v_1. Hence, $\gamma_\mu(\mathbb{Z}_p) \ge 2p+1$.
Now we prove $\gamma_\mu(\mathbb{Z}_p) \le 2p+1$.

Case 1: Let v_0 be the source vertex.
We see that v_0 is adjacent to all vertices. Thus, the number of pebbles needed to cover all vertices is $2p-1$.

Case 2: Let v_1 be the source vertex.
There will be $p-2$ vertices at monophonic distance 1 and one vertex at the distance 2. Thus, to cover all the vertices it will cost $2(p-2)+4+1 = 2p+1$ pebbles. By symmetry the proof follows for the source vertices $v_2, v_3, \cdots, v_{p-1}$. □

5. Conclusion

We determined the monophonic cover pebbling number of cycles, square of cycles, shadow graph of cycles, complete graphs, Jahangir graphs, fan graphs, zero divisor graphs and unit graphs. For the future research we can find the monophonic cover pebbling number of network-related graphs, product graphs and prove the NP-completeness of monophonic cover pebbling number.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *Journal of Algebra* **217**(2) (1999), 434 – 447, DOI: 10.1006/jabr.1998.7840.

[2] R. A. Beeler, T. W. Haynes, M. A. Henning and R. Keaton, Total domination cover rubbling, *Discrete Applied Mathematics* **283** (2020), 133 – 141, DOI: 10.1016/j.dam.2019.12.020.

[3] F. R. K. Chung, Pebbling in hypercubes, *SIAM Journal on Discrete Mathematics* **2**(4) (1989), 467 – 472, DOI: 10.1137/0402041.

[4] B. Crull, T. Cundiff, P. Feltman, G. H. Hurlbert, L. Pudell, Z. Szaniszlo and Z. Tuza, The cover pebbling number of graphs, *Discrete Mathematics* **296**(1) (2005), 15 – 23, DOI: 10.1016/j.disc.2005.03.009.

[5] R. Jayagopal and V. Raju, Domination parameters in shadow graph and path connected graph, *International Journal of Mathematics and Its Applications* **6**(2-B) (2018), 167 – 172.

[6] A. Lourdusamy, I. Dhivviyanandam and S. K. Iammal, Monophonic pebbling number and t-pebbling number of some graphs, *AKCE International Journal of Graphs and Combinatorics* **19**(2) (2022), 108 – 111, DOI: 10.1080/09728600.2022.2072789.

[7] A. Lourdusamy, S. K. Iammal and I. Dhivviyanandam, Detour pebbling number on some commutative ring graphs, *Communications in Mathematics and Applications* **14**(1) (2023), 323 – 331, DOI: 10.26713/cma.v14i1.2018.

[8] H. R. Maimani, M. R. Pournaki and S. Yassemi, Weakly perfect graphs arising from rings, *Glasgow Mathematical Journal* **52**(3) (2010), 417 – 425, DOI: 10.1017/S0017089510000108.

[9] M. E. Subido and I. S. Aniversario, The cover pebbling number of the join of some graphs, *Applied Mathematical Science* **8**(86) (2014), 4275 – 4283, DOI: 10.12988/ams.2014.45377.