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Research Article

Some Fixed Point Results on Generalized Weak Contractions in Partial Metric Spaces With an Application

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Abstract. In this paper, we consider generalized τ-ψ-φ weak contraction and generalized τ-φ weak contraction mappings in the setting of partial metric spaces and verify the existence of a fixed point on such spaces. Our results extend and strengthen various known results in this direction. We present an example showing the significance of our result. We also show that our result can be applied to the existence of solutions of a differential equation.

Keywords. Generalized τ-ψ-φ weak contraction mapping, τ-φ Weak contraction mapping, Partial metric space

Mathematics Subject Classification (2020). Primary: 47H10; Secondary: 54H25

1. Introduction

Matthews [11, 12] introduced a partial metric space that allows for non-zero self-distance, which is an interesting expansion of the distance function. Several authors generalized fixed point theorems on this space, as seen in e.g. Karapınar *et al.* [8], Kumam *et al.* [10], Nazam *et al.* [14] and the references cited therein.

In 1997, Alber and Guerre-Delabrieer [1] first established the concept of φ-weak contraction. They demonstrated the existence of fixed points for single-valued mappings that meet the weak contraction condition in Hilbert spaces. Rhoades [15] shown that Alber and Guerre-Delabrieer's result remains valid in complete metric spaces. Samet *et al.* [16] introduced α-ψ contractive type mappings, which expand and generalize the Banach contraction principle. Karapınar *et al*. [7] extended the concept of α-(ψ-φ) Contractive mappings to quasi-partial metric spaces. Further, Mebawondu and Mebawondu [13] introduced the notion of (ψ,φ)-Suzuki type mapping and established some results for this class of mappings. Recently, Baiz *et al*. [2, 3] presented the τ-ψ contraction, generalized τ-ψ contraction mappings and established several results for such mappings.

Following basic result is due to Rhoades [15].

Theorem 1.1. *Let* $(\mathcal{U},\eth)$ *be a complete metric space and* $\Gamma:\mathcal{U}\to\mathcal{U}$ *be a self-map satisfying the inequality*

$$\eth(\Gamma\mu,\Gamma\nu)\le\eth(\mu,\nu)-\varphi(\eth(\mu,\nu)), \tag{1.1}$$

for all $\mu,\nu\in\mathcal{U}$*, where* $\varphi:[0,\infty)\to[0,\infty)$ *is continuous and non-decreasing functions with* $\varphi(t)=0$ *if and only if* $t=0$*. Then* Γ *has a unique fixed point in* $\mathcal{U}$*.*

Dutta and Choudhury [6] introduced (ψ-φ) weak contraction, an extension of weak contraction in complete metric spaces, as follows:

Theorem 1.2. *Let* $(\mathcal{U},d)$ *be a complete metric space and* $\Gamma:\mathcal{U}\to\mathcal{U}$ *be a self-map satisfying the inequality*

$$\psi(d(\Gamma\mu,\Gamma\nu))\le\psi(d(\mu,\nu))-\varphi(d(\mu,\nu)), \tag{1.2}$$

for all $\mu,\nu\in\mathcal{U}$*, where* $\psi,\varphi:[0,\infty)\to[0,\infty)$ *are both continuous and non-decreasing functions with* $\psi(t)=\varphi(t)=0$ *if and only if* $t=0$*. Then* Γ *has a unique fixed point in* $\mathcal{U}$*.*

Recently, Karapınar *et al*. [8] established following fixed point theorem for mappings satisfying rational type contractive conditions using auxiliary functions in partial metric spaces.

Theorem 1.3. *Let* $(\mathcal{U},\varrho)$ *be a complete partial metric space and* $\Gamma:\mathcal{U}\to\mathcal{U}$ *be a self-map satisfying the inequality*

$$\psi(\varrho(\Gamma\mu,\Gamma\nu))\le\psi(M(\mu,\nu))-\varphi(M(\mu,\nu)), \tag{1.3}$$

for all $\mu,\nu\in\mathcal{U}$*, where* M *is given by*

$$M(\mu,\nu)=\max\left\{\varrho(\nu,\Gamma\nu)\frac{1+\varrho(\mu,\Gamma\mu)}{1+\varrho(\mu,\nu)},\varrho(\mu,\nu)\right\} \tag{1.4}$$

and $\psi:[0,\infty)\to[0,\infty)$ *is continuous monotone non-decreasing function with* $\psi(t)=0$ *if and only if* $t=0$ *and* $\varphi:[0,\infty)\to[0,\infty)$ *is lower semi-continuous function with* $\varphi(t)=0$ *if and only if* $t=0$*. Then* Γ *has a unique fixed point in* $\mathcal{U}$*.*

Samet *et al*. [16] presented α-admissible mappings and demonstrated the following results:

Definition 1.4. *Let* $\Gamma:\mathcal{U}\to\mathcal{U}$ *and* $\alpha:\mathcal{U}\times\mathcal{U}\to[0,\infty)$*.* Γ *is said to* α*-admissible if*

$$\alpha(\mu,\nu)\ge1\Rightarrow\alpha(\Gamma\mu,\Gamma\nu)\ge1,$$

for all $\mu,\nu\in\mathcal{U}$*.*

Definition 1.5. *Let* $(\mathfrak{U},\mathfrak{d})$ *be a metric space and* $\Gamma:\mathfrak{U}\to\mathfrak{U}$ *be a self mapping.* Γ *is said to be an* α-ψ *contractive mappings if there exists two functions* $\alpha:\mathfrak{U}\times\mathfrak{U}\to[0,\infty)$ *and* $\psi\in\Psi$ *such that*

$$\alpha(\mu,\nu)\mathfrak{d}(\Gamma\mu,\Gamma\nu)\le\psi(,\mathfrak{d}(\mu,\nu)),$$

for all $\mu,\nu\in\mathfrak{U}$.

Theorem 1.6. *Let* $(\mathfrak{U},\mathfrak{d})$ *be a complete metric space and* $\Gamma:\mathfrak{U}\to\mathfrak{U}$ *be an* α-ψ *contractive mappings satisfying the conditions:*

(i) Γ *is* α*-admissible;*

(ii) *there exists* $\mu_0\in\mathfrak{U}$ *such that* $\alpha(\mu_0,\Gamma\mu_0)\ge1$*;*

(iii) Γ *is continuous or* $\{\mu_p\}$ *is a sequence in* $\mathfrak{U}$ *such that* $\alpha(\mu_p,\mu_{p+1})\ge1$*, for all* p *and* $\mu_p\to\mu\in\mathfrak{U}$ *as* $p\to\infty$*, then* $\alpha(\mu_p,\mu)\ge1$.

Then Γ *has a fixed point.*

Further, Samet *et al*. [16] added the condition that for all $\mu,\nu\in\mathfrak{U}$, there exists $u\in\mathfrak{U}$ such that $\alpha(\mu,u)\ge1$ and $\alpha(\nu,u)\ge1$ to hypotheses of above theorems to assure the uniqueness of the fixed point.

Recently, Baiz *et al*. [2] introduced the concept of generalized τ-ψ-contraction mappings and established the following generalization of Theorem 1.6 in rectangular quasi-b metric spaces.

Definition 1.7. *Let* $(\mathfrak{U},\mathfrak{d})$ *be a rectangular quasi b metric space and* $\Gamma:\mathfrak{U}\to\mathfrak{U}$ *be a self mapping.* Γ *is said to be an* τ-ψ *contraction mapping if there exists* $\tau>1$ *and* $\psi\in\Psi$ *such that*

$$\tau\mathfrak{d}(\Gamma\mu,\Gamma\nu)\le\psi(M(\mu,\nu)),$$

for all $\mu,\nu\in\mathfrak{U}$*, where*

$$M(\mu,\nu)=\max\left\{d(\mu,\nu),\frac{(\mu,\Gamma\mu)(\nu,\Gamma\nu)}{1+(\mu,\Gamma\nu)+(\nu,\Gamma\mu)},d(\mu,\Gamma\mu),d(\nu,\Gamma\nu)\right\}. \tag{1.5}$$

Theorem 1.8. *Let* $(\mathfrak{U},\mathfrak{d})$ *be a complete rectangular quasi b metric space and* $\Gamma:\mathfrak{U}\to\mathfrak{U}$ *be a generalized* τ-ψ *contraction mapping satisfying the conditions:*

(i) *there exists* $\mu_p\in\mathfrak{U}$ *such that* $\mu_{p+1}=\Gamma\mu_p=\Gamma^p(\mu_0)$ *for all* $p\ge0$;

(ii) Γ *is continuous or* $\{\mu_p\}$ *is a sequence in* $\mathfrak{U}$ *such that* $\mu_p\to\mu\in\mathfrak{U}$ *as* $p\to\infty$.

Then Γ *has a unique fixed point.*

Matthews [11, 12] presented generalization of the metric space as follows:

Definition 1.9. *Let* $\mathfrak{U}$ *be a non-empty set. A function* $\varrho:\mathfrak{U}\times\mathfrak{U}\to[0,\infty)$ *is said to be a partial metric on* $\mathfrak{U}$ *if the following conditions hold:*

(i) $\mu=\nu\iff\varrho(\mu,\mu)=\varrho(\nu,\nu)=\varrho(\mu,\nu)$;

(ii) $\varrho(\mu,\mu)\le\varrho(\mu,\nu)$;

(iii) $\varrho(\mu,\nu)=\varrho(\nu,\mu)$;

(iv) $\varrho(\mu,\nu)\le\varrho(\mu,\delta)+\varrho(\delta,\nu)-\varrho(\delta,\delta)$,

for all $\mu,\nu,\delta\in\mathfrak{U}$.

The set $\mathcal{U}$ *equipped with the metric* ϱ *defined above is called a partial metric space and it is denoted by* $(\mathcal{U},\varrho)$ *(in short PMS). Each partial metric* ϱ *on* $\mathcal{U}$ *generates a* Γ_0 *topology* τ_ϱ *on* $\mathcal{U}$*, which has a base of the family of open* ϱ*-balls*

$$\{B_\varrho(\mu,\epsilon):\mu\in\mathcal{U},\epsilon>0\},$$

where

$$B_\varrho(\mu,\epsilon)=\{\nu\in\mathcal{U}:\varrho(\mu,\nu)<\varrho(\mu,\mu)+\epsilon\},$$

for all $\mu\in\mathcal{U}$ *and* $\epsilon>0$.

Lemma 1.10 ([12])**.** *Let* $(\mathcal{U},\varrho)$ *be a partial metric space:*

(a) *a sequence* $\{\mu_p\}$ *in* $(\mathcal{U},\varrho)$ *converges to a point* $\mu\in\mathcal{U}\iff\varrho(\mu,\mu)=\lim_{p\to\infty}\varrho(\mu_p,\mu)$,

(b) *if* $\lim_{q,p\to\infty}\varrho(\mu_p,\mu_q)$ *exists and finite. Then the sequence* $\{\mu_p\}$ *is a Cauchy sequence in* $(\mathcal{U},\varrho)$,

(c) $(\mathcal{U},\varrho)$ *is complete if every Cauchy* $\{\mu_p\}$ *in* $\mathcal{U}$ *converges to a point* $\mu\in\mathcal{U}$*, such that*

$$\varrho(\mu,\mu)=\lim_{q,p\to\infty}\varrho(\mu_q,\mu_p)=\lim_{p\to\infty}\varrho(\mu_p,\mu)=\varrho(\mu,\mu).$$

Lemma 1.11 ([12], [11])**.** *Let* ϱ *be a partial metric on* $\mathcal{U}$*, then the function* $d_\varrho:\mathcal{U}\times\mathcal{U}\to\mathbb{R}^+$ *such that*

$$d_\varrho(\mu,\nu)=2\varrho(\mu,\nu)-\varrho(\mu,\mu)-\varrho(\nu,\nu)$$

is metric on $\mathcal{U}$ *and* $(\mathcal{U},d_\varrho)$ *is metric space.*
Let $(\mathcal{U},\varrho)$ *be a partial metric space. Then*

(1) *a sequence* $\{\mu_n\}$ *in* $(\mathcal{U},\varrho)$ *is a Cauchy sequence* $\Leftrightarrow$ $\{\mu_n\}$ *is a Cauchy sequence in the metric space* $(\mathcal{U},d_\varrho)$,

(2) *the metric space* $(\mathcal{U},\varrho)$ *is complete* $\iff$ $(\mathcal{U},d_\varrho)$ *is complete. Moreover,*

$$\lim_{p\to\infty}d_\varrho(\mu_p,\mu)=0\iff\varrho(\mu,\mu)=\lim_{p\to\infty}\varrho(\mu_p,\mu)=\lim_{p,q\to\infty}\varrho(\mu_p,\mu_q).$$

Lemma 1.12 ([17])**.** *Assume that* $\mu_p\to\delta$ *as* $p\to\infty$ *in a partial metric space* $(\mathcal{U},\varrho)$ *such that* $\varrho(\delta,\delta)=0$. *Then*

$$\lim_{p\to\infty}\varrho(\mu_p,\nu)=\varrho(\delta,\nu),$$

for every $\nu\in\mathcal{U}$.

Lemma 1.13 ([5])**.** *If* $\{\mu_p\}$ *with* $\lim_{p\to\infty}\varrho(\mu_p,\mu_{p+1})=0$ *is not a Cauchy sequence in* $(\mathcal{U},\varrho)$*, and two sequences* $\{i(n)\}$ *and* $\{j(n)\}$ *of positive integers such that* $i(n)>j(n)>n$*, then following four sequences*

$$\varrho(\mu_{j(n)},\mu_{i(n)+1}),\ \varrho(\mu_{j(n)},\mu_{i(n)}),\ \varrho(\mu_{j(n)-1},\mu_{i(n)+1}),\ \varrho(\mu_{j(n)-1},\mu_{i(n)})$$

tend to $\epsilon^+>0$ *when* $n\to\infty$.

Lemma 1.14 ([4])**.** *Let* $(\mathcal{U},\varrho)$ *be a partial metric space,*

(1) *if* $\varrho(\mu,\nu)=0$ *then* $\mu=\nu$,

(2) *if* $\mu\neq\nu$ *then* $\varrho(\mu,\nu)>0$.

Definition 1.15 ([9,13]). *Let $\Phi = \{\varphi : [0,\infty) \to [0,\infty)\}$ be the set of all non-decreasing continuous function such that $\varphi(t) > 0$ for $t > 0$ and $\varphi(t) = 0 \Longleftrightarrow t = 0$.*

2. Main Results

Definition 2.1. *Let $(\mathcal{U},\varrho)$ be a partial metric space and $\Gamma : \mathcal{U} \to \mathcal{U}$ be a given self map. We say that Γ is generalized τ-ψ-φ-weak contraction mapping if there exists $\tau > 1$ and $\psi, \varphi \in \Phi$ such that for all $\mu, \nu \in \mathcal{U}$, we have*

$$\tau\psi(\varrho(\Gamma\mu,\Gamma\nu)) \le \psi(\widetilde{\Omega}(\mu,\nu))) - k\varphi(\widetilde{\Omega}(\mu,\nu)), \tag{2.1}$$

where $0 < k \le 1$, and

$$\widetilde{\Omega}(\mu,\nu) = \max\{\varrho(\mu,\nu), \varrho(\mu,\Gamma\mu), \varrho(\nu,\Gamma\nu)\}. \tag{2.2}$$

Theorem 2.2. *Let $(\mathcal{U},\varrho)$ be a complete partial metric space and $\Gamma : \mathcal{U} \to \mathcal{U}$ be self mapping. Suppose the following conditions are satisfied:*

(i) Γ *is generalized τ-ψ-φ-weak contraction mapping*;

(ii) Γ *is continuous or if $\{\mu_p\}$ is a sequence in $\mathcal{U}$ such that $\mu_p \to \mu$ as $p \to \infty$.*

Then Γ has a unique fixed point in $\mathcal{U}$.

Proof. Let μ_0 be an arbitrary point. Consider a sequence $\{\mu_p\}$ in $\mathcal{U}$ such that $\mu_{p+1} = \Gamma\mu_p$, for all $p \in \mathbb{N}$.

If $\mu_p = \mu_{p+1}$ for some $p \in \mathbb{N}$, then μ_p is a fixed point of Γ, completing the existence proof. Suppose $\mu_p \ne \mu_{p+1}$ for every $p \in \mathbb{N}$.

Now as $\tau > 1$, from (2.1) we have

$$\begin{aligned}\psi(\varrho(\mu_p,\mu_{p+1})) &= \psi(\varrho(\Gamma\mu_{p-1},\Gamma\mu_p)) \\ &\le \tau\psi(\varrho(\Gamma\mu_{p-1},\Gamma\mu_p)) \\ &\le \psi(\widetilde{\Omega}(\mu_{p-1},\mu_p)) - k\varphi(\widetilde{\Omega}(\mu_{p-1},\mu_p)),\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}\widetilde{\Omega}(\mu_{p-1},\mu_p) &= \max\{\varrho(\mu_{p-1},\mu_p), \varrho(\mu_{p-1},\Gamma\mu_{p-1}), \varrho(\mu_p,\Gamma\mu_p))\} \\ &= \max\{\varrho(\mu_{p-1},\mu_p), \varrho(\mu_{p-1},\mu_p), \varrho(\mu_p,\mu_{p+1}))\} \\ &= \max\{\varrho(\mu_{p-1},\mu_p), \varrho(\mu_p,\mu_{p+1})\}.\end{aligned} \tag{2.4}$$

Replacing (2.4) in (2.3), we get

$$\psi(\varrho(\mu_p,\mu_{p+1})) \le \psi(\max\{\varrho(\mu_p,\mu_{p+1}), \varrho(\mu_{p-1},\mu_p)\}) - k\varphi(\max\{\varrho(\mu_p,\mu_{p+1}), \varrho(\mu_{p-1},\mu_p)\})$$

Now, if $\varrho(\mu_p,\mu_{p+1}) > \varrho(\mu_{p-1},\mu_p)$ then

$$\psi(\varrho(\mu_p,\mu_{p+1})) \le \psi(\varrho(\mu_p,\mu_{p+1})) - k\varphi(\varrho(\mu_p,\mu_{p+1})),$$

i.e., $\varphi(\varrho(\mu_p,\mu_{p+1})) = 0$ then $\varrho(\mu_p,\mu_{p+1}) = 0$ which is a contradiction. Therefore,

$$\varrho(\mu_p,\mu_{p+1}) \le \varrho(\mu_{p-1},\mu_p) \tag{2.5}$$

and

$$\psi(\varrho(\mu_p,\mu_{p+1})) \le \psi(\varrho(\mu_{p-1},\mu_p)) - k\varphi(\varrho(\mu_{p-1},\mu_p)), \tag{2.6}$$

for all p.

Then from (2.5) we get that $\{\varrho(\mu_p,\mu_{p+1}) : p \in \mathbb{N}\}$ is a non negative nonincreasing sequence real numbers. Hence it is convergent to a real number, therefore there exists $\epsilon_0 \ge 0$ such that

$$\lim_{p\to\infty} \varrho(\mu_p,\mu_{p+1}) = \epsilon_0 .$$

Let $\epsilon_0 > 0$. Then taking the limit $p \to \infty$ in (2.6) we get

$$\psi(\epsilon_0) \le \psi(\epsilon_0) - k\varphi(\epsilon_0),$$

i.e., $\varphi(\epsilon_0) = 0$ then ϵ_0. This is contradiction. Hence

$$\lim_{p\to\infty} \varrho(\mu_p,\mu_{p+1}) = 0 . \tag{2.7}$$

Now, we show that $\{\mu_p\}$ is a Cauchy sequence in $\mathcal{U}$, i.e., we prove that

$$\lim_{p,q\to\infty} \varrho(\mu_p,\mu_q) = 0.$$

We prove it by contradiction.
Let

$$\lim_{p,q\to\infty} \varrho(\mu_p,\mu_q) \ne 0.$$

Then sequences in Lemma 1.13 tends to $\epsilon^+ > 0$, when $n \to \infty$.
Thus, we can see that

$$\lim_{n\to\infty} \varrho(\mu_{j(n)},\mu_{i(n)}) = \epsilon^+ . \tag{2.8}$$

Further corresponding to $j(n)$, we can choose $i(n)$ in such a way that it is smallest integer with $i(n) > j(n) > n$. Then

$$\lim_{n\to\infty} \varrho(\mu_{i(n)-1},\mu_{j(n)}) = \epsilon^+ . \tag{2.9}$$

Again,

$$\varrho(\mu_{j(n)-1},\mu_{i(n)-1}) \le \varrho(\mu_{j(n)-1},\mu_{i(n)}) + \varrho(\mu_{i(n)},\mu_{i(n)-1}) - \varrho(\mu_{i(n)},\mu_{i(n)}).$$

Letting $n \to \infty$ and using Lemma 1.13, we get

$$\lim_{n\to\infty} \varrho(\mu_{j(n)-1},\mu_{i(n)-1}) = \epsilon^+ . \tag{2.10}$$

Now in (2.1) replacing μ by $\mu_{i(n)}$ and ν by $\mu_{j(n)}$ respectively, we get

$$\begin{aligned}\psi(\varrho(\mu_{i(n)},\mu_{j(n)})) &= \psi(\varrho(\Gamma\mu_{i(n)-1},\Gamma\mu_{j(n)-1}))\\ &\le \tau\psi(\varrho(\Gamma\mu_{i(n)-1},\Gamma\mu_{j(n)-1}))\\ &\le \psi(\widetilde{\Omega}(\mu_{i(n)-1},\mu_{j(n)-1})) - \varphi(\widetilde{\Omega}(\mu_{i(n)-1},\mu_{j(n)-1})),\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}\widetilde{\Omega}(\mu_{i(n)-1},\mu_{j(n)-1}) &= \max\{\varrho(\mu_{i(n)-1},\mu_{j(n)-1}),\varrho(\mu_{i(n)-1},\Gamma\mu_{i(n)-1}),\varrho(\mu_{j(n)-1},\Gamma\mu_{j(n)-1})\}\\ &= \max\{\varrho(\mu_{i(n)-1},\mu_{j(n)-1}),\varrho(\mu_{i(n)-1},\mu_{i(n)}),\varrho(\mu_{j(n)-1},\mu_{j(n)})\}.\end{aligned} \tag{2.12}$$

Letting $n \to \infty$ in (2.12) and using (2.8), (2.9), (2.10) and Lemma 1.13, we get

$$\lim_{n\to\infty} \widetilde{\Omega}(\mu_{i(n)-1},\mu_{j(n)-1}) = \epsilon^+ . \tag{2.13}$$

Now letting $n \to \infty$ in (2.11) and using (2.13), we get

$$\psi(\epsilon^+) \le \psi(\epsilon^+) - k\varphi(\epsilon^+),$$

i.e., $\varphi(\epsilon^+) = 0$ then $\epsilon^+ = 0$. This is a contradiction, therefore

$$\lim_{p,q\to\infty} \varrho(\mu_p, \nu_q) = 0. \tag{2.14}$$

This implies that $\{\mu_p\}$ is a Cauchy sequence in $\mathcal{U}$. Thus, by Lemma 1.11 this sequence will also Cauchy in the metric space $(\mathcal{U}, d_\varrho)$. In addition, since $(\mathcal{U}, \varrho)$ is complete, $(\mathcal{U}, d_\varrho)$ is also complete. Therefore, the sequence $\{\mu_p\}$ is convergent in the space $(\mathcal{U}, d_\varrho)$, So there exists ω in $\mathcal{U}$ such that $\mu_p \to \omega$ as $p \to \infty$. Again from Lemma 1.10, we get

$$\varrho(\omega,\omega) = \lim_{p\to\infty} \varrho(\mu_p, \omega) = \lim_{q,p\to\infty} \varrho(\mu_p, \mu_q) = 0. \tag{2.15}$$

Now from (2.15) it follows that $\mu_{p+1} \to \omega$ and $\mu_{p+2} \to \omega$ as $p \to \infty$.
If Γ is continuous. Then we have

$$\omega = \lim_{p\to\infty} \mu_{p+1} = \lim_{p\to\infty} \Gamma\mu_p = \Gamma\omega.$$

If Γ is not continuous. We know that $\{\mu_p\}$ is a Cauchy sequence in $\mathcal{U}$ such that $\mu_p \to \omega$ as $n \to \infty$. Assume that $\varrho(\Gamma\omega, \omega) > 0$,

$$\begin{aligned} \psi(\varrho(\mu_{p+1}, \Gamma\omega)) &= \psi(\varrho(\Gamma\mu_p, \Gamma\omega)) \\ &\le \tau\psi(\varrho(\Gamma\mu_p, \Gamma\omega)) \\ &\le \psi(\widetilde{\Omega}(\mu_p, \omega)) - k\varphi(\widetilde{\Omega}(\mu_p, \omega)), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} \widetilde{\Omega}(\mu_p, \omega) &= \max\{\varrho(\mu_p, \omega), \varrho(\mu_p, \Gamma\mu_p), \varrho(\omega, \Gamma\omega)\} \\ &= \max\{\varrho(\mu_p, \omega), \varrho(\mu_p, \mu_{p+1}), \varrho(\omega, \Gamma\omega)\}. \end{aligned} \tag{2.17}$$

Now, taking $p \to \infty$ in (2.17), we get

$$\lim_{p\to\infty} \widetilde{\Omega}(\mu_p, \omega) = \varrho(\omega, \Gamma\omega). \tag{2.18}$$

Taking $p \to \infty$ in (2.16) and using (2.18), we get

$$\psi(\varrho(\omega, \Gamma\omega)) \le \psi(\varrho(\omega, \Gamma\omega)) - k\varphi(\varrho(\omega, \Gamma\omega)),$$

i.e., $\varphi(\varrho(\omega, \Gamma\omega)) = 0$ this implies that $\varrho(\omega, \Gamma\omega) = 0$ which is a contradiction. Therefore, $\Gamma\omega = \omega$, i.e., ω is a fixed point.
Further, suppose ω and ς be two fixed point of Γ. By replacing μ by ω and ν by ς in (2.1), we get

$$\begin{aligned} \psi(\varrho(\omega, \varsigma)) = \psi(\varrho(\Gamma\omega, \Gamma\varsigma)) &\le \tau\varrho(\Gamma\omega, \Gamma\varsigma) \\ &\le \psi(\widetilde{\Omega}(\omega, \varsigma)) - k\varphi(\widetilde{\Omega}(\omega, \varsigma)), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} \widetilde{\Omega}(\omega, \varsigma) &= \max\{\varrho(\omega, \varsigma), \varrho(\omega, \Gamma\omega), \varrho(\varsigma, \Gamma\varsigma)\} \\ &= \max\{\varrho(\omega, \varsigma), \varrho(\omega, \omega), \varrho(\varsigma, \varsigma)\} \\ &= \varrho(\omega, \varsigma). \end{aligned} \tag{2.20}$$

Putting (2.20) in (2.19), we get

$$\psi(\varrho(\omega,\varsigma)) \le \psi(\varrho(\omega,\varsigma)) - k\varphi(\varrho(\omega,\varsigma)),$$

i.e., $\varphi(\varrho(\omega,\varsigma)) = 0$ implies that $\varrho(\omega,\varsigma) = 0$ which is a contradiction. Hence Γ has a unique fixed point. This completes the proof. □

Definition 2.3. *Let $(\mathfrak{U},\rho)$ be a partial metric space and $\Gamma : \mathfrak{U} \to \mathfrak{U}$ be a given self map. We say that Γ is generalized τ-φ-weak contraction mapping if there exists $\tau > 1$ and $\varphi \in \Phi$ such that for all $\mu,\nu \in \mathfrak{U}$, we have*

$$\tau(\rho(\Gamma\mu,\Gamma\nu)) \le \widetilde{\Sigma}(\mu,\nu)) - \varphi(\widetilde{\Sigma}(\mu,\nu)), \tag{2.21}$$

where

$$\widetilde{\Sigma}(\mu,\nu) = \max\left\{\rho(\mu,\nu), \frac{1+\rho(\mu,\Gamma\mu)}{1+\rho(\mu,\nu)}\rho(\mu,\Gamma\mu), \frac{1+\rho(\mu,\Gamma\mu)}{1+\rho(\mu,\nu)}\rho(\nu,\Gamma\nu)\right\}. \tag{2.22}$$

Theorem 2.4. *Let $(\mathfrak{U},\rho)$ be a complete partial metric space and $\Gamma : \mathfrak{U} \to \mathfrak{U}$ be self mapping. Suppose the following conditions are satisfied:*

(i) *Γ is generalized τ-φ-weak contraction mapping;*

(ii) *Γ is continuous or if $\{\mu_p\}$ is a sequence in $\mathfrak{U}$ such that $\mu_p \to \mu$ as $p \to \infty$.*

Then Γ has a unique fixed point in $\mathfrak{U}$.

Proof. Let μ_0 be an arbitrary point. Suppose we have a sequence $\{\mu_p\}$ in $\mathfrak{U}$ such that $\mu_{p+1} = \Gamma\mu_p$ for all $p \in \mathbb{N}$.
If $\mu_p = \mu_{p+1}$ for some $p \in \mathbb{N}$, then μ_p is a fixed point of Γ and the existence part of the proof is finished. Suppose $\mu_p \ne \mu_{p+1}$ for every $p \in \mathbb{N}$.
Now as $\tau > 1$, from (2.21), we have

$$\begin{aligned}\varrho(\mu_p,\mu_{p+1}) = \varrho(\Gamma\mu_{p-1},\Gamma\mu_p) &\le \tau(\varrho(\Gamma\mu_{p-1},\Gamma\mu_p)) \\ &\le \widetilde{\Sigma}(\mu_{p-1},\mu_p) - \varphi(\widetilde{\Sigma}(\mu_{p-1},\mu_p)),\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}\widetilde{\Sigma}(\mu_{p-1},\mu_p) &= \max\left\{\varrho(\mu_{p-1},\mu_p), \frac{1+\varrho(\mu_{p-1},\Gamma\mu_{p-1})}{1+\varrho(\mu_{p-1},\mu_p)}\varrho(\mu_{p-1},\Gamma\mu_{p-1}), \frac{1+\varrho(\mu_{p-1},\Gamma\mu_{p-1})}{1+\varrho(\mu_{p-1},\mu_p)}\varrho(\mu_p,\Gamma\mu_p)\right\} \\ &= \max\left\{\varrho(\mu_{p-1},\mu_p), \frac{1+\varrho(\mu_{p-1},\mu_p)}{1+\varrho(\mu_{p-1},\mu_p)}\varrho(\mu_{p-1},\mu_p), \frac{1+\varrho(\mu_{p-1},\mu_p)}{1+\varrho(\mu_{p-1},\mu_p)}\varrho(\mu_p,\mu_{p+1})\right\} \\ &= \max\{\varrho(\mu_{p-1},\mu_p),\varrho(\mu_p,\mu_{p+1})\}.\end{aligned} \tag{2.24}$$

Replacing (2.24) in (2.23), we get that

$$\varrho(\mu_p,\mu_{p+1}) \le \max\{\varrho(\mu_p,\mu_{p+1}),\varrho(\mu_{p-1},\mu_p)\} - \varphi(\max\{\varrho(\mu_p,\mu_{p+1}),\varrho(\mu_{p-1},\mu_p)\}).$$

Now, if $\varrho(\mu_p,\mu_{p+1}) > \varrho(\mu_{p-1},\mu_p)$ then

$$\varrho(\mu_p,\mu_{p+1}) \le \varrho(\mu_p,\mu_{p+1}) - \varphi(\varrho(\mu_p,\mu_{p+1})) < \varrho(\mu_p,\mu_{p+1})$$

which is a contradiction. Therefore,

$$\varrho(\mu_p,\mu_{p+1}) \le \varrho(\mu_{p-1},\mu_p) \tag{2.25}$$

and

$$\varrho(\mu_p,\mu_{p+1}) \le \varrho(\mu_{p-1},\mu_p) - \varphi(\varrho(\mu_{p-1},\mu_p)), \tag{2.26}$$

for all p.

Then from (2.25) we get that $\{\varrho(\mu_p,\mu_{p+1}) : p \in \mathbb{N}\}$ is a non negative nonincreasing sequence real numbers. Hence it is convergent to a real number, therefore there exists $\epsilon_0 \ge 0$ such that

$$\lim_{p\to\infty} \varrho(\mu_p,\mu_{p+1}) = \epsilon_0 .$$

Let ϵ_0. Then taking the limit $n \to \infty$ in (2.26), we get

$$\epsilon_0 \le \epsilon_0 - \varphi(\epsilon_0) < \epsilon_0.$$

This is contradiction. Hence

$$\lim_{n\to\infty} \varrho(\mu_p,\mu_{p+1}) = 0. \tag{2.27}$$

Now, we show that $\{\mu_p\}$ is a Cauchy sequence in $\mathcal{U}$, i.e., we prove that

$$\lim_{p,q\to\infty} \rho(\mu_p,\mu_q) = 0.$$

We prove it by contradiction.

Let

$$\lim_{p,q\to\infty} \rho(\mu_p,\mu_q) \ne 0.$$

Further corresponding to $j(n)$, we can choose $i(n)$ in such a way that it is smallest integer with $i(n) > j(n) > n$. Then, sequences in Lemma 1.13 tends to $\epsilon^+ > 0$, when $p \to \infty$.
Now in (2.21) replacing μ by $\mu_{i(n)}$ and ν by $\mu_{j(n)}$ respectively, we get

$$\begin{aligned}\rho(\mu_{i(n)},\mu_{j(n)}) &= \rho(\Gamma\mu_{i(n)-1},\Gamma\mu_{j(n)-1})\\ &\le \tau(\rho(\Gamma\mu_{i(n)-1},\Gamma\mu_{j(n)-1}))\\ &\le \widetilde{\Sigma}(\mu_{i(n)-1},\mu_{j(n)-1})) - \varphi(\widetilde{\Sigma}(\mu_{i(n)-1},\mu_{j(n)-1})),\end{aligned} \tag{2.28}$$

where

$$\begin{aligned}\widetilde{\Sigma}(\mu_{i(n)-1},\mu_{j(n)-1}) &= \max\Big\{\rho(\mu_{i(n)-1},\mu_{j(n)-1}), \frac{1+\rho(\mu_{i(n)-1},\Gamma\mu_{i(n)-1})}{1+\rho(\mu_{i(n)-1},\mu_{j(n)-1})}\rho(\mu_{i(n)-1},\Gamma\mu_{i(n)-1}),\\ &\qquad \frac{1+\rho(\mu_{i(n)-1},\Gamma\mu_{i(n)-1})}{1+\rho(\mu_{i(n)-1},\mu_{j(n)-1})}\rho(\mu_{j(n)-1},\Gamma\mu_{j(n)-1})\Big\}\\ &= \max\Big\{\rho(\mu_{i(n)-1},\mu_{j(n)-1}), \frac{1+\rho(\mu_{i(n)-1},\mu_{i(n)})}{1+\rho(\mu_{i(n)-1},\mu_{j(n)-1})}\rho(\mu_{i(n)-1},\mu_{i(n)}),\\ &\qquad \frac{1+\rho(\mu_{i(n)-1},\mu_{i(n)})}{1+\rho(\mu_{i(n)-1},\mu_{j(n)-1})}\rho(\mu_{j(n)-1},\mu_{j(n)})\Big\}.\end{aligned} \tag{2.29}$$

Letting $n \to \infty$ in (2.29) and using (2.8), (2.9), (2.10) and Lemma 1.13 we get

$$\lim_{n\to\infty} \widetilde{\Sigma}(\mu_{i(n)-1},\mu_{j(n)-1}) = \epsilon^+ . \tag{2.30}$$

Now Letting $n \to \infty$ in (2.28) and using (2.30), we get

$$\epsilon^+ \le \epsilon^+ - \varphi(\epsilon^+) < \epsilon^+ .$$

This is a contradiction, therefore

$$\lim_{p,q\to\infty}\rho(\mu_p,\mu_q)=0. \tag{2.31}$$

This implies that $\{\mu_p\}$ is a Cauchy sequence in $\mathcal{U}$. Thus, by Lemma 1.11 this sequence will also Cauchy in the metric space $(\mathcal{U},d_\varrho)$. In addition, since $(\mathcal{U},\varrho)$ is complete, $(\mathcal{U},d_\varrho)$ is also complete. Therefore, the sequence $\{\mu_p\}$ is convergent in the space $(\mathcal{U},d_\varrho)$, So there exists ω in $\mathcal{U}$ such that $\mu_p\to\omega$ as $p\to\infty$. Again from Lemma 1.10, we get

$$\varrho(\omega,\omega)=\lim_{p\to\infty}\varrho(\mu_p,\omega)=\lim_{q,p\to\infty}\varrho(\mu_p,\mu_q)=0. \tag{2.32}$$

If Γ is continuous. Then, we have

$$\omega=\lim_{p\to\infty}\mu_{p+1}=\lim_{p\to\infty}\Gamma\mu_p=\Gamma\mu_p.$$

If Γ is not continuous, We know that $\{\mu_p\}$ is a Cauchy sequence in $\mathcal{U}$ such that $\mu_p\to\omega$ as $p\to\infty$. Assume that $\rho(\Gamma\omega,\omega)>0$.

$$\begin{aligned}\varrho(\mu_{p+1},\Gamma\omega)&=\varrho(\Gamma\mu_p,\Gamma\omega)\\&\le\tau\psi(\varrho(\Gamma\mu_p,\Gamma\omega))\\&\le\widetilde{\Sigma}(\mu_p,\omega)-\varphi(\widetilde{\Sigma}(\mu_p,\omega)),\end{aligned} \tag{2.33}$$

where

$$\begin{aligned}\widetilde{\Sigma}(\mu_p,\omega)&=\max\left\{\varrho(\mu_p,\omega),\frac{1+\varrho(\mu_p,\Gamma\mu_p)}{1+\varrho(\mu_p,\omega)}\varrho(\mu_p,\Gamma\mu_p),\frac{1+\varrho(\mu_p,\Gamma\mu_p)}{1+\varrho(\mu_p,\omega)}\varrho(\omega,\Gamma\omega)\right\}\\&=\max\left\{\varrho(\mu_p,\omega),\frac{1+\varrho(\mu_p,\mu_{p+1})}{1+\varrho(\mu_p,\omega)}\varrho(\mu_p,\mu_{p+1}),\frac{1+\varrho(\mu_p,\mu_{p+1})}{1+\varrho(\mu_p,\omega)}\varrho(\omega,\Gamma\omega)\right\}.\end{aligned} \tag{2.34}$$

Now, taking $p\to\infty$ in (2.34), we get

$$\lim_{p\to\infty}\widetilde{\Sigma}(\mu_p,\omega)=\rho(\omega,\Gamma\omega). \tag{2.35}$$

Taking $p\to\infty$ in (2.33) and using (2.35) we get

$$\varrho(\omega,\Gamma\omega)\le\varrho(\omega,\Gamma\omega))-\varphi(\varrho(\omega,\Gamma\omega))<\varrho(\omega,\Gamma\omega)$$

which is a contradiction. Therefore $\Gamma\omega=\omega$, i.e., ω is a fixed point.
Further, suppose ω and ς be two fixed point of Γ. By replacing μ by ω and ν by ς in (2.21), we get

$$\varrho(\omega,\varsigma)=\varrho(\Gamma\omega,\Gamma\varsigma)\le\tau\varrho(\Gamma\omega,\Gamma\varsigma)\le\psi(\widetilde{\Sigma}(\omega,\varsigma))-k\varphi(\widetilde{\Sigma}(\omega,\varsigma)), \tag{2.36}$$

where

$$\begin{aligned}\widetilde{\Sigma}(\omega,\varsigma)&=\max\left\{\varrho(\omega,\varsigma),\frac{1+\varrho(\omega,\Gamma\omega)}{1+\varrho(\omega,\varsigma)}\varrho(\omega,\Gamma\omega),\frac{1+\varrho(\omega,\Gamma\omega)}{1+\varrho(\omega,\varsigma)}\varrho(\varsigma,\Gamma\varsigma)\right\}\\&=\max\left\{\varrho(\omega,\varsigma),\frac{1+\varrho(\omega,\omega)}{1+\varrho(\omega,\varsigma)}\varrho(\omega,\omega),\frac{1+\varrho(\omega,\omega)}{1+\varrho(\omega,\varsigma)}\varrho(\varsigma,\varsigma)\right\}\\&=\varrho(\omega,\varsigma).\end{aligned} \tag{2.37}$$

Putting (2.37) in (2.36), we get

$$\varrho(\omega,\varsigma)\le\varrho(\omega,\varsigma)-\varphi(\varrho(\omega,\varsigma))<\varrho(\omega,\varsigma)$$

which is a contradiction. Hence Γ has a unique fixed point. This completes the proof. □

Following are consequences of the theorems.

Corollary 2.5. *Let* $(\mathcal{U},\varrho)$ *be a complete partial metric space.* $\Gamma:\mathcal{U}\to\mathcal{U}$ *be self-mapping and* $\tau>1$ *such that*

$$\tau\psi(\varrho(\Gamma\mu,\Gamma\nu))\le\psi(\varrho(\mu,\nu))-\varphi(\varrho(\mu,\nu)), \tag{2.38}$$

for all $\mu,\nu\in\mathcal{U}$ *and* $\psi,\varphi\in\Phi$. *Then* Γ *has a unique fixed point in* $\mathcal{U}$.

Corollary 2.6. *Let* $(\mathcal{U},\varrho)$ *be a complete partial metric space.* $\Gamma:\mathcal{U}\to\mathcal{U}$ *be self-mapping and* $\tau>1$ *such that*

$$\tau\varrho(\Gamma\mu,\Gamma\nu))\le\varrho(\mu,\nu))-\varphi(\varrho(\mu,\nu)), \tag{2.39}$$

for all $\mu,\nu\in\mathcal{U}$ *and* $\varphi\in\Phi$. *Then* Γ *has a unique fixed point in* $\mathcal{U}$.

Example 2.7. Let $\mathcal{U}=[0,1]$ and $\varrho(\mu,\nu)=\max\{\mu,\nu\}$. Then $(\mathcal{U},\varrho)$ is a complete partial metric space. Consider the mapping $\Gamma:\mathcal{U}\to\mathcal{U}$ defined by $\Gamma(\mu)=\frac{\mu}{5}$ and $\psi,\varphi:[0,\infty)\to[0,\infty)$ be such that $\psi(t)=t$, $\varphi(t)=\frac{t}{2}$. Without loss of generality we assume that $\mu\ge\nu$. Then for $\mu,\nu\in[0,1]$ and $\tau=\frac{5}{2}$, we get

$$\begin{aligned}\tau\psi(\varrho(\Gamma\mu,\Gamma\nu))&=\frac{5}{2}\psi\left(\varrho\left(\frac{\mu}{5},\frac{\nu}{5}\right)\right)=\frac{5}{2}\left(\frac{\mu}{5}\right)=\frac{\mu}{2}\\&\le\mu-\frac{\mu}{4}\\&=\psi(\varrho(\mu,\nu))-\frac{1}{2}\varphi(\varrho(\mu,\nu))\\&\le\psi(\widetilde{\Omega}(\mu,\nu))-k\varphi(\widetilde{\Omega}(\mu,\nu)),\end{aligned}$$

where $k=\frac{1}{2}$ and

$$\widetilde{\Omega}(\mu,\nu)=\max\{\varrho(\mu,\nu),\varrho(\mu,\Gamma\mu),\varrho(\nu,\Gamma\nu)\}.$$

Therefore, all the conditions of Theorem 2.2 are satisfied. Hence Γ has a fixed point, which in this case is 0.

3. Application

In this section, we give an application of Theorem 2.4 to the solution of second order differential equation of the form

$$\left.\begin{aligned}&\mu''(t)=-f(t,\mu(t)),\quad t\in I,\\&\mu(0)=\mu(1)=0,\end{aligned}\right\} \tag{3.1}$$

where $I=[0,1]$, $f:I\times\mathbb{R}\to\mathbb{R}$ is a continuous function.
Consider the space $\mathcal{U}=C(I)$ of continuous function defined on I. Define

$$\varrho(\mu,\nu)=\eth(\mu,\nu)+c_p=\sup_{t\in[0,1]}|\mu(t)-\nu(t)|+c_p,$$

for all $\mu,\nu\in\mathcal{U}$, where $\{c_p\}$ is a sequence of positive real numbers satisfying, $c_p\to0$ as $p\to\infty$. It is easy to verify that $(\mathcal{U},\varrho)$ is a complete partial metric space.

It is well-known that the problem (3.1) is equivalent to the integral equation

$$\mu(t)=\int_0^1 g(t,r)f(r,\mu(r))\,dr, \tag{3.2}$$

for all $t\in[0,1]$ where g is the Green function defined by

$$g(t,r)=\begin{cases}(1-t)r, & 0\le r\le t\le 1,\\ (1-r)t, & 0\le t\le r\le 1.\end{cases} \tag{3.3}$$

If $\mu\in C^2(I)$ then $\mu\in C(I)$ is a solution of (3.1) if and only if it is a solution (3.2).

Theorem 3.1. *Let $\mathfrak{U}=C(I)$ and $\Gamma:\mathfrak{U}\to\mathfrak{U}$ be an operator given by*

$$\Gamma\mu(t)=\int_0^1 g(t,r)f(r,x(r))\,dr, \tag{3.4}$$

for all $\mu\in\mathfrak{U}$ and $t\in I=[0,1]$. Suppose for all $u,v\in\mathfrak{U}$ there exists $A\in(0,1)$ such that

$$|f(r,u)-f(r,v)|\le 8A\sigma(|u(t)-v(t)|)|, \tag{3.5}$$

where $\sigma:[0,\infty)\to[0,\infty)$ is a non-decreasing function such that $\sigma(t)=t-\varphi(t)$, $\varphi(t)<t$ and $\varphi\in\Phi$. Then, the second order differential equation (3.1) *has a solution.*

Proof. Following the assumptions of Theorem 3.1, we have

$$\begin{aligned}\varrho(\Gamma\mu,\Gamma\nu)&=\eth(\Gamma\mu,\Gamma\nu)+c_p\\ &=\sup_{t\in[0,1]}|\Gamma\mu(t)-\Gamma\nu(t)|+c_p\\ &=\sup_{t\in[0,1]}\int_0^1 g(t,r)|[f(r,\mu(r))-f(r,\nu(r)]|\,dr+c_p\\ &=\sup_{t\in[0,1]}\int_0^1 g(t,r)8A\sigma(|\mu(r)-\nu(r)|)\,dr+c_p.\end{aligned} \tag{3.6}$$

Since the function σ is non-decreasing, we have

$$\sigma(|\mu(r)-\nu(r)|)\le\sigma\Big(\sup_{t\in[0,1]}|\mu(r)-\nu(r)|\Big)=\sigma(\eth(\mu,\nu)). \tag{3.7}$$

Moreover,

$$\sup_{t\in[0,1]}\int_0^1 g(t,r)\,dr=\frac{1}{8}. \tag{3.8}$$

Taking $p\to\infty$ in (3.6) and using (3.7), (3.8) and fact that $c_p\to 0$ as $p\to\infty$, we get

$$\begin{aligned}&\rho(\Gamma\mu,\Gamma\nu)\le A\sigma(\eth(\mu,\nu)),\\ &\frac{1}{A}\rho(\Gamma\mu,\Gamma\nu)\le\sigma(\eth(\mu,\nu)).\end{aligned}$$

Taking $\frac{1}{A}=\tau>1$, we get

$$\tau\rho(\Gamma\mu,\Gamma\nu)\le\sigma(\eth(\mu,\nu))\le\sigma(\varrho(\mu,\nu))\le\sigma(\widetilde{\Sigma}(\mu,\nu))=\widetilde{\Sigma}(\mu,\nu)-\varphi(\widetilde{\Sigma}(\mu,\nu)).$$

Clearly, all the conditions of Theorem 2.4 are satisfied and so Γ has a unique fixed point. Thus, the system of integral equations (3.2) has a unique solution. □

4. Conclusion

This article introduces a few mappings such that generalized τ-ψ-φ weak contraction and generalized τ-φ weak contraction mappings to determine fixed point results in complete partial metric spaces. In the line of work carried out in this paper, one may consider the deduction of fixed point theorems under various conditions in partial metric spaces, which could be an active subject of research for future scholars in this branch.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] Y. I. Alber and S. Guerre-Delabrieer, Principles of weakly contractive maps in Hilbert spaces, in: *New Results in Operator Theory and Its Applications. Operator Theory: Advances and Applications*, I. Gohberg and Y. Lyubich (editors), Vol. **98**, Birkhäuser, Basel (1997), DOI: 10.1007/978-3-0348-8910-0_2.

[2] A. Baiz, J. Mouline and A. Kari, Fixed point theorems for generalized τ-ψ-contraction mappings in rectangular quasi b-metric spaces, *Advances in Fixed Point Theory* **13** (2023), Article ID 10, DOI: 10.28919/afpt/8114.

[3] A. Baiz, J. Mouline, Y. E. Bekri, A. Faiz and K. Bouzkoura, New fixed point theorems for τ-ψ-contraction mappings in complete rectangular M-metric space, *Advances in Fixed Point Theory* **13** (2023), Article ID 18, DOI: 10.28919/afpt/8190.

[4] S. Chandok, D. Kumar and M. S. Khan, Some results in partial metric space using auxiliary functions, *Applied Mathematics E-Notes* **15** (2015), 233 – 242, URL: https://www.emis.de/journals/AMEN/2015/AMEN-150228.pdf.

[5] D. Đukić, Z. Kadelburg and S. Radenović, Fixed points of Geraghty-type mappings in various generalized metric spaces, *Abstract and Applied Analysis* **2011** (2011), 561245, 13 pages, DOI: 10.1155/2011/561245.

[6] P. N. Dutta and B. S. Choudhury, A generalisation of contraction principle in metric spaces, *Fixed Point Theory and Applications* **2008** (2008), Article number 406368, DOI: 10.1155/2008/406368.

[7] E. Karapınar, L. Gholizadeh, H. Alsulami and M. Noorwali, α-(ψ-φ) Contractive mappings on quasi-partial metric spaces, *Fixed Point Theory and Applications* **2015** (2015), Article number 105, DOI: 10.1186/s13663-015-0352-z.

[8] E. Karapınar, W. Shatanawi and K. Tas, Fixed point theorem on partial metric spaces involving rational expressions, *Miskolc Mathematical Notes*, **14**(1) (2013), 135 – 142, DOI: 10.18514/MMN.2013.471.

[9] M. S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, *Bulletin of the Australian Mathematical Society* **30**(1) (1984), 1 – 9, DOI: 10.1017/S0004972700001659.

[10] P. Kumam, C. Vetro and F. Vetro, Fixed points for weak α-ψ-contractions in partial metric spaces, *Abstract and Applied Analysis* **2013** (2013), 986028, 9 pages, DOI: 10.1155/2013/986028.

[11] S. G. Matthews, Partial metric topology, *Annals of the New York Academy of Sciences* **728** (1994), 183 – 197, DOI: 10.1111/j.1749-6632.1994.tb44144.x.

[12] S. G. Matthews, Partial metric topology, Research Report 2012, Department of Computer Science, University of Warwick, UK (1992).

[13] A. A. Mebawondu and S. I. Mebawondu, Generalized Suzuki (ψ,φ)-contraction in complete metric spaces, *International Journal of Nonlinear Analysis and Applications* **12**(1) (2021), 963 – 978, DOI: 10.22075/ijnaa.2020.15983.1837.

[14] M. Nazam, M. Arshad and C. Park, Fixed point theorems for improved α-Geraghty contractions in partial metric spaces, *Journal of Nonlinear Sciences and Applications* **9**(6) (2016), 4436 – 4449, DOI: 10.22436/jnsa.009.06.83.

[15] B. E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Analysis: Theory, Methods & Applications* **47**(4) (2001), 2683 – 2693, DOI: 10.1016/S0362-546X(01)00388-1.

[16] B. Samet, C. Vetro and P. Vetro, Fixed point theorem for α-ψ contractive type mappings, *Nonlinear Analysis: Theory, Methods & Applications* **75**(4) (2012), 2154 – 2165, DOI: 10.1016/j.na.2011.10.014.

[17] I. Yildirim, A. H. Ansari, M. S. Khan and B. Fisher, Fixed point theorems on partial metric spaces involving rational type expressions with C-class functions, *Turkish Journal of Mathematics and Computer Science* **7** (2017), 1 – 9, URL: https://dergipark.org.tr/en/pub/tjmcs/issue/31829/313306.