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Research Article

Some Properties of (ϵ)-Kenmotsu Manifolds With Quarter-Symmetric Non-Metric Connection

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Abstract. The objective of this paper is to investigate the (ϵ)-Kenmotsu manifolds with quarter-symmetric non-metric connection. We have investigate an (ϵ)-Kenmotsu manifolds admitting the quarter-symmetric non-metric connections satisfying certain conditions. We have further provided the equivalent conditions for Ricci soliton in an (ϵ)-Kenmotsu manifolds to be shrinking or expanding with the quarter-symmetric non-metric connection. We have also investigated ϕ-projectively flat, Quasi-projectively flat and some interesting results. Finally, we have given an example of 3-dimensional (ϵ)-Kenmotsu manifolds with respect to quarter-symmetric non-metric connection.

Keywords. (ϵ)-Kenmotsu manifold, Quarter-symmetric non-metric connection, Ricci soliton, Quasi-projectively flat, ϕ-projectively flat

Mathematics Subject Classification (2020). 53C15, 53C25, 53D15, 53D10, 53C05

1. Introduction

Kenmotsu [10] investigated the geometry of the Kenmotsu manifolds. The concept of semi-symmetric connection on a differentiable manifold was established by Friedmann and Schouten [5] in 1924. The study of semi-symmetric connection in a Riemannian manifold was introduced by Yano [17]. In 1972, a class of contact Riemannian manifolds that satisfied the specific requirements are investigated by Kenmotsu [10]. The conception of (ϵ)-Kenmotsu manifolds with indefinite metric was brought by De and Sarkar [2]. De *et al.* [4] studied $*$-Ricci soliton on (ϵ)-Kenmotsu manifolds. $*$-conformal η-Ricci solitons in (ϵ)-Kenmotsu manifolds has been investigated by Haseeb and Prasad [8]. In recent research on (ϵ)-Kenmotsu manifold with a semi-symmetric metric connection was conducted by Khan *et al*. [9]. Several authors, e.g., Haseeb [7], Haseeb *et al*. [9], Singh *et al*. [13], and Venkatesha and Vishnuvardhana [15]) also studied (ϵ)-Kenmotsu manifold.

The quarter-symmetric non-metric connection is the most recognized connection and the study on Kenmotsu Manifold with quarter-symmetric non-metric ϕ-connection was studied by Singh and Srivastava [12] in 2016. Hamilton introduce the theory of Ricci flow to establish a canonical metric on a smooth manifold in 1982. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}g(t) = -R(t)g(t).$$

A Ricci soliton (g,V,λ) on a Riemannian manifold (M,g) is generalization of an Einstein metric such that it satisfies the following condition (Hamilton [6]):

$$Lvg + 2S + g = 0, \tag{1}$$

where S is the Ricci tensor, L_v is the Lie derivative operator along the vector field V on (M,g) and λ is a real number. The Ricci soliton is said to be shrinking, steady, or expanding according as λ is negative, zero or positive.

The present work is organized as follows: In Section 1 introduction is discussed. Section 2 is equipped with some prerequisites about (ϵ)-Kenmotsu manifolds. Section 3, deals with (ϵ)-Kenmotsu manifold admitting the quarter-symmetric non-metric connection with some interesting results. Section 4 concerned with the study of Ricci soliton on an (ϵ)-Kenmotsu manifold admitting the quarter-symmetric non-metric connection. In Section 5, we study the quasi-projectively flat (ϵ)-Kenmotsu manifold admitting the quarter-symmetric non-metric connection. Section 6 is devoted to the study of ϕ-projectively flat (ϵ)-Kenmotsu manifold with respect to the connection $\widetilde{\nabla}$. In the last section, we have given an example of 3-dimensional (ϵ)-Kenmotsu manifold in the support of our results.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M together with a $(1,1)$-tensor field ϕ, a vector field ξ, η is 1-form and semi-Riemannian metric g is called an (ϵ)-almost contact metric manifold if

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{2}$$

$$g(\xi,\xi) = \epsilon, \quad \epsilon g(X,\xi) = \eta(X), \tag{3}$$

$$g(\phi X,\phi Y) = g(X,Y) - \epsilon\eta(X)\eta(Y), \tag{4}$$

where ϵ is 1 or -1 according as ξ is spacelike or timelike, and the rank of ϕ is $2n$. It is important to mention that in the above definition ξ is never a lightlike vector field. If

$$d\eta(X,Y) = g(\phi X,Y), \tag{5}$$

for all $X,Y \in \chi(M)$, where $\chi(M)$ is a set of all smooth vector fields on M. Then M is an ϵ-contact metric manifold. It follows that

$$\eta(\phi X) = 0, \quad \phi\xi = 0, \tag{6}$$

$$g(X,\phi Y) = -g(\phi X,Y). \tag{7}$$

Moreover, if the manifold satisfies

$$(\nabla_X \phi)Y = -g(X,\phi Y)\xi - \epsilon\eta(Y)\phi X, \tag{8}$$

where ∇ denotes the Riemannian connection of g, then manifold is called an (ϵ)-Kenmotsu manifold (De and Sarkar [2]). On an (ϵ)-Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g,\epsilon)$, the following relations hold:

$$(\nabla_X \xi) = \epsilon(X - \eta(X)\xi), \tag{9}$$

$$(\nabla_X \eta)Y = g(X,Y) - \epsilon\eta(X)\eta(Y), \tag{10}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{11}$$

$$R(\xi,X)Y = \eta(Y)X - \epsilon g(X,Y)\xi, \tag{12}$$

$$R(X,Y)\phi Z = \phi R(X,Y)Z + \epsilon[g(Y,Z)\phi X - g(X,Z)\phi Y + g(X,\phi Z)Y - g(Y,\phi Z)X], \tag{13}$$

$$\eta(R(X,Y)Z) = \epsilon[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)], \tag{14}$$

$$S(X,\xi) = -2n\eta(X), \tag{15}$$

$$S(\phi X,\phi Y) = S(X,Y) + 2n\epsilon\eta(X)\eta(Y), \tag{16}$$

$$S(X,Y) = g(QX,Y), \tag{17}$$

$$Q\xi = -2n\epsilon\xi, \tag{18}$$

where R,S and Q denotes the curvature tensor of type $(1,3)$, the Ricci tensor of type $(0,2)$ and the Ricci operator of type $(1,1)$, respectively.

Definition 2.1. An ϵ-Kenmotsu manifold M is said to be a generalized η-Einstein manifold if the following condition (Yidiz *et al*. [18]):

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + cg(\phi X,Y), \tag{19}$$

holds on M, where a,b and c are smooth functions on M. If $c = 0, b = c = 0$, and $a = c = 0$, then the manifold is called an η-Einstein manifold, an Einstein and a special type of η-Einstein manifold, respectively.

3. (ϵ)-Kenmotsu Manifolds Admitting the Quarter-Symmetric Non-Metric Connections

Let M^{2n+1} be an (ϵ)-Kenmotsu manifold with Levi-Civita connection ∇. A linear connection (Yadav and Suthar [16]) $\widetilde{\nabla}$ is defined by

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\phi X \tag{20}$$

and it is said to be a quarter-symmetric connection non-metric connection if satisfies

$$\widetilde{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y \tag{21}$$

and

$$(\widetilde{\nabla}_X g)(Y,Z) = -(\eta(Y)g(\phi X,Z) + \eta(Z)g(\phi X,Y)). \tag{22}$$

A relation between the curvature tensors $\widetilde{R}$ and R is given as

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{XY} Z. \tag{23}$$

Using (7), (8), (10) and (20), we have

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + 2g(\phi X,Y)\eta(Z)\xi + g(X,Z)\phi Y - g(Y,Z)\phi X, \tag{24}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{XY} Z \tag{25}$$

is the curvature tensor of the Levi-Civita connection ∇.
On contracting (24), we have

$$\widetilde{S}(Y,Z) = S(Y,Z) + g(\phi Y,Z). \tag{26}$$

Replacing X by ξ in (24) and using (3), (6), (12), we have

$$\widetilde{R}(\xi,X)Y = -\epsilon g(X,Y)\xi + \eta(Y)X + \frac{1}{\epsilon}\eta(Y)\phi X. \tag{27}$$

Replacing Z by ξ in (24) and using (2), (3), (11), we have

$$\widetilde{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X - \frac{1}{\epsilon}\eta(Y)\phi X + \frac{1}{\epsilon}\eta(X)\phi Y + 2g(\phi X,Y)\xi. \tag{28}$$

Taking η on both sides of (24) and using (2), (6), (14), we have

$$\eta(\widetilde{R}(X,Y)Z) = \epsilon g(X,Z)\eta(Y) - \epsilon g(Y,Z)\eta(X) + 2\eta(Z)g(\phi X,Y). \tag{29}$$

Replacing Z by ξ in (26) and using (3), (6), (15), we have

$$\widetilde{S}(Y,\xi) = -2n\eta(Y). \tag{30}$$

By virtue of (26), we have

$$\widetilde{Q}(Y) = Q(Y) + \phi Y. \tag{31}$$

Contracting (26), we have

$$\widetilde{r} = r. \tag{32}$$

Theorem 3.1. *In an (ϵ)-Kenmotsu manifolds admitting the quarter-symmetric non-metric connection $\widetilde{\nabla}$, the scalar curvature tensor is invariant with respect to the connections ∇ and $\widetilde{\nabla}$.*

On taking $X = \xi$ in (22) and using (6), we have

$$(\widetilde{\nabla}_\xi g)(Y,Z) = -(\eta(Y)g(\phi\xi,Z) + \eta(Z)g(\phi\xi,Y)),$$

$$(\widetilde{\nabla}_\xi g)(Y,Z) = 0 \tag{33}$$

Hence we have the following:

Corollary 3.1. *Co-variant differentiation of Riemannian metric g with respect to contra-variant vector field ξ vanish identically in a contact metric manifold admitting the quarter-symmetric non-metric connection $\widetilde{\nabla}$.*

The projective curvature tensor (De and Yildiz [3]) $\widetilde{P}$ with respect to the connection $\widetilde{\nabla}$ is defined as

$$\widetilde{P}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{1}{2n}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y]. \tag{34}$$

If $\widetilde{P}$ with $\widetilde{\nabla}$ vanishes, then from (34), we have

$$\widetilde{R}(X,Y)Z = \frac{1}{2n}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y]. \tag{35}$$

By virtue of (24) and (26), we have

$$R(X,Y)Z = \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y + g(\phi Y,Z)X - g(\phi X,Z)Y] \\ -2g(\phi X,Y)\eta(Z)\xi - g(X,Z)\phi Y + g(Y,Z)\phi X. \tag{36}$$

Taking inner product with W in (36), we have

$$g(R(X,Y)Z,W) = \frac{1}{2n}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(\phi Y,Z)g(X,W) - g(\phi X,Z)g(Y,W)] \\ -2g(\phi X,Y)\eta(Z)g(\xi,W) - g(X,Z)g(\phi Y,W) + g(Y,Z)g(\phi X,W). \tag{37}$$

Replacing W by ξ in (37) and using (3), (6), (14), we have

$$S(Y,Z)\eta(X) = [S(X,Z)\eta(Y) - g(\phi Y,Z)\eta(X) + g(\phi X,Z)\eta(Y) + 4n(\epsilon)^2 g(\phi X,Y)\eta(Z)] \\ +2n\epsilon[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \tag{38}$$

Again replacing X by ξ in (38) and using (2), (3), (6), (15), we have

$$S(Y,Z) = -g(\phi Y,Z) - 2n\epsilon g(Y,Z) = -2n\epsilon g(Y,Z) - g(\phi Y,Z). \tag{39}$$

Contracting (39), we have

$$r = -2n\epsilon(2n+1). \tag{40}$$

By virtue of (32) and (40), we have

$$\widetilde{r} = -2n\epsilon(2n+1). \tag{41}$$

Using (39) in (26), we have

$$\widetilde{S}(Y,Z) = -2n\epsilon g(Y,Z). \tag{42}$$

Using (42) in (35), we have

$$\widetilde{R}(X,Y)Z = \epsilon[g(X,Z)Y - g(Y,Z)X]. \tag{43}$$

Taking η on both sides of (43) and using (14), we have

$$\eta(\widetilde{R}(X,Y)Z) = \eta(R(X,Y)Z). \tag{44}$$

Hence, we have the following theorem:

Theorem 3.2. *If the projective curvature tensor $\widetilde{P}$ of $\widetilde{\nabla}$ in an (ϵ)-Kenmotsu manifold admitting $\widetilde{\nabla}$ vanishes then the manifold becomes a generalized η-Einstein manifold.*

Theorem 3.3. *If the projective curvature tensor $\widetilde{P}$ of $\widetilde{\nabla}$ in an (ϵ)-Kenmotsu manifold admitting $\widetilde{\nabla}$ then the curvature tensor, Ricci tensor and scalar curvature with respect to $\widetilde{\nabla}$ are given by* (43)*,* (42) *and* (41)*, respectively.*

Using (24) and (26) in (34), we have

$$\widetilde{P}(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y] + \left[g(\phi X,Y)\eta(Z)\xi + g(X,Z)\phi Y - g(Y,Z)\phi X - \frac{1}{2n}g(\phi X,Y)\right], \tag{45}$$

$$\widetilde{P}(X,Y)Z = P(X,Y)Z + \left[2g(\phi X,Y)\eta(Z)\xi + g(X,Z)\phi Y - g(Y,Z)\phi X - \frac{1}{2n}g(\phi Y,Z)X + \frac{1}{2n}g(\phi X,Z)Y\right], \tag{46}$$

where

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{47}$$

Thus by virtue of (46), we have the following:

Theorem 3.4. *The projective curvature tensor $\widetilde{P}$ of the connection $\widetilde{\nabla}$ in an (ϵ)-Kenmotsu manifold admitting $\widetilde{\nabla}$ coincides with projective curvature tensor P with respect to the Levi-Civita connection ∇ iff*

$$\left[2g(\phi X,Y)\eta(Z)\xi + g(X,Z)\phi Y - g(Y,Z)\phi X - \frac{1}{2n}g(\phi Y,Z)X + \frac{1}{2n}g(\phi X,Z)Y\right] = 0, \tag{48}$$

holds for arbitrary vector fields X, Y and Z.

Replacing Z by ξ in (46) and using (2), (3), (6), we have

$$\widetilde{P}(X,Y)\xi = P(X,Y)\xi + \left[2g(\phi X,Y)\xi - \frac{1}{\epsilon}\widetilde{T}(X,Y)\right]. \tag{49}$$

Hence, we have the following theorem:

Theorem 3.5. *An (ϵ)-Kenmotsu manifold is ξ-projectively flat with respect to $\widetilde{\nabla}$ iff the manifold is also ξ-projectively flat with respect to ∇ provided $\left[2g(\phi X,Y)\xi - \frac{1}{\epsilon}\widetilde{T}(X,Y)\right] = 0$.*

4. Ricci Soliton on an (ϵ)-Kenmotsu Manifold Admitting the Quarter-Symmetric Non-Metric Connection

Let (g,ξ,λ) be the Ricci soliton on an (ϵ)-Kenmotsu manifold admitting $\widetilde{\nabla}$. Then from (1), we have

$$(\widetilde{L}_\xi g)(X,Y) + 2\widetilde{S}(X,Y) + 2\lambda g(X,Y) = 0. \tag{50}$$

Now

$$(\widetilde{L}_\xi g)(X,Y) = g(\widetilde{\nabla}_X\xi,Y) + g(X,\widetilde{\nabla}_Y\xi). \tag{51}$$

Using (3), (7), (9) and (20) in (51), we have

$$(\widetilde{L}_\xi g)(X,Y) = 2\epsilon g(X,Y) - 2\eta(X)\eta(Y). \tag{52}$$

Using (26) and (52) in (50), we have

$$S(X,Y) = -(\epsilon+\lambda)g(X,Y) + \eta(X)\eta(Y) - g(\phi X,Y). \tag{53}$$

On contracting (53), we have

$$r = -(2n+1)(\epsilon+\lambda)+\epsilon^3. \tag{54}$$

Hence, we have the following theorem:

Theorem 4.1. *If* (g,ξ,λ) *is a Ricci soliton on an* (ϵ)*-Kenmotsu manifold admitting* $\widetilde{\nabla}$ *then the manifold is a generalized* η*-Einstein manifold.*

Replacing Y by ξ in (53) and using (2), (3), (6), (15), we have

$$S(X,\xi) = -\frac{\lambda}{\epsilon}\eta(X). \tag{55}$$

By virtue of (15) and (55), we have

$$\lambda = 2n\epsilon. \tag{56}$$

Hence, we have the following theorem:

Theorem 4.2. *A Ricci soliton* (g,ξ,λ) *on an* (ϵ)*-Kenmotsu manifold admitting* $\widetilde{\nabla}$ *is either expanding or shrinking.*

Let (g,V,λ) be the Ricci soliton on an (ϵ)-Kenmotsu manifold admitting $\widetilde{\nabla}$. Then from (1), we have

$$(\widetilde{L}_V g)(X,Y)+2\widetilde{S}(X,Y)+2\lambda g(X,Y) = 0, \tag{57}$$

where $\widetilde{L}$ is the Lie derivative along the vector field V on M^{2n+1} admitting $\widetilde{\nabla}$.
By virtue of (20), we have

$$(\widetilde{L}_V g)(X,Y) = (L_V g)(X,Y). \tag{58}$$

Using (26) and (58) in (57), we have

$$(\widetilde{L}_V g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)+2g(\phi X,Y) = 0. \tag{59}$$

If (g,V,λ) is a Ricci soliton on an (ϵ)-Kenmotsu manifold admitting $\widetilde{\nabla}$ then (1) holds.
Hence from (1) and (59), we have:

Theorem 4.3. *A Ricci soliton* (g,ξ,λ) *on an* (ϵ)*-Kenmotsu manifold is invariant with respect to* $\widetilde{\nabla}$ *iff*

$$2g(\phi X,Y) = 0, \tag{60}$$

holds for arbitrary vector fields X *and* Y*.*

Let (g,V,λ) be the Ricci soliton on an (ϵ)-Kenmotsu manifold M^{2n+1} $(\phi,\xi,\eta,g,\epsilon)$ admitting $\widetilde{\nabla}$ such that V is pointwise collinear with ξ, i.e., $V = b\xi$, where b is a function then (57) holds and follows that

$$bg(\widetilde{\nabla}_X\xi,Y)+(Xb)\eta(Y)+bg(X,\widetilde{\nabla}_Y\xi)+(Yb)\eta(X)+2\widetilde{S}(X,Y)+2\lambda g(X,Y) = 0. \tag{61}$$

Using (20) and (26) in (61), we have

$$(Xb)\eta(Y)+(Yb)\eta(X)+2S(X,Y)+2(\epsilon b+\lambda)g(X,Y)-2b\eta(X)\eta(Y)+2g(\phi X,Y) = 0. \tag{62}$$

Replacing Y by ξ in (62) and using (2), (3), (6) and (15), we have

$$(Xb)+\left(\frac{2\lambda}{\epsilon}-4n+(\xi b)\right)\eta(X)=0. \tag{63}$$

Again replacing X by ξ in (63) and using (2), we have

$$\xi b=\left[2n-\frac{\lambda}{\epsilon}\right]. \tag{64}$$

By virtue of (64), above equation takes the form

$$db=\left[2n-\frac{\lambda}{\epsilon}\right]\eta. \tag{65}$$

Applying d on (65), we have

$$\left[2n-\frac{\lambda}{\epsilon}\right]d\eta=0. \tag{66}$$

Since $d\eta\neq 0$, then (66) implies that $\lambda=2\epsilon n$. Consequently, from (65) we obtain $db=0$, i.e., b is constant.

Hence we have the following theorem:

Theorem 4.4. *If (g,V,λ) be a Ricci soliton on an (ϵ)-Kenmotsu manifold M^{2n+1} admitting $\widetilde{\nabla}$ such that V=bξ, then V is a constant multiple of ξ and the Ricci soliton is either expanding or shrinking.*

5. Quasi-Projectively Flat (ϵ)-Kenmotsu Manifold Admitting the Quarter-Symmetric Non-Metric Connection

Definition 5.1. An (ϵ)-Kenmotsu manifold admitting $\widetilde{\nabla}$ then the manifold M^{2n+1} $(\phi,\xi,\eta,g,\epsilon)$ is said to be quasi-projectively flat with respect to $\widetilde{\nabla}$ if

$$g(\widetilde{P}(\phi X,Y)Z,\phi W)=0, \tag{67}$$

where $\widetilde{P}$ is the projective curvature tensor with respect to $\widetilde{\nabla}$.

Taking inner product with W in (34), we have

$$g(\widetilde{P}(X,Y)Z,W)=g(\widetilde{R}(X,Y)Z,W)-\frac{1}{2n}[\widetilde{S}(Y,Z)g(X,W)-\widetilde{S}(X,Z)g(Y,W)]. \tag{68}$$

Replacing X by ϕX and W by ϕW in (68), we have

$$g(\widetilde{P}(\phi X,Y)Z,\phi W)=g(\widetilde{R}(\phi X,Y)Z,\phi W)-\frac{1}{2n}[\widetilde{S}(Y,Z)g(\phi X,\phi W)-\widetilde{S}(\phi X,Z)g(Y,\phi W)]. \tag{69}$$

Using (67) in (69), we have

$$g(\widetilde{R}(\phi X,Y)Z,\phi W)=-\frac{1}{2n}[\widetilde{S}(Y,Z)g(\phi X,\phi Y)-\widetilde{S}(\phi X,Z)g(Y,\phi W)]. \tag{70}$$

Using (24) and (26) in (70), we have

$$\begin{aligned}g(R(\phi X,Y)Z,\phi W)=\frac{1}{2n}&[S(Y,Z)g(\phi X,\phi W)+g(\phi Y,Z)(\phi X,\phi W)-S(\phi X,Z)g(Y,\phi W)\\&-g(X,Z)(\phi Y,W)+g(\phi Y,W)g(X,\xi)\eta(Z)]-g(\phi X,Z)(\phi Y,\phi W)\\&-g(Y,Z).\ g(X\phi W).\end{aligned} \tag{71}$$

Let $(e_1, e_2, e_3, \dots, e_{2n}, \xi)$ be a local orthonormal basis of vector fields in M^{2n+1} then $(\phi e_1, \phi e_2, \phi e_3, \dots, \phi e_{2n}, \xi)$ is also a local orthonormal basis of vector fields in M^{2n+1}. Putting $X = W = e_i$ in (71) and taking summation over i, $i \in [1, 2n]$, we have

$$\sum_{i=1}^{2n} g(R(\phi e_i, Y)Z, \phi e_i) = \frac{1}{2n} \sum_{i=1}^{2n} \Big[S(Y,Z)g(\phi e_i, \phi e_i) + g(\phi Y, Z)(\phi e_i, \phi e_i) - S(\phi e_i, Z)g(Y, \phi e_i) - g(e_i, Z)(\phi Y, e_i) + g(\phi Y, e_i)g(e_i, \xi)\eta(Z) \Big] - \sum_{i=1}^{2n} g(\phi e_i, Z)(\phi Y, \phi e_i) - \sum_{i=1}^{2n} g(Y,Z).\ g(e_i \phi e_i). \tag{72}$$

Also,

$$\sum_{i=1}^{2n} g(R(\phi e_i, Y)Z, \phi e_i) = S(Y,Z) + g(Y,Z), \tag{73}$$

$$\sum_{i=1}^{2n} S(\phi e_i, Z)g(\phi e_i, Y) = S(Y,Z), \tag{74}$$

$$\sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n \tag{75}$$

and

$$\sum_{i=1}^{2n} g(e_i, \phi e_i) = trace(\phi) = 0. \tag{76}$$

Using (73), (74), (75) and (76) in (72), we have

$$S(Y,Z) = -2ng(Y,Z) - g(\phi Y, Z). \tag{77}$$

Hence we have the following theorem:

Theorem 5.1. *A quasi-projectively flat (ϵ)-Kenmotsu manifold with respect to the connection $\tilde{\tilde{\nabla}}$ is a generalized η-Einstein manifold.*

6. ϕ-Projectively Flat (ϵ)-Kenmotsu Manifold Admitting the Quarter-Symmetric Non-Metric Connection

Definition 6.1. An (ϵ)-Kenmotsu manifold M^{2n+1} $(\phi, \xi, \eta, g, \epsilon)$ with respect to $\tilde{\nabla}$ is said to be ϕ-projectively flat if

$$\phi^2(\tilde{P}(\phi X, \phi Y)\phi Z) = 0. \tag{78}$$

where $\tilde{P}$ is the projective curvature tensor with respect to $\tilde{\nabla}$.

It is easy to show that $\phi^2(\tilde{P}(\phi X, \phi Y)\phi Z) = 0$ holds iff

$$g(\tilde{P}(\phi X, \phi Y)\phi Z, \phi W) = 0. \tag{79}$$

for all $X, Y, Z, W \in \mathfrak{X}(M^{2n+1})$.

Replacing Y by ϕY and Z by ϕZ in (69), we have

$$g(\tilde{P}(\phi X, \phi Y)\phi Z, \phi W) = g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi W) - \frac{1}{2n}[\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi W)]. \tag{80}$$

Using (79) in (80), we have

$$g(\widetilde{R}(\phi X,\phi Y)\phi Z,\phi W)=\frac{1}{2n}[\widetilde{S}(\phi Y,\phi Z)g(\phi X,\phi W)-\widetilde{S}(\phi X,\phi Z)g(\phi Y,\phi W)]. \tag{81}$$

Using (24) and (26) in (81), we have

$$\begin{aligned}g(R(\phi X,\phi Y)\phi Z,\phi W)=\frac{1}{2n}&[S(\phi Y,\phi Z)g(\phi X,\phi W)-g(Y,\phi Z)g(\phi X,\phi W)\\&-S(\phi X,\phi Z)g(\phi Y,\phi W)+g(X,\phi Z)g(\phi Y,\phi W)]\\&+g(\phi X,\phi Z)Y-g(\phi X,\phi Z)\eta(Y)\xi-g(\phi Y,\phi Z)X\\&+g(\phi Y,\phi Z)\eta(X)\xi.\end{aligned} \tag{82}$$

Let $(e_1,e_2,e_3,\dots,e_{2n},\xi)$ be a local orthonormal basis of vector fields in the manifold M^{2n+1} then $(\phi e_1,\phi e_2,\phi e_3,\dots,\phi e_{2n},\xi)$ is also a local orthonormal basis of the manifold M^{2n+1}. Replacing X by e_i and W by e_i in (82) and applying summation over $i\in[1,2n]$, we have

$$S(\phi Y,\phi Z)=2n(\epsilon^2-2n-1)g(\phi Y,\phi Z)+2ng(Y,Z)-\frac{2n}{\epsilon}\eta(Y)\eta(Z)-(1-2n)g(\phi Y,Z). \tag{83}$$

Using (4) and (16) in (83), we have

$$S(Y,Z)=2n(\epsilon^2-2n)g(Y,Z)-2n\epsilon\left(\frac{1}{\epsilon^2}+\epsilon^2-2n\right)\eta(Y)\eta(Z)-(1-2n)g(\phi Y,Z). \tag{84}$$

Theorem 6.1. *A ϕ-projectively flat (ϵ)-Kenmotsu manifold admitting the connection $\widetilde{\nabla}$ is a generalized η-Einstein manifold.*

7. Example of (ϵ)-Kenmotsu Manifold

Let $M=[(x,y,z)\in R^3 : z>0]$ be a 3-dimensional manifold, where (x,y,z) are the standard coordinates in R^3. Choosing vector fields [13]

$$e_1=z\frac{\partial}{\partial x},\quad e_2=z\frac{\partial}{\partial y},\quad e_3=-z\frac{\partial}{\partial z}, \tag{85}$$

are linearly independent at every point of M. The Riemannian metric g is defined by

$$\begin{aligned}&g(e_i,e_j)=0,\quad i\neq j,\ i,j=1,2,3.\\&g(e_1,e_1)=\epsilon,\ \ g(e_2,e_2)=\epsilon,\ \ g(e_3,e_3)=\epsilon,\end{aligned} \tag{86}$$

where $\epsilon=1$ or -1. Suppose η be the 1-form defined as $\eta(X)=\epsilon g(X,\xi)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1)=-e_2,\ \ \phi(e_2)=e_1,\ \ \phi(e_3)=0. \tag{87}$$

By linearity property of ϕ and g, we have

$$\eta(e_3)=\eta(\xi),\ \ \phi^2X=-X+\eta(X)e_3,\ \ g(\phi X,\phi Y)=g(X,Y)-\epsilon\eta(X)\eta(Y), \tag{88}$$

for all $X,Y\in\chi(M)$.

Consider ∇ be the Levi-Civita connection with Riemannian metric g, then we have

$$[e_1,e_2]=0,\ \ [e_1,e_3]=\epsilon e_1,\ \ [e_2,e_3]=\epsilon e_2. \tag{89}$$

Koszul's formula is given as

$$2g(\nabla_X Y,Z) = Xg(Y,Z) + Yg(Z,X) - Z\hat{g}(X,Y) - g([Y,Z],X) + g([Z,X],Y) + g([X,Y],Z), \quad (90)$$

for arbitrary vector fields $X,Y,Z \in \chi(M)$.
By virtue of (90), we have

$$\left.\begin{array}{lll} \nabla_{e_1}e_1 = -\epsilon e_3, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = \epsilon e_1, \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = -\epsilon e_3, & \nabla_{e_2}e_3 = \epsilon e_2, \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = 0. \end{array}\right\} \quad (91)$$

In view of above results, the manifold satisfies

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi).$$

Now for $X = X^1 e_1 + X^2 e_2 + X^3 e_3$, we have

$$\nabla_X \xi = \epsilon(X^1 e_1 + X^2 e_2)$$

and

$$\epsilon(X - \eta(X)\xi) = \epsilon(X^1 e_1 + X^2 e_2),$$

where X^1, X^2, X^3 are scalars.
Hence for $\xi = e_3$ the manifold $(M^{2n+1}, \phi, \xi, \eta, g, \epsilon)$ under consideration example is an (ϵ)-Kenmotsu manifolds.
By virtue of (3), (20) and (91), we have

$$\left.\begin{array}{lll} \tilde{\nabla}_{e_1}e_1 = -\epsilon e_3, & \tilde{\nabla}_{e_1}e_2 = 0, & \tilde{\nabla}_{e_1}e_3 = \epsilon e_1 - e_2, \\ \tilde{\nabla}_{e_2}e_1 = 0, & \tilde{\nabla}_{e_2}e_2 = -\epsilon e_3, & \tilde{\nabla}_{e_2}e_3 = e_2(\epsilon - 1), \\ \tilde{\nabla}_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \tilde{\nabla}_{e_3}e_3 = 0. \end{array}\right\} \quad (92)$$

In view of (21), the torsion tensor $\tilde{T}$ with respect to $\tilde{\nabla}$ as follows:

$$\tilde{T}(e_i, e_i) = 0, \quad \text{for all } i = 1,2,3,$$
$$\tilde{T}(e_1, e_2) = 0, \; \tilde{T}(e_1, e_3) = -e_2, \; \tilde{T}(e_2, e_3) = e_1.$$

Also we have

$$(\tilde{\nabla}_{e_1} g)(e_2, e_3) = \epsilon, \;\; (\tilde{\nabla}_{e_2} g)(e_3, e_1) = -\epsilon, \;\; (\tilde{\nabla}_{e_3} g)(e_1, e_2) = 0.$$

Hence the manifold is an (ϵ)-Kenmotsu manifold with respect to the connection $\tilde{\nabla}$. The curvature tensor $R(e_i, e_j)e_k$; $i,j,k = 1,2,3$ of ∇ can be calculated by using (25), (89) and (91), we have

$$\left.\begin{array}{lll} R(e_1,e_2)e_3 = 0, & R(e_1,e_3)e_3 = \epsilon^2 e_1, & R(e_2,e_3)e_2 = -\epsilon^2 e_3, \\ R(e_3,e_1)e_1 = -\epsilon^2 e_3, & R(e_2,e_1)e_1 = -\epsilon^2 e_2, & R(e_2,e_3)e_3 = -\epsilon^2 e_2, \\ R(e_2,e_3)e_1 = 0, & R(e_1,e_2)e_2 = \epsilon^2 e_1, & R(e_3,e_1)e_2 = 0. \end{array}\right\} \quad (93)$$

Along with $R(e_i, e_i)e_i = 0$, for all $i = 1,2,3$. In view of above calculation, we verify (11), (12), (13) and (14).
The $\tilde{R}(e_i, e_j)e_k$, $i,j,k = 1,2,3$ of $\tilde{\nabla}$ can be calculated by using (23), (89) and (92), we have

$$\left.\begin{array}{lll} \tilde{R}(e_1,e_2)e_3 = -\epsilon e_3, & \tilde{R}(e_1,e_3)e_3 = \epsilon^2 e_1 - \epsilon e_2, & \tilde{R}(e_2,e_3)e_2 = -\epsilon^2 e_3, \\ \tilde{R}(e_3,e_1)e_1 = -\epsilon^2 e_3, & \tilde{R}(e_2,e_1)e_1 = -\epsilon e_2(\epsilon - 1), & \tilde{R}(e_2,e_3)e_3 = -\epsilon e_2(\epsilon - 1), \\ \tilde{R}(e_2,e_3)e_1 = 0, & \tilde{R}(e_1,e_2)e_2 = -\epsilon^2 e_1 - \epsilon e_2, & \tilde{R}(e_3,e_1)e_2 = 0. \end{array}\right\} \quad (94)$$

Along with $\widetilde{R}(e_i,e_i)e_i = 0$, for all $i = 1,2,3$.
The Ricci tensor $S(e_j,e_k)$, for all $j,k = 1,2,3$ of ∇ is given by using (93), we have

$$S(e_j,e_k) = \sum_{i=1}^{3} g(R(e_i,e_j)e_k,e_i).$$

It follows that:

$$S(e_1,e_1) = S(e_2,e_2) = -2\epsilon^3, \quad S(e_3,e_3) = 0. \tag{95}$$

Along with $S(e_j,e_k) = 0$, for all $j,k = 1,2,3$ $(j \neq k)$. In view of (95), we verify (15) and (16).
By virtue of (95), we can calculate r with respect to the connection ∇ as

$$r = \sum_{i=1}^{3} S(e_i,e_i) = -4\epsilon^3. \tag{96}$$

The $\widetilde{S}(e_j,e_k)$, for all $j,k = 1,2,3$ of $\widetilde{\nabla}$ can also be calculated by using (94) as under:

$$\widetilde{S}(e_j,e_k) = \sum_{i=1}^{3} g(\widetilde{R}(e_i,e_j)e_k,e_i).$$

It follows that

$$\widetilde{S}(e_1,e_1) = -2\epsilon^3 + \epsilon^2, \quad \widetilde{S}(e_2,e_2) = -2\epsilon^3, \quad \widetilde{S}(e_3,e_3) = -2\epsilon^3 + \epsilon^2. \tag{97}$$

Along with $\widetilde{S}(e_j,e_k) = 0$, for all $j,k = 1,2,3$ $(j \neq k)$. In view of (97) the scalar curvature $\widetilde{r}$ with respect to the connection $\widetilde{\nabla}$ can be calculated as under:

$$\widetilde{r} = \sum_{i=1}^{3} S(e_i,e_i) = -6\epsilon^3 + 2\epsilon^2. \tag{98}$$

By virtue of (32) and (54), we have

$$\lambda = \frac{\epsilon^2(7\epsilon - 2) - \epsilon(2n+1)}{(2n+1)} \tag{99}$$

Thus the Ricci soliton (g,ξ,λ) on an (ϵ)-Kenmotsu manifold admitting the connection $\widetilde{\nabla}$ is always shrinking. Hence Theorem 4.2 is verified.

8. Conclusion

During the study of this manuscript, we have studied some interesting results given in Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4, Theorem 3.5, Theorem 4.1, Theorem 4.2, Theorem 4.3, Theorem 4.4, Theorem 5.1, Theorem 6.1. Further, researchers in future can apply this quarter-symmetric non-metric connection for finding new results on several manifolds like Hyperbolic Kenmotsu manifolds, β-Kenmotsu manifolds, Para-Kenmotsu manifolds, Lorentzian α-Sasakian, Lorentzian β-Kenmotsu manifolds etc.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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