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Research Article

On the Bihyperbolic *k*-Fibonacci Numbers

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Abstract. In this paper, we extend the concept of bihyperbolic numbers to the k-Fibonacci numbers. We will find the main formulas that affect these numbers, such as the recurrence formulas and the sum of the first n terms, as well as the identities of Binet, Catalan and d'Ocagne. We finish the study by finding the generating function of the bihyperbolic k-Fibonacci numbers.

Keywords. k-Fibonacci numbers, Hiperbolic and bihyperbolic numbers, Convolution, Binet, Catalan and d'Ocagne identities, Generating function

Mathematics Subject Classification (2020). 11B39, 11C20

1. Introduction

In this section, we will review the concepts of hyperbolic and bihyperbolic numbers, as well as the k-Fibonacci numbers and some of their properties.

1.1 Hyperbolic Numbers

Hyperbolic numbers [6, 7],[1] can be defined as follows: if x and y are two real numbers, a bihyperbolic number is $z = x + jy$ with $j^2 = 1$, $j \neq 1$.

The set of hyperbolic numbers is denoted as C^+ or H_1. The hyperbolic numbers are also called double numbers, split complex numbers and perplex numbers. For two hyperbolic numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ it is possible to define the following operations:

[1] See URL: https://en.wikipedia.org/wiki/Split-complex_number.

- *Addition*: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$.
- *Multiplication*: $z_1 z_2 = (x_1 x_2 + y_1 y_2) + j(x_1 y_2 + x_2 y_1)$.

The conjugate of $z = x + jy$ is $\overline{z} = x - jy$ and satisfies similar properties to usual complex conjugate. Namely, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{(z_1 z_2)} = \overline{z_1}\ \overline{z_2}$, $\overline{(\overline{z})} = z$.
The modulus of a bihyperbolic number $z = x + jy$ is $\|z\|^2 = z\overline{z} = x^2 - y^2$. Then, $\|z\| = \sqrt{x^2 - y^2}$ if $|x| > |y|$, or $\|z\| = \sqrt{y^2 - x^2}$ if $|y| > |x|$.
A bihyperbolic number is invertible if and only if its modulus is nonzero, $\|z\| \neq 0$. The multiplicative inverse of an invertible element is given by $z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{\overline{z}}{\|z\|^2}$. Then, the inverse number of z, if exists ($\|z\| \neq 0$), is $z^{-1} = \frac{x}{x^2 - y^2} - j\frac{y}{x^2 - y^2}$. So, the division of two hyperbolic numbers, if exists, is $\frac{z_1}{z_2} = \frac{x_1 x_2 - y_1 y_2}{x_2^2 - y_2^2} - j\frac{x_1 y_2 - x_2 y_1}{x_2^2 - y_2^2}$

1.2 Bihyperbolic Numbers

Now, we will extend the concept of hiperbolic numbers to the bihyperbolic numbers.

Definition 1. Let H_2 be the set of bihyperbolic numbers defined by

$$z = x_0 + j_1 x_1 + j_2 x_2 + j_3 x_3,$$

where $x_0, x_1, x_2, x_3 \in R$ and $j_1, j_2, j_3 \notin R$ are operators such that $j_1^2 = j_2^2 = j_3^2 = 1$ and $j_1 j_2 = j_2 j_1 = j_3$, $j_1 j_3 = j_3 j_1 = j_2$, $j_2 j_3 = j_3 j_2 = j_1$.

The addition and multiplication on H_2 are commutative and associative. Also, the multiplication is distributive over addition. Hence $(H_2, +, \cdot)$ is a commutative ring. In this paper we will study the bihyperbolic k-Fibonacci numbers.

1.3 k-Fibonacci Numbers

One of the more studied integer sequences is the Fibonacci sequence [5, 8], and it has been generalized in many ways, as [4]. Here, we use the following one-parameter generalization of the Fibonacci sequence [2, 3].

Definition 2. For any integer number $k \geq 1$, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n \in \mathbf{N}}$ is defined recurrently by:

$$F_{k,0} = 0,\ F_{k,1} = 1 \text{ and } F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad \text{for } n \geq 1.$$

The first k-Fibonacci numbers are

$$\{F_{k,0}, F_{k,1}, F_{k,2}, F_{k,3}, F_{k,4}, F_{k,5}, \ldots\} = \{0, 1, k, k^2 + 1, k^3 + 2k, k^4 + 3k^2 + 1, \ldots\}.$$

Note that for $k = 1$ the classical Fibonacci sequence is obtained and for $k = 2$ it is the Pell sequence.

The characteristic equation of the definition is $r^2 = kr + 1$ whose solutions are $\sigma_1 = \frac{k + \sqrt{k^2 + 4}}{2}$ and $\sigma_2 = \frac{k - \sqrt{k^2 + 4}}{2}$, that verify $\sigma_1 \cdot \sigma_2 = -1$, $\sigma_1 + \sigma_2 = k$, $\sigma_1 - \sigma_2 = \sqrt{k^2 + 4}$, $\sigma^2 = k\sigma + 1$, $\sigma_1 > 0$, $\sigma_2 < 0$.
Generating function of the k-Fibonacci numbers is $f(k, x) = \frac{x}{1 - kx - x^2}$.

For the properties of the k-Fibonacci numbers, see [2,3].

$$Binet\ Identity:\ F_{k,n}=\frac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2}, \tag{1.1}$$

$$Sum:\ \sum_{i=0}^{n}F_{k,i}=\frac{1}{k}(F_{k,n+1}+F_{k,n}-1), \tag{1.2}$$

$$Convolution:\ F_{k,a+b}=F_{k,a}F_{k,b-1}+F_{k,a+1}F_{k,b}. \tag{1.3}$$

2. On the Bihyperbolic k-Fibonacci Numbers

In [1], the bihyperbolic Fibonacci, Jacobsthal and Pell numbers are studied. We will extend this concept to the k-Fibonacci numbers.

According to Definition 1, the nth byperbolic k-Fibonacci number $BhF_{k,n}$ is defined as

$$BhF_{k,n}=F_{k,n}+j_1F_{k,n+1}+j_2F_{k,n+2}+j_3F_{k,n+3}. \tag{2.1}$$

Theorem 1 (Recurrence Relation). *Let $n\geq 2$ be an integer. Then $BhF_{k,n}=k\,BhF_{k,n-1}+BhF_{k,n-2}$, with initial conditions*

$$BhF_{k,0}=F_{k,0}+j_1F_{k,1}+j_2F_{k,2}+j_3F_{k,3}=j_1+j_2k+j_3(k^2+1),$$
$$BhF_{k,1}=F_{k,1}+j_1F_{k,2}+j_2F_{k,3}+j_3F_{k,4}=1+j_1k+j_2(k^2+1)+j_3(k^3+2k).$$

Proof. Taking into account formula (2.1) and Definition of k-Fibonacci numbers:

$$\begin{aligned}
k\cdot BhF_{k,n-1}+BhF_{k,n-2}&=k(F_{k,n-1}+j_1F_{k,n}+j_2F_{k,n+1}+j_3F_{k,n+2})\\
&\quad+(F_{k,n-2}+j_1F_{k,n-1}+j_2F_{k,n}+j_3F_{k,n+1})\\
&=(kF_{k,n-1}+F_{k,n-2})+j_1(kF_{k,n}+F_{k,n-1})\\
&\quad+j_2(kF_{k,n+1}+F_{k,n})+j_3(kF_{k,n+2}+F_{k,n+1})\\
&=F_{k,n}+j_1F_{k,n+1}+j_2F_{k,n+2}+j_3F_{k,n+3}\\
&=BhF_{k,n}.
\end{aligned}$$

So, the sequence of bihyperbolic k-Fibonacci numbers $\{BhF_{k,n}\}_{n\in N}$ is also a k-Fibonacci sequence, with initial conditions $BhF_{k,0}$, $BhF_{k,1}$.

In the next subsection, we give a recurrrence relation between the bihyperbolic k-Fibonacci numbers.

2.1 Recurrence Relation

Let $n\geq 0$ be an integer. The bihyperbolic k-Fibonacci numbers verify the recurrence relation

$$BhF_{k,n}-j_1BhF_{k,n+1}-j_2BhF_{k,n+2}+j_3BhF_{k,n+3}=k^2(F_{k,n+4}+F_{k,n+2}).$$

Proof.

$$\begin{aligned}
&BhF_{k,n}-j_1BhF_{k,n+1}-j_2BhF_{k,n+2}+j_3BhF_{k,n+3}\\
&\quad=F_{k,n}+j_1F_{k,n+1}+j_2F_{k,n+2}+j_3F_{k,n+3}-j_1(F_{k,n+1}+j_1F_{k,n+2}+j_2F_{k,n+3}+j_3F_{k,n+4})\\
&\qquad-j_2(F_{k,n+2}+j_1F_{k,n+3}+j_2F_{k,n+4}+j_3F_{k,n+5})+j_3(F_{k,n+3}+j_1F_{k,n+4}+j_2F_{k,n+5}+j_3F_{k,n+6})
\end{aligned}$$

$$= F_{k,n} + j_1 F_{k,n+1} + j_2 F_{k,n+2} + j_3 F_{k,n+3} - j_1 F_{k,n+1} - F_{k,n+2} - j_3 F_{k,n+3} - j_2 F_{k,n+4}$$
$$- j_2 F_{k,n+2} - j_3 F_{k,n+3} - F_{k,n+4} - j_1 F_{k,n+5} + j_3 F_{k,n+3} + j_2 F_{k,n+4} + j_1 F_{k,n+5} + F_{k,n+6}$$
$$= F_{k,n} - F_{k,n+2} - F_{k,n+4} + F_{k,n+6}$$
$$= k F_{k,n+5} - k F_{k,n+1}$$
$$= k(F_{k,n+5} - F_{k,n+1})$$
$$= k(k F_{k,n+4} + F_{k,n+3} - F_{k,n+3} + k F_{k,n+2})$$
$$= k^2 (F_{k,n+4} + F_{k,n+2}). \qquad \square$$

In the next theorem, we study the Binet formula for these numbers.

Theorem 2 (Binet Identity)**.** *For $n \in N$,*

$$BhF_{k,n} = \frac{A\sigma_1^n - B\sigma_2^n}{\sigma_1 - \sigma_2}, \tag{2.2}$$

where $A = 1 + j_1\sigma_1 + j_2\sigma_1^2 + j_3\sigma_1^3$ and $B = 1 + j_1\sigma_2 + j_2\sigma_2^2 + j_3\sigma_2^3$.

Proof. In [3], it is proven that $\sigma^n = F_{k,n}\sigma + F_{k,n-1}$, for $\sigma = \sigma_{1,2}$. Using this formula and the Binet Identity (Formula (1.1)) for the k-Fibonacci numbers, $F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$,

$$\begin{aligned} BhF_{k,n} &= F_{k,n} + j_1 F_{k,n+1} + j_2 F_{k,n+2} + j_3 F_{k,n+3} \\ &= \frac{1}{\sigma_1 - \sigma_2}(\sigma_1^n - \sigma_2^n + j_1(\sigma_1^{n+1} - \sigma_2^{n+1}) + j_2(\sigma_1^{n+2} - \sigma_2^{n+2}) + j_3(\sigma_1^{n+3} - \sigma_2^{n+3})) \\ &= \frac{1}{\sigma_1 - \sigma_2}(\sigma_1^n(1 + j_1\sigma_1 + j_2\sigma_1^2 + j_3\sigma_1^3) - \sigma_2^n(1 + j_1\sigma_2 + j_2\sigma_2^2 + j_3\sigma_2^3)) \\ &= \frac{A\sigma_1^n - B\sigma_2^n}{\sigma_1 - \sigma_2}, \end{aligned}$$

where $A = 1 + j_1\sigma_1 + j_2\sigma_1^2 + j_3\sigma_1^3$ and $B = 1 + j_1\sigma_2 + j_2\sigma_2^2 + j_3\sigma_2^3$.

Other form of expressing A and B is the following:

$$\begin{aligned} 1 + j_1\sigma + j_2\sigma^2 + j_3\sigma^3 &= 1 + j_1\sigma + j_2(k\sigma + 1) + j_3((k^2+1)\sigma + k) \\ &= (1 + j_2 + k j_3) + (j_1 + k j_2 + (k^2+1) j_3)\sigma. \end{aligned}$$

If $F_{k,-n} = (-1)^{n+1} F_{k,n}$, last formula is $BhF_{k,-1} + BhF_{k,0}\sigma$, and A and B take the form $A = BhF_{k,-1} + BhF_{k,0}\sigma_1$ and $B = BhF_{k,-1} + BhF_{k,0}\sigma_2$. $\square$

2.2 Some Properties of A and B

Taking into account the preceding concepts, $A = (1 + j_2 + j_3 k) + \left(j_1 + j_2 k + j_3(k^2+1)\right)\sigma_1$ and $B = (1 + j_2 + j_3 k) + \left(j_1 + j_2 k + j_3(k^2+1)\right)\sigma_2$, and $\sigma_1\sigma_2 = -1$ it is

$$\begin{aligned} A - B &= (j_1 + j_2 k + j_3(k^2+1))(\sigma_1 - \sigma_2) \\ &= BhF_{k,0}\sqrt{k^2+4}, \\ A + B &= 2(1 + j_2 + j_3 k) + (j_1 + j_2 k + j_3(k^2+1))k \\ &= 2 + j_1 k + j_2(k^2+2) + j_3(k^3+3k) \end{aligned}$$

$$
\begin{aligned}
&= 1+(F_{k,1}+j_1F_{k,2}+j_2F_{k,3}+j_3F_{k,4})+j_2+k\,j_3\\
&= (1+j_2+k\,j_3)+BhF_{k,1},\\
A\cdot B &= (1+j_2+j_3k)^2+(1+j_2+j_3k)\left(j_1+j_2k+j_3(k^2+1)\right)k-(j_1+j_2k+j_3(k^2+1))^2\\
&= 1+1+k^2+2j_2+2j_3k+2j_1k+j_1k+j_2k^2+j_3(k^3+k)+j_3k+k^2+j_1(k^3+k)\\
&\quad +j_2k^2+j_1k^3+(k^4+k^2)-(1+k^2+(k^4+2k^2+1)+2j_3k+2j_2(k^2+1)+2j_1(k^3+k))\\
&= 2j_1k+j_3(k^3+2k)\\
&= 2j_1F_{k,2}+j_3F_{k,4}.
\end{aligned}
$$

Catalan Identity for the k-Fibonacci numbers is [2] $F_{k,n-r}F_{k,n+r}-F_{k,n}^2=(-1)^{n+r-1}F_{k,r}^2$. We will apply this formula to find the Catalan Identity for the bihyperbolic k-Fibonacci numbers.

Theorem 3 (Catalan Identity)**.** *Let $n\geq r\geq 0$ be integers. Then*

$$BhF_{k,n-r}BhF_{k,n+r}-BhF_{k,n}^2=(-1)^{n+r-1}(AB)F_{k,r}^2,$$

where $(AB)=2k\,j_1+(k^3+2k)j_3$.

Proof. Because the Binet Identity for the bihyperbolic k-Fibonacci numbers and $\sigma_1\sigma_2=-1$,

$$BhF_{k,n-r}=\frac{A\sigma_1^{n-r}-B\sigma_2^{n-r}}{\sigma_1-\sigma_2};\quad BhF_{k,n+r}=\frac{A\sigma_1^{n+r}-B\sigma_2^{n+r}}{\sigma_1-\sigma_2},$$

$$BhF_{k,n-r}\cdot BhF_{k,n+r}=\frac{1}{(\sigma_1-\sigma_2)^2}\left(A^2\sigma_1^{2n}+B^2\sigma_2^{2n}-(AB)(-1)^n\left(\left(\frac{\sigma_1}{\sigma_2}\right)^r+\left(\frac{\sigma_2}{\sigma_1}\right)^r\right)\right),$$

$$BhF_{k,n}^2=\frac{1}{(\sigma_1-\sigma_2)^2}(A^2\sigma_1^{2n}+B^2\sigma_2^{2n}-2(AB)(-1)^n),$$

$$
\begin{aligned}
BhF_{k,n-r}\cdot BhF_{k,n+r}-BhF_{k,n}^2 &= \frac{1}{(\sigma_1-\sigma_2)^2}(AB)(-1)^{n-1}((-1)^r(\sigma_1^{2r}+\sigma_2^{2r})-2)\\
&= (AB)(-1)^{n-1}(-1)^r\frac{(\sigma_1^r-\sigma_2^r)^2}{(\sigma_1-\sigma_2)^2}\\
&= (AB)(-1)^{n+r-1}F_{k,r}^2.
\end{aligned}
$$

Corollary 1. *If $r=1$,*

$$BhF_{k,n-1}\cdot BhF_{k,n+1}-BhF_{k,n}^2=(AB)(-1)^n=(2k\,j_1+(k^3+2k)j_3)(-1)^n.$$

2.3 A New Relation

Between the bihyperbolic k-Fibonacci numbers and the k-Fibonacci numbers the following relation exists.

For $n\geq 0$,

$$BhF_{k,n}=(1+j_2+k\,j_3)F_{k,n}+BhF_{k,0}F_{k,n+1}. \tag{2.3}$$

Proof. Taking into account $BhF_{k,0}=j_1+j_2k+j_3(k^2+1)$ and $\sigma^r=F_{k,r}\sigma+F_{k,r-1}$, from the Binet Identity for the k-Fibonacci numbers (Formula (1.1)),

$$1+j_1\sigma+j_2\sigma^2+j_3\sigma^3=1+j_1\sigma+j_2(k\,\sigma+1)+j_3\left((k^2+1)\sigma+k\right)$$

$$
\begin{aligned}
&= (1+j_2+j_3k)+(j_1+j_2k+j_3(k^2+1))\sigma \\
&= (1+j_2+j_3k)+BhF_{k,0}\,\sigma\,, \\
BhF_{k,n} &= (1+j_2+j_3\,k)\frac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2}+BhF_{k,0}\,\frac{\sigma_1^{n+1}-\sigma_2^{n+1}}{\sigma_1-\sigma_2} \\
&= (1+j_2+j_3\,k)F_{k,n}+BhF_{k,0}F_{k,n+1}\,.
\end{aligned}
$$

On the other hand, $F_{k,-r}=(-1)^{r+1}F_{k,r}\to 1+j_2j_3\,k=F_{,-1}+j_1F_{k,0}+j_2F_{k,1}+j_3F_{k,2}=BhF_{k,-1}$. So, $BhF_{k,n}=BhF_{k,-1}F_{k,n}+BhF_{k,0}F_{k,n+1}$. □

2.4 Sum of the Bihyperbolic k-Fibonacci Numbers

For $n\geq 0$,

$$
\sum_{i=0}^{n}BhF_{k,i}=\frac{1}{k}\left(BhF_{k,n+1}+BhF_{k,n}-1+\sum_{i=1}^{3}(F_{k,i}-F_{k,i-1})j_i\right).
$$

Proof. From equation (2.3) and taking into account formula (1.2):

$$
\begin{aligned}
\sum_{i=0}^{n}BhF_{k,i} &= (1+j_2+k\,j_3)\sum_{i=0}^{n}F_{k,i}+BhF_{k,0}\sum_{i=0}^{n}F_{k,i+1} \\
&= (1+j_2+k\,j_3)\frac{1}{k}(F_{k,n+1}+F_{k,n}-1)+BhF_{k,0}\left(\frac{1}{k}(F_{k,n+1}+F_{k,n}-1)+F_{k,n+1}-F_{k,0}\right) \\
&= (1+j_2+k\,j_3)\frac{1}{k}(F_{k,n+1}+F_{k,n}-1)+BhF_{k,0}\frac{1}{k}(F_{k,n+1}+F_{k,n}+k\,F_{k,n+1}-1) \\
&= (1+j_2+k\,j_3)\frac{1}{k}(F_{k,n+1}+F_{k,n}-1)+BhF_{k,0}\left(\frac{1}{k}(F_{k,n+2}+F_{k,n+1}-1)\right) \\
&= \frac{1}{k}((1+j_2+k\,j_3)F_{k,n+1}+BhF_{k,0}F_{k,n+2}+(1+j_2+k\,j_3)F_{k,n} \\
&\quad +BhF_{k,0}F_{k,n+1}-((1+j_2+k\,j_3)+BhF_{k,0})) \\
&= \frac{1}{k}(BhF_{k,n+1}+BhF_{k,n}-(1+F_{k,0}j_1+F_{k,1}j_2+F_{k,2}j_3)+(F_{k,1}j_1+F_{k,2}j_2+F_{k,3}j_3)) \\
&= \frac{1}{k}(BhF_{k,n+1}+BhF_{k,n}-BhF_{k,-1}+BhF_{k,0}).
\end{aligned}
$$

□

Theorem 4 (D'Ocagne Identity)**.** *Let $m\geq n\geq 0$ be integer numbers. Then*

$$
BhF_{k,m}BhF_{k,n+1}-BhF_{k,m+1}BhF_{k,n}=(-1)^n(AB)F_{k,m-n}\,.
$$

Proof. Using the Binet Identity for the bihyperbolic k-Fbibonacci numbers (formula (2.2)) and taking into account $\sigma_1\cdot\sigma_2=-1$,

$$
\begin{aligned}
&BhF_{k,m}BhF_{k,n+1}-BhF_{k,m+1}BhF_{k,n} \\
&= \frac{1}{(\sigma_1-\sigma_2)^2}((A\sigma_1^m-B\sigma_2^m)(A\sigma_1^{n+1}-B\sigma_2^{n+1})-(A\sigma_1^{m+1}-B\sigma_2^{m+1})(A\sigma_1^n-B\sigma_2^n)) \\
&= \frac{1}{(\sigma_1-\sigma_2)^2}(AB)(-\sigma_1^m\sigma_2^{n+1}-\sigma_1^{n+1}\sigma_2^m+\sigma_1^{m+1}\sigma_2^n+\sigma_1^n\sigma_2^{m+1}) \\
&= \frac{1}{(\sigma_1-\sigma_2)^2}(AB)(\sigma_1^m\sigma_2^n(\sigma_1-\sigma_2)-\sigma_1^n\sigma_2^m(\sigma_1-\sigma_2))
\end{aligned}
$$

$$= (AB)\frac{\sigma_1^m\sigma_2^n - \sigma_1^n\sigma_2^m}{\sigma_1 - \sigma_2}$$
$$= (AB)(-1)^n\frac{\sigma_1^{m-n} - \sigma_2^{m-n}}{\sigma_1 - \sigma_2}$$
$$= (-1)^n(AB)F_{k,m-n}\,.$$

Using convolution Formula (1.3) of the k-Fibonacci numbers, we will try a similar formula for bihyperbolic k-Fibonacci numbers.

2.5 Convolution Formula

If n,m are integer numbers $n,m \geq 0$ it is verified

$$BhF_{k,n+m} = F_{k,m}BhF_{k,n+1} + F_{k,m-1}BhF_{k,n}.$$

Proof. From the definition, and taking into account the convolution formula for the k-Fibonacci numbers (1.3):

$$\begin{aligned}
BhF_{k,n+m} &= F_{k,n+m} + j_1F_{k,n+m+1} + j_2F_{k,n+m+2} + j_3F_{k,n+m+3}\\
&= F_{k,n+1}F_{k,m} + F_{k,n}F_{k,m-1} + j_1(F_{k,n+2}F_{k,m} + F_{k,n+1}F_{k,m-1})\\
&\quad + j_2(F_{k,n+3}F_{k,m} + F_{k,n+2}F_{k,m-1}) + j_3(F_{k,n+4}F_{k,m} + F_{k,n+3}F_{k,m-1})\\
&= F_{k,m}(F_{k,n+1} + j_1F_{k,n+2} + j_2F_{k,n+3} + j_3F_{k,n+4})\\
&\quad + F_{k,m-1}(F_{k,n} + j_1F_{k,n+1} + j_2F_{k,n+2} + j_3F_{k,n+3})\\
&= F_{k,m}BhF_{k,n+1} + F_{k,m-1}BhF_{k,n}
\end{aligned}$$

2.6 Generating Function

Following the same process as in [3], we will find the generating function of the bihyperbolic k-Fibonacci numbers.

Let $bhf(x)$ be the generating function of the sequence of bihyperbolioc k-Fibonacci numbers, $\{BhF_{k,n}\}_{n\in N}$. Then, taking into account that $BhF_{k,m} = k\,BhF_{k,m-1} + BhF_{k,m-2}$ for $m \geq 2$,

$$\begin{aligned}
bhf(x) &= BhF_{k,0} + BhF_{k,1}x + BhF_{k,2}x^2 + BhF_{k,3}x^3 + \cdots + BhF_{k,n}x^n + \cdots,\\
k\,bhf(x)x &= k\,BhF_{k,0}x + k\,BhF_{k,1}x^2 + k\,BhF_{k,2}x^3 + \cdots + k\,BhF_{k,n-1}x^n + \cdots,\\
x^2bhf(x) &= BhF_{k,0}\,x^2 + BhF_{k,1}\,x^3 + \cdots + BhF_{k,n-2}x^n + \cdots,\\
bhf(x)(1 - k\,x - x^2) &= BhF_{k,0} + \left(BhF_{k,1} - k\,BhF_{k,0}\right)x^2\,.
\end{aligned}$$

On the other hand

$$\begin{aligned}
BhF_{k,1} - k\,BhF_{k,0} &= F_{k,1} + (F_{k,2} - k\,F_{k,1})j_1 + (F_{k,3} - k\,F_{k,2})j_2 + (F_{k,4} - k\,F_{k,3})j_3)\\
&= F_{k,1} + F_{k,0}j_1 + F_{k,1}j_2 + F_{k,2}j_3\\
&= 1 + j_2 + k\,j_3\\
&= BhF_{k,-1}\,.
\end{aligned}$$

So, the generating function of the bihyperbolic k-Fibonacci numbers is

$$bhf(x) = \frac{j_1 + k\,j_2 + (k^2+1)j_3 + (1 + j_2 + k\,j_3)x}{1 - k\,x - x^2}\,.$$

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] D. Bród, A. Szynal-Liana and I. Włoch, On some combinatorial properties of bihyperbolic numbers of the Fibonacci type, *Mathematical Methods in the Applied Sciences* **44**(6) (2021), 4607 – 4615, DOI: 10.1002/mma.7054.

[2] S. Falcón and Á. Plaza, On the Fibonacci k-numbers, *Chaos, Solitons & Fractals* **32**(5) (2007), 1615 – 1624, DOI: 10.1016/j.chaos.2006.09.022.

[3] S. Falcón and Á. Plaza, The k-Fibonacci sequence and the Pascal 2-triangle, *Chaos, Solitons & Fractals* **33**(1) (2007), 38 – 49, DOI: 10.1016/j.chaos.2006.10.022.

[4] V. E. Hoggat, *Fibonacci and Lucas Numbers*, Houghton Mifflin Company, Palo Alto, USA, iv + 92 pages (1979).

[5] A. F. Horadam, A generalized Fibonacci sequence, *The American Mathematical Monthly* **68**(5) (1961), 455 – 459, DOI: 10.1080/00029890.1961.11989696.

[6] D. S. Kulyabov, A. V. Korolkova and M. N. Gevorkyan, Hyperbolic numbers as Einstein numbers, *Journal of Physics: Conference Series* **1557**(1) (2020), 012027, DOI: 10.1088/1742-6596/1557/1/012027.

[7] G. Sobczyk, The hyperbolic number plane, *The College Mathematical Journal* **26**(4) (1995), 268 – 280, DOI: 10.2307/2687027.

[8] S. Vajda, *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications*, Ellis Horwood Limited, Chichester (1989).