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Research Article

Fixed Points of Meromorphic Functions Concerning Exponential and Linear Difference Polynomials

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Abstract. In this paper, we analyse the correlation between fixed points of finite order transcendental meromorphic functions, exponential polynomials and linear difference polynomials. Let $\mathbf{P}_d(z,f)$ is a difference polynomial in f and $\phi(z)=\sum_{j=1}^{k}P_j^*e^{Q_j^*}$ is an exponential polynomial in z. We look upon the zeroes as well as growth of $f^n\mathbf{P}_d(z,f)+\phi(z)$. Our results will extend the findings of Fang *et al*. (Value distribution of meromorphic functions concerning rational functions and differences, *Advances in Difference Equations* **2020** (2020), Article number: 692) for linear and exponential difference polynomials.

Keywords. Meromorphic function, Linear difference polynomials, Exponential polynomials, Fixed points

Mathematics Subject Classification (2020). 30D35, 30D45

1. Introduction

In 20th century, Nevanlinna theory has emerged as one of the key advancements in complex analysis. In complex plane, consider f as a meromorphic function, which is a finite order. We suppose that fundamentals of Nevanlinna theory are known by readers as in (see Hayman [3], Lo [5], and Yang and Hua [9]).

Consider a non-constant meromorphic function f of order

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ T_0(r,f)}{\log r}.$$

The exponent of convergence of zeroes and poles of f are respectively,

$$\lambda = \overline{\lim_{r\to\infty}} \frac{\log^+ N_0\left(r,\frac{1}{f}\right)}{\log r}$$

and

$$\lambda\left(\frac{1}{f}\right) = \overline{\lim_{r\to\infty}} \frac{\log^+ N_0(r,f)}{\log r}.$$

The exponent of convergence of fixed points of f is

$$\tau(f) = \overline{\lim_{r\to\infty}} \frac{\log N_0\left(r,\frac{1}{f-z}\right)}{\log r}.$$

For $a \in \mathbb{C}\cup\{\infty\}$, Nevanlinna's deficiency of f is

$$\delta(a,f) = \underline{\lim_{r\to\infty}} \frac{m_0\left(r,\frac{1}{f-a}\right)}{T_0(r,f)} = 1 - \overline{\lim_{r\to\infty}} \frac{N_0\left(r,\frac{1}{f-a}\right)}{T_0(r,f)}.$$

Replace $N_0\left(r,\frac{1}{f-a}\right)$ by $N_0(r,f)$, when $a=\infty$.

We denote $\mathsf{G} = \{f(z) : f$ is a transcendental meromorphic function of finite order$\}$,

$\mathbf{P}_d(z,f)$ be difference polynomial in f with $d \le n-2$, $n \ge 2$,

$\phi(z) = P_1^* e^{Q_1^*} + P_2^* e^{Q_2^*} + \cdots + P_k^* e^{Q_k^*} = \sum_{u=1}^k P_u^* e^{Q_u^*}$ exponential polynomial in z, with $P_u^*(z)$ and $Q_u^*(z)$, $u = 1,2,\ldots,k$ of degree $q_* = \max\{P_u^*(z), Q_u^*(z); u = 1,2,\ldots,k\}$ are polynomials in z,

BeV as Borel exceptional value.

In 2015, Wu *et al*. [8] considered the functions in the annuli and obtained the following result:

Theorem 1.1 ([8]). *Consider meromorphic function f in annuli, then one of f or f' has several fixed points infinitely.*

In 2016, Zhang *et al*. [10], proved the following result by considering $\lambda\left(\frac{1}{f}\right) < \rho(f)$.

Theorem 1.2 ([10]). *Consider $f(z) \in \mathsf{G}$ and $a, c(\neq 0) \in \mathbb{C}$ with $\lambda(f-a) < \rho(f)$ then*

$$\max\{\tau(f), \tau(\Delta_c f)\} = \rho(f),$$
$$\max\{\tau(f), \tau(f_c(z)\} = \rho(f),$$
$$\max\{\tau(\Delta_c f), \tau(f_c(z))\} = \rho(f).$$

In 2019, Wu and Wu [7], obtained the results for $\Delta_c f(z)$, with order of f as an integer. In the same year, Lan and Chen [4], studied the relationship among the convergence of exponent of fixed points of f, forward differences and its shift. Also, Chen and Zheng [1], investigated the results for higher order functions and also for a non-zero polynomials with Borel exceptional value.

Theorem 1.3 ([7]). *Let $f \in G$, $c(\neq 0) \in \mathbb{C}$ so that $\Delta_c f \neq 0$. If $a \in \mathbb{C}$ with $\delta(\infty, f) = 1$ and $\delta(a, f) > 0$, then $\Delta_c f$ have infinite several fixed points and $\tau(\Delta_c f) = \rho(f)$.*

Theorem 1.4 ([4]). *Let $f \in G$ with $\delta(a, f) > 0$, $c(\neq 0) \in \mathbb{C}$ hence*

(i) $\max\{\tau(f), \tau(\Delta_c f)\} = \rho(f)$,

$\max\{\tau(f_c(z)), \tau(\Delta_c f)\} = \rho(f)$, $(\Delta_c(f - z) \neq 0)$,

(ii) $\max\{\tau(f), \tau(f_c(z))\} = \rho(f)$,

(iii) $\tau(f) = \tau(f_c(z)) = \tau(\Delta_c f) = \rho(f)$

with $f_c(z) \neq f(z)$.

Theorem 1.5 ([1]). *Consider $f \in G$, $c \in \mathbb{C} - \{0\}$, $n \in N$. Suppose $a \in \mathbb{C}$ is a BeV of $f(z)$ then*

$$\max\{\tau(f), \tau(\Delta_c^n f)\} = \rho(f),$$

$$\max\{\tau(f), \tau(f_{nc}(z)\} = \rho(f),$$

$$\max\{\tau(\Delta_c^n f), \tau(f_{nc}(z))\} = \rho(f).$$

Theorem 1.6 ([1]). *Consider $c \in \mathbb{C} - \{0\}$, $f \in G$, $m \in \mathbb{N}$, $P(z) = P_m z^m + P_{m-1} z^{m-1} + \cdots + P_0$ be a non-zero polynomial so that $P_t \in \mathbb{C}$ with $P_m \neq 0$ $(t = 0, 1, \ldots, m)$. If $a \in \mathbb{C}$ is a BeV then*

$$\max\{\lambda(f - P(z)), \lambda(\Delta_c f - P(z))\} = \rho(f),$$

$$\max\{\lambda(f - P(z)), \lambda(f_c(z) - P(z)\} = \rho(f),$$

$$\max\{\lambda(\Delta_c f - P(z)), \lambda(f_c(z) - P(z))\} = \rho(f).$$

In 2020, Fang *et al*. [2], investigated the outcomes on fixed points of meromorphic function for non-constant rational function.

Theorem 1.7 ([2]). *Let $f \in G$, $a, c(\neq 0) \in \mathbb{C}$ so that $\lambda(f - a) < \rho(f)$. Consider a non-constant rational function R_1 in f. Hence*

$$\max\{\lambda(f - R_1), \lambda(\Delta_c^n f - R_1)\} = \rho(f),$$

$$\max\{\lambda(f - R_1), \lambda(f_{nc}(z) - R_1)\} = \rho(f),$$

$$\max\{\lambda(\Delta_c^n f - R_1)\lambda(f_{nc}(z) - R_1)\} = \rho(f).$$

Inspired by the above results, We investigate the relation among the meromorphic functions fixed points, linear difference polynomials, exponential polynomials and attain the subsequent outcomes.

1.1 Lemmas

Lemma 1.1 ([7]). *Consider $f \in G$, then for $n \in \mathbb{Q}^+$*

$$m_0\left(r, \frac{\Delta_c f}{f}\right) = S_0(r, f).$$

Lemma 1.2 ([7]). *Consider* $f \in G$ *and* $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ *be a polynomial with constants* a_0 $(\neq 0)$, $a_1, \dots, a_n$. *Hence*

$$T_0(r, P(f)) = nT_0(r, f) + S_0(r, f).$$

Lemma 1.3 ([6]). *If* $f \in G$, *then*

$$N_0(r, f(z+c)) = N_0(r, f) + S_0(r, f),$$
$$T_0(r, f(z+c)) = T_0(r, f) + S_0(r, f).$$

Lemma 1.4 ([1]). *Let* $H(z)$ *be a meromorphic function and* $c(\neq 0)$ *a constant. A polynomial* $h(z)$, $\deg(h(z)) \geq 1$. *Suppose* $\rho(H) < \rho(e^h)$ *then*

$$T_0(r, H) = S_0(r, e^h),$$
$$T_0(r, H(z+c)) = S_0(r, e^h),$$
$$T_0(r, e^{h(z+c)-h}) = S_0(r, e^h).$$

Lemma 1.5 ([1]). *Consider a finite order entire functions* $\mathcal{A}_0(z), \dots, \mathcal{A}_n(z)$, *with* $\rho = \max\{\rho(\mathcal{A}_k : 0 \leq k \leq n\}$ *then*

$$\mathcal{A}_n f(z+c_n) + \cdots + \mathcal{A}_0 f(z) = 0,$$

for any solution of f *we get* $\rho(f) \geq \rho + 1$.

Lemma 1.6 ([1]). *Suppose* $g_1, g_2, \dots, g_m$ *are entire and* $f_1, f_2, \dots, f_m$ *are meromorphic functions to satisfy:*

(i) $\sum_{l=1}^{m} f_l e^{g_l} \not\equiv 0$;

(ii) *for* $1 \leq l < k \leq m$; $g_l - g_k$ *are non constant;*

(iii) $T_0(r, f_l) = o\{T_0(r, e^{g_h - g_k})\}$ *as* $r \to \infty$, $1 \leq l \leq m$, $1 \leq h < k \leq m$, *then*, $f_l(z) = 0$, $l = 1, 2, \dots, m$.

Lemma 1.7 ([10]). *Consider a meromorphic function* $h(z)$ *to satisfy*

$$\overline{N}_0(r, h) + \overline{N}_0\left(r, \frac{1}{h}\right) = S_0(r, h).$$

Let $\chi(z) = \frac{a_0(z)h(z)^p + a_1(z)h(z)^{p-1} + \cdots + a_p(z)}{b_0(z)h(z)^q + b_1(z)h(z)^{q-1} + \cdots + b_q(z)}$, *where* $a_p(z), b_q(z)$ $(i = 0, 1, \dots, p, j = 0, 1, \dots, q)$ *small functions of* h, *with* a_0, b_0, a_p *and* $b_q \neq 0$.
Hence $\lambda(\chi) = \rho(h)$ *if* $T_0(r, \chi) \geq T_0(r, h) + S_0(r, h)$, $p \geq q$.

Lemma 1.8 ([10]). *Let* f *be a meromorphic function having a non-reducible rational function* $R^{**}(z)$. *Then*

$$R^{**}(z, f(z)) = \frac{\sum_{a=0}^{s_1} \alpha_a f^a}{\sum_{b=0}^{s_2} \beta_b f^b},$$

having coefficients $\alpha_a(z)$, $a = 0, 1, \dots, s_1$ *and* $\beta_b(z)$, $b = 0, 1, \dots, s_2$ *so that*

$$T_0(r, \alpha_a) = S_0(r, f), \quad a = 0, 1, \dots, s_1,$$
$$T_0(r, \beta_b) = S_0(r, f), \quad b = 0, 1, \dots, s_2$$

then

$$T_0(r, R^{**}(z, f(z))) = \max\{s_1, s_2\} T_0(r, f) + S_0(r, f).$$

Lemma 1.9 ([1]). *Let $f(z) \in G$ with $\zeta \in \mathbb{C} - \{0\}$, $\rho(f) < \infty$. Hence for $\epsilon > 0$*

$$T_0(r, f(z+\zeta)) = T_0(r, f) + O(r^{\rho+\epsilon-1}) + O(\log r).$$

Lemma 1.10 ([1]). *Consider f with $b, \zeta(\neq 0) \in \mathbb{C}$, $\lambda(f-b) < \infty$. Hence for $\epsilon > 0$*

$$N_0\left(r, \frac{1}{f(z+\zeta)-b}\right) = N_0\left(r, \frac{1}{f(z)-b}\right) + O(r^{\lambda(f-b)+\epsilon-1}) + O(\log r).$$

2. Main Results

Theorem 2.1. *Let $f \in G$, $a, c, m, s \in \mathbb{C} - \{0\}$ so that $\lambda(f-a) < \rho(f)$.*
Let $X(z,f) = \sum_{i=0}^{l} a_i(z) f(z+s_i)$ a linear difference polynomial of f. If $b_0 \in \mathbb{C}$ is BeV of $f(z)$ then

$$\max\{\tau(f - X(z,f)), \tau(\Delta_c^m f - X(z,f))\} = \rho(f),$$
$$\max\{\tau(f - X(z,f)), \tau(f(z+mc) - X(z,f))\} = \rho(f),$$
$$\max\{\tau(\Delta_c^m f - X(z,f)), \tau(f(z+mc) - X(z,f))\} = \rho(f).$$

Proof. Let us assume $\tau(f - X(z,f)) < \rho(f)$ then we show that $\tau\{(\Delta_c^m f - X(z,f))\} = \rho(f)$.
But $\lambda(f-a) < \rho(f)$ and $X(z,f)$ is a linear difference polynomial. Hence

$$\frac{f(z) - X(z,f)}{f(z) - b_0} = \kappa e^P, \tag{2.1}$$

where $\deg(P) = \rho(f) = p$, $\rho(\kappa) < \rho(f)$ with $\kappa(\neq 0, \infty)$ as a meromorphic function.
Thus, $T_0(r, \kappa) = S_0(r, e^P)$, $T_0(r, f) = T_0(r, e^P) + S_0(r, f)$.
From (2.1)

$$f = b_0 + \frac{X(z,f) - b_0}{1 - \kappa(z) e^{P(z)}}. \tag{2.2}$$

Thus

$$\begin{aligned}
\Delta_c^m f &= \Delta_c^m (f - b_0) \\
&= \sum_{h=0}^{m} (-1)^h C_m^h (f(z + (m-h)c) - b_0) \\
&= \sum_{h=0}^{m} (-1)^h C_m^h \left[\frac{X(z+(m-h)c, f) - b_0}{1 - \kappa(z+(m-h)c) e^{P(z+(m-h)c)}}\right] \\
&= \sum_{h=0}^{m} (-1)^h C_m^h \left[\frac{\sum_{i=0}^{l} a_i(z+(m-h)c) f(z+(m-h)c+s_i) - b_0}{1 - \kappa(z+(m-h)c) e^{P(z+(m-h)c)}}\right] \\
&= \frac{\sum_{h=0}^{m} (-1)^h C_m^h \left[\sum_{i=0}^{l} a_i(z+(m-h)c) f(z+(m-h)c+s_i) - b_0\right] \cdot H}{\prod_{h=0}^{m} \left[1 - \kappa(z+(m-h)c) e^{P(z+(m-h)c)}\right]} \\
&= \frac{\sum_{h=1}^{m} A_{m,h}(z) e^{hP(z)} + \sum_{h=1}^{m} B_{m,h}(z) e^{P(z)}}{\sum_{h=1}^{m+1} A_{m,m+h}(z) e^{hP(z)} + 1},
\end{aligned} \tag{2.3}$$

where

$$H = \prod_{k \neq h}^{m} (1 - \kappa(z+(m-k)c) e^{P(z+(m-k)c)}),$$

$$A_{m,h}(z) = \left[(-1)^m \left(\sum_{h=0}^{m} (-1)^h C_m^h \left[\sum_{i=0}^{l} a_i(z+(m-h)c) f(z+(m-h)c+s_i)\right] - b_0\right)\right]$$

$$\cdot\left[\sum_{k\neq h}^{m} e^{P(z+(m-k)c)-P(z)}\kappa(z+(m-k)c)\right],$$

$$B_{m,h}(z)=\sum_{h=0}^{m}\left[\sum_{i=0}^{l} a_i(z+(m-h)c)f(z+(m-h)c)+s_i)-b_0\right]e^{-P(z)},$$

$$A_{m,m+h}(z)=(-1)^{m+1}\prod_{h=0}^{m} e^{P(z+(m-h)c)-hP(z)}\kappa(z+(m-h)c).$$

From (2.3)

$$\Delta_c^m f(z)-X(z,f)=\frac{\sum_{h=1}^{m}A_{m,h}(z)e^{hP(z)}+\sum_{h=1}^{m}B_{m,h}(z)e^{P(z)}}{\sum_{h=1}^{m+1}A_{m,m+h}(z)e^{hP(z)}+1}-X(z,f)$$

$$=\frac{\sum_{h=1}^{m}A_{m,h}(z)e^{hP(z)}+\sum_{h=1}^{m}B_{m,h}(z)e^{P(z)}-H_1}{\sum_{h=1}^{m+1}A_{m,m+h}(z)e^{hP(z)}+1}, \tag{2.4}$$

where $H_1=X(z,f)\sum_{h=1}^{m+1}A_{m,m+h}(z)e^{hP(z)}-X(z,f)$.

By Lemma 1.4 and from (2.3) and (2.4) we observe $\Delta_c^m f(z)$ and $\Delta_c^m f(z)-X(z,f)$ as rational functions in e^P and the coefficients as small functions of $e^{P(z)}$. Hence

$$\rho(A_{m,h})\leq\max\{\rho(\kappa),\rho(e^{P(z+(m-k)c)-P})\}<\rho(P)=p,\quad h=1,2,\dots,2m+1,\ \ k=1,2,\dots,m.$$

Therefore, e^P has small functions $A_{m,h}$ $(h=1,2,\dots,2m+1)$. Then, to justify $A_{m,1}$, rephrase it as

$$A_{m,1}(z)=-\sum_{h=0}^{m}\sum_{k\neq h}^{m}(-1)^k C_m^k\left[\sum_{i=0}^{l}a_i(z+(m-k)c)f(z+(m-k)c+s_i)-b_0\right]$$
$$\cdot e^{P(z+(m-h)c)-P(z)}\kappa(z+(m-h)c).$$

Clearly, we have

$$\rho(e^{P(z+(m-h)c)-P(z)})=p-1,\quad h=0,1,2,\dots,m-1$$

and

$$\rho(e^{P(z+mc)-P(z)})>\rho(e^{P(z+(m-h)c)-P(z)}).$$

Obviously

$$\sum_{k=0}^{m-1}(-1)^k C_m^k\left[\sum_{i=0}^{l}a_i(z+(m-k)c)f(z+(m-k)c+s_i)-b_0\right]\neq 0.$$

Thus, if $A_{m,h}(z)=0$, from Lemma 1.5, $\rho(\kappa)=p$, this contradicts $\rho(\kappa)<p$.
Therefore, $A_{m,h}(z)\neq 0$ holds.
By Lemma 1.6 and $A_{m,h}(z)\neq 0$, we obtain

$$\sum_{h=1}^{m}A_{m,h}(z)e^{P(z)}\neq 0.$$

Thus, from (2.3) one can get

$$T_0(r,\Delta_c^m f(z))\geq T_0(r,e^{P(z)})+S_0(r,e^{P(z)}).$$

Consequently

$$T_0(r,\Delta_c^m f(z)-X(z,f))\geq T_0(r,e^{P(z)})+S_0(r,e^{P(z)}). \tag{2.5}$$

From (2.4), (2.5) and Lemma 1.7

$$\tau(\Delta_c^m f(z) - X(z,f)) = \lambda(\Delta_c^m f(z) - X(z,f)) = \rho(e^{P(z)}) = \rho(f) = p, \tag{2.6}$$

which is

$$\max\{\tau(f(z) - X(z,f)), \tau(\Delta_c^m f(z) - X(z,f))\} = \rho(f).$$

From (2.2)

$$\begin{aligned} f(z+mc) - X(z,f) &= b_0 + \frac{X(z+mc,f) - b_0}{1 - \kappa(z+mc)e^{P(z+mc)}} - X(z,f) \\ &= \frac{(X(z,f) - b_0)\kappa(z+mc)e^{P(z+mc)} + X(z+mc,f) - X(z,f)}{1 - \kappa(z+mc)e^{P(z+mc)}}. \end{aligned} \tag{2.7}$$

Since

$$(X(z,f) - b_0)\kappa(z+mc) + [X(z+mc,f) - X(z,f)] \cdot \kappa(z+mc) = [X(z+mc,f) - b_0] \cdot \kappa(z+mc) \neq 0.$$

Therefore, $f(z+mc) - X(z,f)$ is a non-reducible rational function in $e^{P(z+mc)}$.
But

$$\rho(\kappa(z+mc)) = \rho(\kappa) < \rho(e^{P(z+mc)}) = \rho(e^{P(z)}).$$

Using Lemma 1.4

$$T_0(r, \kappa(z+mc)) = S_0(r, e^{P(z+mc)}). \tag{2.8}$$

Using (2.7), (2.8), and Lemma 1.8

$$T_0(r, f(z+mc) - X(z,f)) = T_0(r, e^{P(z+mc)}) + S_0(r, e^{P(z+mc)}).$$

Using Lemma 1.7 one can get

$$\tau(f(z+mc) - X(z,f)) = \lambda(f(z+mc) - X(z,f)) = \rho(e^{P(z+mc)}) = \rho(e^{P(z)}) = \rho(f)$$

which is

$$\max\{\tau(f(z) - X(z,f)), \tau(f(z+mc) - X(z,f))\} = \rho(f).$$

Next, suppose

$$\tau(f(z+mc) - X(z,f)) = \lambda(f(z+mc) - X(z,f)) < \rho(f)$$

then to show

$$\tau(\Delta_c^m f(z) - X(z,f)) = \rho(f).$$

Represent

$$\omega(z) = \frac{f(z+mc) - X(z,f)}{f(z+mc) - b_0} \tag{2.9}$$

and hence

$$f(z) = \frac{b_0 - X(z-mc,f)}{\omega(z-mc) - 1} + b_0.$$

From (2.9) and Lemma 1.9

$$\rho(\omega(z)) = \rho(f(z+mc)) = \rho(\omega(z-mc)) = \rho(f).$$

Since $\lambda(f - b_0) < \rho(f)$ and using Lemma 1.10 we get

$$\lambda\left(\frac{1}{\omega}\right) = \lambda(f(z+mc) - b_0) = \lambda(f - b_0) < \rho(f) = \rho(\omega).$$

Also,

$$\lambda(\omega) = \lambda(f(z+mc) - X(z,f)) = \tau(f(z+mc) - X(z,f)) < \rho(f) = \rho(\omega).$$

In accordance with $\omega(z)$ has 0 and ∞ as Borel exceptional values. Succeeding the similar steps as in (2.1) to (2.6) we get

$$\tau(\Delta_c^m f(z) - X(z,f)) = \lambda(\Delta_c^m f(z) - X(z,f)) = \rho(f).$$

Thus

$$\max\{\tau(\Delta_c^m f(z) - X(z,f)), \tau(f(z+mc) - X(z,f))\} = \rho(f).$$

Hence the proof. □

Theorem 2.2. *Let's define $f \in G$ in $A(R) = \{z : \frac{1}{R} < |z| < R\}$, $1 < R \le \infty$. Suppose that $\mathbf{P}_d(z,f)$, $\phi(z)$ be not zero and $a, b_1, c \in \mathbb{C} - \{0\}$ with $\delta(a,f) > 0$, $\delta(\infty, f) = 1$ then $\mathcal{X}_1 = f^n \mathbf{P}_d(z,f) + \phi(z)$ has several fixed points infinitely satisfying $\tau(\Delta_c \mathcal{X}_1) = \rho(f)$.*

Proof. Let

$$\frac{1}{f^n} = \frac{\mathcal{X}_1}{b_1 f^n} - \frac{\Delta_c(\mathcal{X}_1 - b_1)}{b_1 f^n} \frac{\mathcal{X}_1 - b_1}{\Delta_c(\mathcal{X}_1 - b_1)}. \tag{2.10}$$

This leads to

$$m_0\left(r, \frac{1}{f^n}\right) \le m_0\left(r, \frac{\mathcal{X}_1}{b_1 f^n}\right) + m_0\left(r, \frac{\Delta_c(\mathcal{X}_1 - b_1)}{b_1 f^n}\right) + m_0\left(r, \frac{\mathcal{X}_1 - b_1}{\Delta_c(\mathcal{X}_1 - b_1)}\right). \tag{2.11}$$

By Nevanlinna first fundamental theorem

$$m_0\left(r, \frac{1}{f^n}\right) = T_0(r, f^n) - N_0\left(r, \frac{1}{f^n}\right) + O(1). \tag{2.12}$$

Next

$$\begin{aligned} m_0\left(r, \frac{\mathcal{X}_1}{b_1 f^n}\right) &= m_0\left(r, \frac{f^n \mathbf{P}_d(z,f) + \phi(z)}{b_1 f^n}\right) \\ &\le m_0(r, \mathbf{P}_d(z,f)) + m_0(r, \phi(z)) + m_0\left(r, \frac{1}{f^n}\right) + O(1) \\ &\le d m_0(r,f) + m_0(r, \phi(z)) + m_0\left(r, \frac{1}{f^n}\right) + O(1) \\ &\le (n-2) m_0(r,f) + q_* k m_0(r,f) + n m_0(r,f) + O(1) \\ &\le (2n + q_* k - 2) m_0(r,f) + S_0(r,f) \\ &\le (2n + q_* k - 2) T_0(r,f) + S_0(r,f) \end{aligned} \tag{2.13}$$

and

$$m_0\left(r, \frac{\Delta_c(\mathcal{X}_1 - b_1)}{b_1 f^n}\right) \le m_0\left(r, \frac{\Delta_c(\mathcal{X}_1 - b_1)}{\mathcal{X}_1 - b_1}\right) + m_0\left(r, \frac{\mathcal{X}_1 - b_1}{b_1 f^n}\right) + O(1).$$

From Lemma 1.1

$$m_0\left(r, \frac{\Delta_c(\mathcal{X}_1 - b_1)}{\mathcal{X}_1 - b_1}\right) = S_0(r,f).$$

Using Lemma 1.3

$$m_0\left(r, \frac{\mathcal{X}_1 - b_1}{b_1 f^n}\right) = m_0\left(r, \frac{f^n \mathbf{P}_d(z,f) + \phi(z) - b_1}{b_1 f^n}\right)$$

$$\begin{aligned}
&\leq m_0\left(r,\frac{f^n\mathbf{P}_d(z,f)}{b_1f^n}\right)+m_0\left(r,\frac{\phi(z)-b_1}{b_1f^n}\right)+O(1)\\
&\leq m_0(r,\mathbf{P}_d(z,f))+m_0\left(r,\phi(z)-b_1\right)+m_0\left(r,\frac{1}{f^n}\right)+O(1)\\
&\leq dm_0(r,f)+q_*km_0(r,f)+nm_0(r,f)+O(1)\\
&\leq (d+q_*k+n)m_0(r,f)+O(1). \qquad (2.14)
\end{aligned}$$

Again using Lemma 1.3

$$\begin{aligned}
m_0\left(r,\frac{\mathcal{X}_1-b_1}{\Delta_c(\mathcal{X}_1-b_1)}\right)&=m_0\left(r,\frac{\Delta_c(\mathcal{X}_1-b_1)}{\mathcal{X}_1-b_1}\right)+N_0\left(r,\frac{\Delta_c(\mathcal{X}_1-b_1)}{\mathcal{X}_1-b_1}\right)-N_0\left(r,\frac{\mathcal{X}_1-b_1}{\Delta_c(\mathcal{X}_1-b_1)}\right)+O(1)\\
&\leq m_0\left(r,\frac{\Delta_c(\mathcal{X}_1-b_1)}{\mathcal{X}_1-b_1}\right)+N_0(r,\Delta_c(\mathcal{X}_1-b_1))+N_0\left(r,\frac{1}{\mathcal{X}_1-b_1}\right)\\
&\quad -N_0(r,\mathcal{X}_1-b_1)-N_0\left(r,\frac{1}{\Delta_c(\mathcal{X}_1-b_1)}\right)+O(1)\\
&\leq m_0\left(r,\frac{\Delta_c(\mathcal{X}_1-b_1)}{\mathcal{X}_1-b_1}\right)+(n+d+q_*k)N_0(r,f)+N_0\left(r,\frac{1}{\mathcal{X}_1-b_1}\right)\\
&\quad -\overline{N}_0(r,f)-N_0\left(r,\frac{1}{\Delta_c(\mathcal{X}_1-b_1)}\right)+O(1)\\
&\leq (n+d+q_*k)N_0(r,f)+N_0\left(r,\frac{1}{\mathcal{X}_1-b_1}\right)-\overline{N}_0(r,f)\\
&\quad -N_0\left(r,\frac{1}{\Delta_c(\mathcal{X}_1-b_1)}\right)+S_0(r,f). \qquad (2.15)
\end{aligned}$$

Using (2.12) to (2.15) and Lemma 1.2 in (2.11)

$$\begin{aligned}
nT_0(r,f)&\leq nN_0\left(r,\frac{1}{f}\right)+(2n+q_*k-2)T_0(r,f)+(n+d+q_*k)T_0(r,f)\\
&\quad +N_0\left(r,\frac{1}{\mathcal{X}_1-b_1}\right)-\overline{N}_0(r,f)-N_0\left(r,\frac{1}{\Delta_c(\mathcal{X}_1-b_1)}\right)+S_0(r,f)\\
&\leq nN_0\left(r,\frac{1}{f}\right)+(3n+2q_*k-2+d)T_0(r,f)+N_0\left(r,\frac{1}{\mathcal{X}_1-b_1}\right)\\
&\quad -\overline{N}_0(r,f)-N_0\left(r,\frac{1}{\Delta_c(\mathcal{X}_1-b_1)}\right)+S_0(r,f). \qquad (2.16)
\end{aligned}$$

Indicating $y=f-a$, by (2.16) we get

$$\begin{aligned}
nT_0(r,f)&\leq nT_0(r,y)+O(1)\\
&\leq nN_0\left(r,\frac{1}{y}\right)+(3n+2q_*k-2+d)T_0(r,y)+N_0\left(r,\frac{1}{y-b_1}\right)\\
&\quad -\overline{N}_0(r,y)-N_0\left(r,\frac{1}{\Delta_c(y-b_1)}\right)+S_0(r,f)\\
&\leq nN_0\left(r,\frac{1}{f-a}\right)+(3n+2q_*k-2+d)T_0(r,f)+N_0\left(r,\frac{1}{f-b_1}\right)\\
&\quad -\overline{N}_0(r,f)-N_0\left(r,\frac{1}{\Delta_c(f-b_1)}\right)+S_0(r,f), \qquad (2.17)
\end{aligned}$$

whereas $\delta(a,f)>0$ and $\delta(\infty,f)=1$, then for $0<\theta<1$

$$N_0\left(r,\frac{1}{f-a}\right)<\theta T_0(r,f). \tag{2.18}$$

Using (2.17) and (2.18) we can get

$$[2(1-q_*k)-d-(2+\theta)n]T_0(r,f)\le N_0\left(r,\frac{1}{f-b_1}\right)-\overline{N}_0(r,f)-N_0\left(r,\frac{1}{\Delta_c(f-b_1)}\right)+S_0(r,f).$$

This gives contradiction. Hence $\mathcal{X}_1$ has several fixed points infinitely, satisfying $\tau(\Delta_c\mathcal{X}_1)=\rho(f)$. □

3. Conclusion

The results of this study add a new dimension to existing findings of Fang *et al*. [2] by mainly concentrating on exponential and linear difference polynomials.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] H.-Y. Chen and X.-M. Zheng, Fixed points of meromorphic functions and their higher order differences and shifts, *Open Mathematics* **17**(1) (2019), 677 – 688, DOI: 10.1515/math-2019-0054.

[2] M. Fang, D. Yang and D. Liu, Value distribution of meromorphic functions concerning rational functions and differences, *Advances in Difference Equations* **2020** (2020), Article number: 692, DOI: 10.1186/s13662-020-03150-6.

[3] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford (1964).

[4] S. Lan and Z. Chen, On fixed points of meromorphic functions $f(z)$ and $f(z+c)$, $\Delta_c f(z)$, *Acta Mathematica Scientia* **39** (2019), 1277 – 1289, DOI: 10.1007/s10473-019-0507-9.

[5] Y. Lo, Value distribution of meromorphic functions together with their derivatives, in: *Value Distribution Theory*, Springer, Berlin — Heidelberg (1993), DOI: 10.1007/978-3-662-02915-2_4.

[6] Z. Wu, An inequality on the difference polynomials of meromorphic functions and its application, *Journal of Mathematical Inequalities* **15**(1) (2021), 349 – 356, DOI: 10.7153/jmi-2021-15-26.

[7] Z. Wu and J. Wu, Fixed points of differences of meromorphic functions, *Advances in Difference Equations* **2019** (2019), Article number: 453, DOI: 10.1186/s13662-019-2386-8.

[8] Z. Wu, Z. Xuan and Y. Chen, Some new results on fixed points of meromorphic functions defined in annuli, *Journal of Function Spaces* **2015** (2015), Article ID 426576, 7 pages, DOI: 10.1155/2015/426576.

[9] C.-C. Yang and X. Hua, Uniqueness and value-sharing of meromorphic function, *Annales Academiæ Scientiarum Fenniae Mathematica* **22** (1997), 395 – 406, URL: https://www.acadsci.fi/mathematica/Vol22/yang.pdf.

[10] R. R. Zhang and Z. X. Chen, Fixed points of meromorphic functions and of their differences, divided differences and shifts, *Acta Mathematica Sinica, English Series* **32** (2016), 1189 – 1202, DOI: 10.1007/s10114-016-4286-0.