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Research Article

On Some Properties of the Degenerate Exponential Function

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Abstract. In this paper, we study some limit properties and inequalities involving the degenerate exponential function. Utilizing analytical methods, we also obtain some monotonic properties involving the function.

Keywords. Degenerate exponential function, Increasing, Decreasing, Inequality

Mathematics Subject Classification (2020). 33B10, 33B99

1. Introduction

As defined in [3], [4], [5], [6], [7], the degenerate exponential function is given as

$$f(t) = (1+\lambda t)^{\frac{1}{\lambda}}, \tag{1.1}$$

for $\lambda \in (0,\infty)$ and $t \in \mathbb{R}$.

This should not be confused with the degenerate exponential function defined by Nantomah [9] as

$$g(t) = (1+\lambda)^{\frac{t}{\lambda}}, \tag{1.2}$$

for $\lambda \in (0,\infty)$ and $t \in \mathbb{R}$.

It is clear that taking the limit of $f(t)$ as $\lambda \to 0$, then $f(t) \to e^t$. Its range is the set of positive real numbers.

The first derivative of the function (1.1) is given as

$$f'(t) = (1+\lambda t)^{\frac{1}{\lambda}-1} > 0, \tag{1.3}$$

for all $t \in (-\infty, \infty)$. This implies that the degenerate exponential function (1.1) is increasing on $t \in (-\infty, \infty)$.

Plot of the degenerate exponential function (1.1) for some values of λ, is shown in Figure 1.

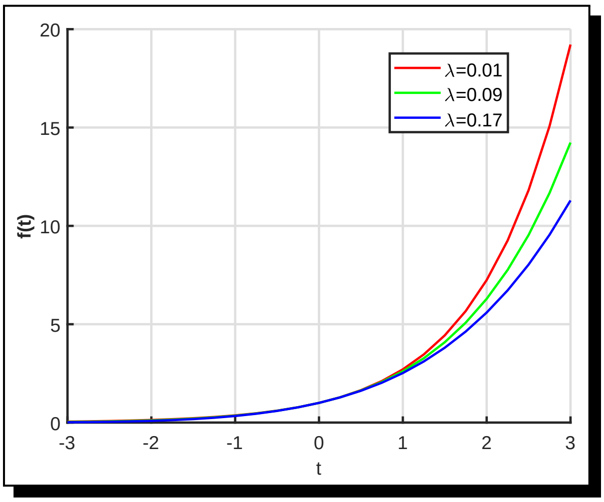

Figure 1. Plot of $f(t)$

Later on, motivated by the degenerate exponential function, a good number of researchers have introduced degenerate versions of many special functions.

Kim and Kim [4] introduced the degenerate gamma function, the degenerate hyperbolic functions and the degenerate Laplace transform and studied some of their properties. Kim *et al*. [6] introduced the modified degenerate gamma function. Nantomah [8] established several properties satisfied by the modified degenerate gamma function. Recently, Nantomah [9] introduced the degenerate exponential integral function and established some of its properties. Also, Akel *et al*. [1] introduced the degenerate gamma matrix function, the degenerate zeta matrix function, the degenerate diagamma matrix function, the degenerate polygamma matrix function and the degenerate Gauss hypergeometric matrix function. The critical role played by the degenerate exponential function in the introduction and the study of properties of the above mentioned functions makes its properties worth studying.

In this paper, we establish some monotonic and limit properties as well as inequlities involving the degenerate exponential function.

2. Results

Proposition 2.1. *The limit of the n-th derivative of the degenerate exponential function as* $\lambda \to 0$ *is given as*

$$\lim_{\lambda \to 0} \left[(1+\lambda t)^{\frac{1}{\lambda}-n} \prod_{k=0}^{n-1} (1-\lambda k) \right] = e^t, \tag{2.1}$$

for $\lambda \in (0,\infty)$, $n \in \mathbb{N}$ *and* $t \in (-\infty, \infty)$.

Proof. Differentiating (1.1), we have

$$f'(t) = (1+\lambda t)^{\frac{1}{\lambda}-1},$$
$$f''(t) = (1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-2},$$
$$f'''(t) = (1-2\lambda)(1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-3},$$
$$f^{(4)}(t) = (1-3\lambda)(1-2\lambda)(1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-4}.$$

Continuing the process n number of times, gives

$$f^{(n)}(t) = (1+\lambda t)^{\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1-\lambda k).$$

Also, let

$$y = \lim_{\lambda\to 0}\left[(1+\lambda t)^{\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1-\lambda k)\right]. \tag{2.2}$$

Taking the natural logarithm on both sides of (2.2), we have

$$\begin{aligned}
\ln y &= \lim_{\lambda\to 0}\ln\left[(1+\lambda t)^{\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1-\lambda k)\right]\\
&= \lim_{\lambda\to 0}\left(\frac{1}{\lambda}-n\right)\ln(1+\lambda t)+\lim_{\lambda\to 0}\ln\prod_{k=0}^{n-1}(1-\lambda k)\\
&= \lim_{\lambda\to 0}\frac{(1-\lambda n)\ln(1+\lambda t)}{\lambda}+\sum_{k=0}^{n-1}\lim_{\lambda\to 0}\ln(1-\lambda k)\\
&= \lim_{\lambda\to 0}\frac{(1-\lambda n)\ln(1+\lambda t)}{\lambda}+\sum_{k=0}^{n-1}\ln(1)\\
&= \lim_{\lambda\to 0}\left[-n\ln(1+\lambda t)+\frac{(1-\lambda n)t}{1+\lambda t}\right]\\
&= \lim_{\lambda\to 0}-n\ln(1+\lambda t)+\lim_{\lambda\to 0}+\frac{t-\lambda nt}{1+\lambda t}\\
&= t.
\end{aligned}$$

Therefore,

$$y = e^t.$$

This concludes the proof. □

Lemma 2.1. *The inequality*

$$(1+\lambda t)^{\frac{1}{\lambda}}-\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}] > 1-\ln 2, \tag{2.3}$$

holds for all $t,\lambda\in(0,\infty)$.

Proof. Let $h(t) = (1+\lambda t)^{\frac{1}{\lambda}}-\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]$. Thus,

$$h'(t) = (1+\lambda t)^{\frac{1}{\lambda}-1}-\frac{(1+\lambda t)^{\frac{1}{\lambda}-1}}{1+(1+\lambda t)^{\frac{1}{\lambda}}}$$

$$= (1+\lambda t)^{\frac{1}{\lambda}-1}\left[1-\frac{1}{1+(1+\lambda t)^{\frac{1}{\lambda}}}\right]$$
$$= \frac{(1+\lambda t)^{\frac{2}{\lambda}-1}}{1+(1+\lambda t)^{\frac{1}{\lambda}}}$$
$$> 0.$$

This shows that $h(t)$ is increasing on $(0,\infty)$. Therefore, we have

$$h(t) > \lim_{t\to 0} h(t) = 1-\ln 2,$$

yielding the results (2.3). □

Theorem 2.1. *The inequality*

$$(1+\lambda t)^{-\frac{1}{\lambda}} \geq 1-t \tag{2.4}$$

is valid for all $t\in[0,\infty)$ *and* $\lambda\in(0,\infty)$. *Equality holds when* $t=0$.

Proof. Let $f(t)=(1+\lambda t)^{-\frac{1}{\lambda}}+t-1$. Thus,

$$f'(t) = -(1+\lambda t)^{-\frac{1}{\lambda}-1}+1$$
$$= -\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}+1}}+1$$
$$\geq 0.$$

This shows that $f(t)$ is increasing. Thus,

$$f(t) > \lim_{t\to 0} f(t) = 0,$$

yielding the results (2.4). □

Theorem 2.2. *For all* $t,\lambda\in(0,\infty)$, *the function*

$$h(t) = [1+(1+\lambda t)^{\frac{1}{\lambda}}]^{1+(1+\lambda t)^{-\frac{1}{\lambda}}}$$

is increasing. Consequently, the inequality

$$2\ln 2 < \frac{(1+\lambda t)^{\frac{1}{\lambda}}}{1+(1+\lambda t)^{\frac{1}{\lambda}}} < \ln[1+(1+\lambda t)^{\frac{1}{\lambda}}], \tag{2.5}$$

is satisfied.

Proof. Let

$$\omega(t) = \ln h(t) = \frac{[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{1}{\lambda}}}\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}],$$

for all $t,\lambda\in(0,\infty)$. Then, we have

$$\omega'(t) = \frac{\{(1+\lambda t)^{\frac{2}{\lambda}-1}-[1+(1+\lambda t)^{\frac{1}{\lambda}}](1+\lambda t)^{\frac{1}{\lambda}-1}\}\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{2}{\lambda}}} + \frac{[1+(1+\lambda t)^{\frac{1}{\lambda}}](1+\lambda t)^{\frac{1}{\lambda}-1}}{(1+\lambda t)^{\frac{1}{\lambda}}[1+(1+\lambda t)^{\frac{1}{\lambda}}]}$$
$$= \frac{\{(1+\lambda t)^{\frac{2}{\lambda}-1}-(1+\lambda t)^{\frac{1}{\lambda}-1}-(1+\lambda t)^{\frac{2}{\lambda}-1}\}\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{2}{\lambda}}} + \frac{1}{(1+\lambda t)}$$

$$
\begin{aligned}
&= \frac{1}{(1+\lambda t)} - \frac{(1+\lambda t)^{\frac{1}{\lambda}-1}\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{2}{\lambda}}} \\
&= \frac{1}{(1+\lambda t)} - \frac{\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{1}{\lambda}+1}} \\
&= \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}+1}}\{(1+\lambda t)^{\frac{1}{\lambda}} - \ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]\} \\
&> 0,
\end{aligned}
$$

which follows from Lemma 2.1. This implies $\omega(t)$ is increasing and consequently, $h(t)$ is also increasing. Thus, we have

$$
\lim_{t\to 0}\omega(t) = \lim_{t\to 0}\frac{[1+(1+\lambda t)^{\frac{1}{\lambda}}]\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{1}{\lambda}}} = 2\ln 2\ ;
$$

$$
\begin{aligned}
\lim_{t\to\infty}\omega(t) &= \lim_{t\to\infty}\frac{[1+(1+\lambda t)^{\frac{1}{\lambda}}]\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{1}{\lambda}}} \\
&= \lim_{t\to\infty}\frac{(1+\lambda t)^{\frac{1}{\lambda}-1}\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}] + \frac{(1+\lambda t)^{\frac{1}{\lambda}-1}}{1+(1+\lambda t)^{\frac{1}{\lambda}}}[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{1}{\lambda}-1}} \\
&= \lim_{t\to\infty}\left\{\frac{(1+\lambda t)^{\frac{1}{\lambda}-1}\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{1}{\lambda}-1}} + \frac{\frac{(1+\lambda t)^{\frac{1}{\lambda}-1}[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{1+(1+\lambda t)^{\frac{1}{\lambda}}}}{(1+\lambda t)^{\frac{1}{\lambda}-1}}\right\} \\
&= \lim_{t\to\infty}\{\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]+1\} \\
&= \infty.
\end{aligned}
$$

For all $t,\lambda \in (0,\infty)$, since $\omega(t)$ is increasing, then we have

$$
2\ln 2 = \lim_{t\to 0}\omega(t) < \omega(t) < \lim_{t\to\infty}\omega(t) = \infty.
$$

This completes the proof. □

Theorem 2.3. *For $t,\lambda \in (0,\infty)$, the inequality*

$$
\ln 2 - \frac{1}{2} < \frac{(1+\lambda t)^{\frac{1}{\lambda}}}{1+(1+\lambda t)^{\frac{1}{\lambda}}} < \ln[1+(1+\lambda t)^{\frac{1}{\lambda}}], \tag{2.6}
$$

holds.

Proof. Let

$$
\Phi(t) = \ln[1+(1+\lambda t)^{\frac{1}{\lambda}}] - \frac{(1+\lambda t)^{\frac{1}{\lambda}}}{1+(1+\lambda t)^{\frac{1}{\lambda}}}. \tag{2.7}
$$

Then,

$$
\Phi'(t) = \frac{(1+\lambda t)^{\frac{1}{\lambda}-1}}{1+(1+\lambda t)^{\frac{1}{\lambda}}} - \frac{(1+\lambda t)^{\frac{1}{\lambda}-1}}{[1+(1+\lambda t)^{\frac{1}{\lambda}}]^2}
$$

$$= \frac{(1+\lambda t)^{\frac{1}{\lambda}-1}}{1+(1+\lambda t)^{\frac{1}{\lambda}}}\left[1-\frac{1}{1+(1+\lambda t)^{\frac{1}{\lambda}}}\right]$$
$$= \frac{(1+\lambda t)^{\frac{2}{\lambda}-1}}{[1+(1+\lambda t)^{\frac{1}{\lambda}}]^2}$$
$$> 0.$$

This implies that $\Phi(t)$ is increasing on $(0,\infty)$. Furthermore,

$$\lim_{t\to 0}\Phi(t) = \lim_{t\to 0}\left\{\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}] - \frac{(1+\lambda t)^{\frac{1}{\lambda}}}{1+(1+\lambda t)^{\frac{1}{\lambda}}}\right\} = \ln 2 - \frac{1}{2};$$
$$\lim_{t\to\infty}\Phi(t) = \lim_{t\to\infty}\left\{\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}] - \frac{(1+\lambda t)^{\frac{1}{\lambda}}}{1+(1+\lambda t)^{\frac{1}{\lambda}}}\right\}$$
$$= \lim_{t\to\infty}\left\{\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}] - \frac{1}{\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}}+1}\right\}$$
$$= \infty.$$

Thus, for $t,\lambda \in (0,\infty)$, we have

$$\ln 2 - \frac{1}{2} = \lim_{t\to 0}\Phi(t) < \Phi(t) < \lim_{t\to\infty}\Phi(t) = \infty,$$

which gives the desired inequality (2.6). This concludes the proof. □

Theorem 2.4. *The function*

$$g(t) = [1+(1+\lambda t)^{\frac{1}{\lambda}}]^{(1+\lambda t)^{-\frac{1}{\lambda}}}$$

is decreasing. As a results, the inequality

$$\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}] < \ln 2(1+\lambda t)^{\frac{1}{\lambda}}, \tag{2.8}$$

is valid for all $t,\lambda \in (0,\infty)$.

Proof. Let $\mu(t) = \ln g(t) = \frac{\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{1}{\lambda}}}$, for all $t,\lambda \in (0,\infty)$. Then, we have

$$\mu'(t) = \frac{\frac{(1+\lambda t)^{\frac{1}{\lambda}-1}}{1+(1+\lambda t)^{\frac{1}{\lambda}}}(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{\frac{1}{\lambda}-1}\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{2}{\lambda}}}$$
$$= \frac{\frac{(1+\lambda t)^{\frac{2}{\lambda}-1}}{1+(1+\lambda t)^{\frac{1}{\lambda}}} - (1+\lambda t)^{\frac{1}{\lambda}-1}\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{2}{\lambda}}}$$
$$= \frac{1}{(1+\lambda t)+(1+\lambda t)^{\frac{1}{\lambda}+1}} - \frac{\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{1}{\lambda}+1}}$$
$$= \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}+1}}\left\{\frac{(1+\lambda t)^{\frac{1}{\lambda}}}{1+(1+\lambda t)^{\frac{1}{\lambda}}} - \ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]\right\}$$
$$< 0,$$

which follows from (2.7). Hence $\mu(t)$ is decreasing and consequently, $g(t)$ is also decreasing. In addition,

$$\lim_{t\to 0}\mu(t)=\lim_{t\to 0}\frac{\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{1}{\lambda}}}=\ln 2\ ;$$

$$\begin{aligned}\lim_{t\to\infty}\mu(t)&=\lim_{t\to\infty}\frac{\ln[1+(1+\lambda t)^{\frac{1}{\lambda}}]}{(1+\lambda t)^{\frac{1}{\lambda}}}\\&=\lim_{t\to\infty}\frac{\frac{(1+\lambda t)^{\frac{1}{\lambda}-1}}{1+(1+\lambda t)^{\frac{1}{\lambda}}}}{(1+\lambda t)^{\frac{1}{\lambda}-1}}\\&=\lim_{t\to\infty}\frac{1}{1+(1+\lambda t)^{\frac{1}{\lambda}}}\\&=0.\end{aligned}$$

Since $\mu(t)$ is decreasing, we have the following. For all $t,\lambda\in(0,\infty)$, we have

$$0=\lim_{t\to\infty}\mu(t)<\mu(t)<\lim_{t\to 0}\mu(t)=\ln 2,$$

which yields the desired result (2.8). □

3. Conclusion

We have established some limit and monotonic properties involving the degenerate exponential function. Inequalities involving the degenerate exponential function have also been obtained. Many fields in mathematics will benefit from these established properties.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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