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Research Article

Pseudosymmetric Almost α-Cosymplectic (κ,μ,ν)-Spaces Admitting Einstein Solitons

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Abstract. This paper attempts to characterize cases of an almost α-cosymplectic (κ,μ,ν)-space admitting Einstein sloitons to be concircular Ricci pseudosymmetry, projective Ricci pseudosymmetry, W_1-curvature and the W_2-curvature Ricci pseudo symmetric.

Keywords. Almost α-cosymplectic (κ,μ,ν)-Space, Einstein soliton, Pseudosymmetric manifold

Mathematics Subject Classification (2020). 53C15, 53C25, 53D25

1. Introduction

An almost contact manifold is an odd-dimensional manifold $\widetilde{M}^{2n+1}$ which carries a field ϕ of endomorphism of the tangent space, a vector field ξ called characteristic, and a 1-form η-satisfying:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1.1}$$

where I denote the identity mapping of tangent space of each point at M. From (1.1), it follows:

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \operatorname{rank}(\phi) = 2n. \tag{1.2}$$

An almost contact manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta)$ is said to be normal if the tensor field $N = [\phi,\phi] + 2d\eta \otimes \xi = 0$, where $[\phi,\phi]$ denote the Nijenhuis tensor field of ϕ. It is well known that any almost contact manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta)$ has a Riemannian metric such that

$$g(\phi\mathcal{Y}_1, \phi\mathcal{Y}_2) = g(\mathcal{Y}_1, \mathcal{Y}_2) - \eta(\mathcal{Y}_1)\eta(\mathcal{Y}_2), \tag{1.3}$$

for any vector fields $\mathcal{Y}_1, \mathcal{Y}_2$ on $\widetilde{M}$ (Carriazo and Martín-Molina [4]). A metric g satisfying these conditions is termed a compatible metric. When manifold $\widetilde{M}^{2n+1}$, along with the structure (ϕ,η,ξ,g), possesses this property, it is termed an almost contact metric manifold, denoted as $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. The 2-form Φ of $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$ is defined as $\Phi(\mathcal{Y}_1,\mathcal{Y}_2) = g(\phi\mathcal{Y}_1,\mathcal{Y}_2)$ and is referred to as the fundamental form of $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. If an almost contact metric manifold satisfies the conditions where both η and Φ are closed (i.e., $d\eta = d\Phi = 0$), then it is termed a cosymplectic manifold (Dacko and Olszak [7]).

An almost α-cosymplectic manifold is a manifold defined for any real number α, characterized by the following properties:

$$d\eta = 0, \quad d\Phi = 2\alpha\eta\wedge\Phi. \tag{1.4}$$

A normal almost α-cosymplectic manifold is referred to as an α-cosymplectic manifold. This terminology is used when an almost α-cosymplectic manifold satisfies certain additional conditions or possesses specific properties that make it fully α-cosymplectic (Hamilton [8]).

On a contact metric manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$, where the tensor h is defined as $2h = L_\xi\phi$, the following inequalities hold:

$$\widetilde{\nabla}_{\mathcal{Y}_1}\xi = -\phi\mathcal{Y}_1 - \phi h\mathcal{Y}_1, \quad h\phi + \phi h = 0, \quad \operatorname{tr} h = \operatorname{tr}\phi h = 0, \quad h\xi = 0, \tag{1.5}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}^{2n+1}$ (Catino and Mazzieri [5]).

In Hamilton [8], the authors investigated almost α-cosymplectic (κ,μ,ν)-spaces, exploring various conditions and provided an example within a three-dimensional setting.

Expanding on the concept beyond generalized (κ,μ)-spaces, in Ye [13], the notion of (κ,μ,ν)-contact metric manifold was introduced. It is described as follows.

The equation is given as:

$$\widetilde{R}(\mathcal{Y}_1,\mathcal{Y}_2)\xi = \eta(\mathcal{Y}_2)[\kappa I + \mu h + \nu\phi h]\mathcal{Y}_1 - \eta(\mathcal{Y}_1)[\kappa I + \mu h + \nu\phi h]\mathcal{Y}_2, \tag{1.6}$$

This equation involves the Riemannian curvature tensor $\widetilde{R}$ of the manifold $\widetilde{M}^{2n+1}$, where $\mathcal{Y}_1$ and $\mathcal{Y}_2$ represent vector fields on $\widetilde{M}^{2n+1}$. Additionally, κ,μ and ν are smooth functions defined on $\widetilde{M}^{2n+1}$.

In their research, they established an intrinsic connection between this type of manifold and the harmonicity of the Reeb vector on contact metric 3-manifolds. Interestingly, certain studies have explored manifolds satisfying condition (1.6) without relying on a contact metric structure. Dacko and Olszak [7] introduced the concept of almost cosymplectic (κ,μ,ν)-spaces. In their framework, an almost cosymplectic manifold is considered to satisfy equation (1.6), yet with κ,μ and ν functions exclusively varying in the direction of ξ. Subsequent works, in Dacko and Olszak [6], have presented additional examples that fall within this category of manifold structures.

Proposition 1.1. *Given $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ an almost α-cosymplectic (κ,μ,ν)-space, then*

$$h^2 = (\kappa+\alpha^2)\phi^2, \quad \phi h + h\phi = 0, \tag{1.7}$$

$$\xi(\kappa) = 2(\kappa+\alpha^2)(\nu - 2\alpha), \tag{1.8}$$

$$\begin{aligned}\widetilde{R}(\xi,\mathcal{Y}_1)\mathcal{Y}_2 &= \kappa[g(\mathcal{Y}_1,\mathcal{Y}_2)\xi - \eta(\mathcal{Y}_2)\mathcal{Y}_1] + \mu[g(h\mathcal{Y}_1,\mathcal{Y}_2)\xi - \eta(\mathcal{Y}_2)h\mathcal{Y}_1]\\ &\quad + \nu[g(\phi h\mathcal{Y}_1,\mathcal{Y}_2)\xi - \eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1],\end{aligned} \tag{1.9}$$

$$(\widetilde{\nabla}_{\mathcal{Y}_1}\phi)\mathcal{Y}_2 = g(\alpha\phi\mathcal{Y}_1 + h\mathcal{Y}_1, \mathcal{Y}_2)\xi - \eta(\mathcal{Y}_2)(\alpha\phi\mathcal{Y}_1 + h\mathcal{Y}_1), \tag{1.10}$$

$$\widetilde{\nabla}_{\mathcal{Y}_1}\xi = -\alpha\phi^2\mathcal{Y}_1 - \phi h\mathcal{Y}_1, \tag{1.11}$$

for all vector fields $\mathcal{Y}_1$, $\mathcal{Y}_2$ *on* $\widetilde{M}^{2n+1}$ (*Carriazo and V. Martín-Molina* [4]).

Absolutely, the fixed points of the Yamabe flow represent metrics within a given conformal class that possess a constant scalar curvature. This flow was initially investigated in the 1980s, outlined in unpublished notes by Richard Hamilton. He proposed the conjecture that, regardless of the initial metric chosen, this flow would converge toward a conformal metric showcasing a constant scalar curvature. The validity of this conjecture in the context of locally conformally flat cases was confirmed by Ye [13]. Hamilton originally introduced the concept of the Yamabe flow as a means to construct Yamabe metrics specifically on compact Riemannian manifolds.

Catino and Mazzieri [5] introduced the Einstein soliton which generates self-similar solutions to Einstein flow, given by

$$\frac{\partial}{\partial t}g(t) + 2\left(S - \frac{\tau}{2}g\right) = 0,$$

where S is Ricci tensor, g is Riemannian metric and τ is the scalar curvature of a semi-Riemannian manifold (M,g).

The equation of the Einstein soliton is given by

$$L_{\mathcal{Y}_5}g + +2S + (2\lambda - \tau)g = 0, \tag{1.12}$$

where $L_{\mathcal{Y}_5}$ is the Lie derivative along the vector field $\mathcal{Y}_5$, λ is a real constant. Also, it is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$, respectively (Blaga [3]).

In the rest of this paper, we will denote the Einstein soliton by $(M, g, \mathcal{Y}_5, \lambda)$.

A Riemannian manifold is categorized as semi-symmetry type (or Ricci semi-symmetry type) under specific conditions imposed on the generalized quasi-conformal curvature tensor $\mathbb{W}$ and the Ricci tensor S. For semi-symmetry type, it is expressed as $R(\mathcal{Y}_1, \mathcal{Y}_2) \cdot \mathbb{W} = 0$, where the "$\cdot$" symbolizes $R(\mathcal{Y}_1, \mathcal{Y}_2)$ acting on $\mathbb{W}$ as a derivation. Conversely, for Ricci semi-symmetry type, the condition is $\mathbb{W}(\mathcal{Y}_1, \mathcal{Y}_2) \cdot S = 0$, indicating that $\mathbb{W}(\mathcal{Y}_1, \mathcal{Y}_2)$ acts on S in a similar derivative manner.

A smooth vector field $\mathcal{Y}_5$ is called a potential field of the Ricci soliton. A Ricci soliton on a semi-Riemannian manifold $(M, g, \mathcal{Y}_5, \lambda)$ is said to be shrinking, steady or expanding according to $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively.

Over the past two decades, the study of Ricci solitons has captivated the interest of numerous mathematicians, especially gaining prominence after Perelman employed Ricci solitons to resolve the longstanding Poincaré conjecture. In [12], Sharma delved into the investigation of Ricci solitons within the realm of contact geometry. Since then, the exploration of Ricci solitons within contact metric manifolds has been a focal point for various mathematicians, leading to diverse studies and analyses in this field (see references, [1, 9–11]).

Motivated by the above studies, in the present paper is to study Ricci pseudosymmetric α-cosymplectic (κ, μ, ν)-spaces whose metric admit Ricci soliton. We find some results among valuables α, κ, μ and ν which contribute differential geometry. For a Riemannian manifold (M^n, g), invariant of a concircular transformation is the concircular curvature tensor C, the W_3-curvature, the Weyly projective curvature tensors are, respectively, given by

$$C(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_3 = R(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_3 - \frac{\tau}{n(n-1)}\{g(\mathcal{Y}_2,\mathcal{Y}_3)\mathcal{Y}_1 - g(\mathcal{Y}_1,\mathcal{Y}_3)\mathcal{Y}_2\}, \tag{1.13}$$

$$W_1(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_3 = R(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_3 + \frac{1}{n-1}\{S(\mathcal{Y}_2,\mathcal{Y}_3)\mathcal{Y}_1 - S(\mathcal{Y}_1,\mathcal{Y}_3)\mathcal{Y}_2\}, \tag{1.14}$$

$$P(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_3 = R(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_3 - \frac{1}{n-1}\{S(\mathcal{Y}_2,\mathcal{Y}_3)\mathcal{Y}_1 - S(\mathcal{Y}_1,\mathcal{Y}_3)\mathcal{Y}_2\} \tag{1.15}$$

and

$$W_2(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_3 = R(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_3 - \frac{1}{n-1}\{g(\mathcal{Y}_2,\mathcal{Y}_3)Q\mathcal{Y}_1 - g(\mathcal{Y}_1,\mathcal{Y}_3)Q\mathcal{Y}_2\}, \tag{1.16}$$

for all $\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_3 \in \Gamma(TM)$, where R, Q and τ denote the Riemannian curvature tensor, Ricci operator the scalar curvature of M^n, respectively. Indeed, when a Riemannian manifold exhibits a vanishing concircular curvature tensor, it signifies a property of constant curvature. The concircular curvature tensor serves as an indicator, highlighting the extent to which a Riemannian manifold deviates from possessing a constant curvature. Its absence or vanishing nature implies that the manifold maintains a constant curvature throughout.

For a $(0,k)$-type tensor field T, $k \geq 1$ and a $(0,2)$-type tensor field A on a Riemannian manifold (M,g), $Q(A,T)$-tensor field is defined by

$$\begin{aligned} Q(A,T)(X_1,X_2,\dots,X_k;\mathcal{Y}_1,\mathcal{Y}_2) = &-T((\mathcal{Y}_1\wedge_A\mathcal{Y}_2)X_1,X_2,\dots,X_k) \\ &-\dots - T(X_1,X_2,\dots X_{k-1},(\mathcal{Y}_1\wedge_A\mathcal{Y}_2)X_k), \end{aligned} \tag{1.17}$$

for all $X_1,X_2,\dots,X_k,\mathcal{Y}_1,\mathcal{Y}_2 \in \Gamma(TM)$ (Atceken *et al.* [2]), where $\wedge_A$-endomrphism is given by

$$(\mathcal{Y}_1\wedge_A\mathcal{Y}_2)\mathcal{Y}_3 = A(\mathcal{Y}_2,\mathcal{Y}_3)\mathcal{Y}_1 - A(\mathcal{Y}_1,\mathcal{Y}_3)\mathcal{Y}_2. \tag{1.18}$$

2. Yamabe Solitons on Almost α-Cosymplectic (κ,μ,ν)-Space

Now, let (g,ξ,λ) be an Einstein soliton on α-cosymplectic (κ,μ,ν)-space. Then, we obtain

$$\begin{aligned} (L_\xi g)(\mathcal{Y}_1,\mathcal{Y}_2) &= g(\nabla_{\mathcal{Y}_1}\xi,\mathcal{Y}_2) + g(\mathcal{Y}_1,\nabla_{\mathcal{Y}_2}\xi) \\ &= g(-\alpha^2\phi\mathcal{Y}_1 - \phi h\mathcal{Y}_1,\mathcal{Y}_2) + g(\mathcal{Y}_1,-\alpha^2\phi\mathcal{Y}_2 - \phi h\mathcal{Y}_2) \\ &= 2\alpha g(\phi\mathcal{Y}_1,\phi\mathcal{Y}_2) - 2g(\phi h\mathcal{Y}_1,\mathcal{Y}_2). \end{aligned} \tag{2.1}$$

From (2.1) and (1.12), we have

$$(L_\xi g)(\mathcal{Y}_1,\mathcal{Y}_2) + 2S(\mathcal{Y}_1,\mathcal{Y}_2) + (2\lambda - \tau)g(\mathcal{Y}_1,\mathcal{Y}_2) = 0,$$

that is,

$$\alpha g(\phi\mathcal{Y}_1,\phi\mathcal{Y}_2) - g(\phi h\mathcal{Y}_1,\mathcal{Y}_2) + S(\mathcal{Y}_1,\mathcal{Y}_2) + \left(\lambda - \frac{\tau}{2}\right)g(\mathcal{Y}_1,\mathcal{Y}_2) = 0,$$

for all $\mathcal{Y}_1,\mathcal{Y}_2 \in \Gamma(TM)$. Thus, we have

$$S(\mathcal{Y}_1,\mathcal{Y}_2) = g(\phi h\mathcal{Y}_1,\mathcal{Y}_2) - \alpha g(\phi\mathcal{Y}_1,\phi\mathcal{Y}_2) + \left(\frac{\tau}{2} - \lambda)g(\mathcal{Y}_1,\mathcal{Y}_2\right). \tag{2.2}$$

Thus for $\mathcal{Y}_2 = \xi$, we have

$$S(\mathcal{Y}_1,\xi) = \left(\frac{\tau}{2} - \lambda\right)\eta(\mathcal{Y}_1). \tag{2.3}$$

In view of (1.6), by direct calculations, we can derive for α-cosymplectic (κ,μ,ν) space M^{2n+1},

$$S(\mathcal{Y}_1,\xi) = 2n\kappa\eta(\mathcal{Y}_1). \tag{2.4}$$

Hence, from (2.3) and (2.4), we have

$$\lambda = \frac{\tau}{2} - 2n\kappa. \tag{2.5}$$

From (1.6) and (1.14), we obtain

$$\begin{aligned} C(\mathcal{Y}_1,\mathcal{Y}_2)\xi &= \left(\kappa - \frac{\tau}{2n(2n+1)}\right)\{\eta(\mathcal{Y}_2)\mathcal{Y}_1 - \eta(\mathcal{Y}_1)\mathcal{Y}_2\} \\ &\quad + \mu\{\eta(\mathcal{Y}_2)h\mathcal{Y}_1 - \eta(\mathcal{Y}_1)h\mathcal{Y}_2\} + \nu\{\eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1 - \eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2\} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \eta(C(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_3) &= \left(\kappa - \frac{\tau}{2n(2n+1)}\right) g(\eta(\mathcal{Y}_1)\mathcal{Y}_2 - \eta(\mathcal{Y}_2)\mathcal{Y}_1, \mathcal{Y}_3) \\ &\quad + \mu g(\eta(\mathcal{Y}_1)h\mathcal{Y}_2 - \eta(\mathcal{Y}_2)h\mathcal{Y}_1, \mathcal{Y}_3) + \nu g(\eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2 - \eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1, \mathcal{Y}_3). \end{aligned} \tag{2.7}$$

Let us suppose that the concircular Ricci-pseudosymmetric almost α-cosymplectic (κ,μ,ν) space admitting Einstein soliton. Then there exists a function L_c on M such that

$$C \cdot S = L_c Q(g,S),$$

that is mean

$$(C(\mathcal{Y}_1,\mathcal{Y}_2)\cdot S)(\mathcal{Y}_4,\mathcal{Y}_5) = L_c Q(g,S)(\mathcal{Y}_4,\mathcal{Y}_5;\mathcal{Y}_1,\mathcal{Y}_2),$$

for all $\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_4,\mathcal{Y}_5 \in (\Gamma(T\widetilde{M})$, that is,

$$S(C(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_4,\mathcal{Y}_5) + S(\mathcal{Y}_4, C(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_5) = L_c\{S((\mathcal{Y}_1 \wedge_g \mathcal{Y}_2)\mathcal{Y}_4,\mathcal{Y}_5) + S(\mathcal{Y}_4,(\mathcal{Y}_1 \wedge_g \mathcal{Y}_2)\mathcal{Y}_5)\},$$

which yields to for $\mathcal{Y}_4 = \xi$,

$$\begin{aligned} &S(C(\mathcal{Y}_1,\mathcal{Y}_2)\xi,\mathcal{Y}_5) + S(C(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_5,\xi) \\ &\quad = L_c\{S(\eta(\mathcal{Y}_2)\mathcal{Y}_1 - \eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5) + S(\xi, g(\mathcal{Y}_2,\mathcal{Y}_5)\mathcal{Y}_1 - g(\mathcal{Y}_1,\mathcal{Y}_5)\mathcal{Y}_2)\}. \end{aligned} \tag{2.8}$$

By using (2.4), (2.6) and (2.7), we have

$$\begin{aligned} &\left(\kappa - \frac{\tau}{2n(2n+1)}\right) S(\eta(\mathcal{Y}_2)\mathcal{Y}_1 - \eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5) + \mu S(\eta(\mathcal{Y}_2)h\mathcal{Y}_1 - \eta(\mathcal{Y}_1)h\mathcal{Y}_2,\mathcal{Y}_5) \\ &+ \nu S(\eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1 - \eta(\mathcal{Y}_1)\phi h,\mathcal{Y}_5) + 2n\kappa\eta(C(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_5) \\ &\quad = L_c\{S(\eta(\mathcal{Y}_2)\mathcal{Y}_1 - \eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5) + 2n\kappa\eta(g(\mathcal{Y}_2,\mathcal{Y}_5)\mathcal{Y}_1 - g(\mathcal{Y}_1,\mathcal{Y}_5)\mathcal{Y}_2)\}. \end{aligned}$$

Taking account of (2.2), we get

$$\begin{aligned} &L_c\{S(\eta(\mathcal{Y}_2)\mathcal{Y}_1 - \eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5) + 2n\kappa g(\eta(\mathcal{Y}_1)\mathcal{Y}_2 - \eta(\mathcal{Y}_2)\mathcal{Y}_1,\mathcal{Y}_5)\} \\ &\quad = 2n\kappa\left(\kappa - \frac{\tau}{2n(2n+1)}\right) g(\eta(\mathcal{Y}_1)\mathcal{Y}_2 - \eta(\mathcal{Y}_2)\mathcal{Y}_1,\mathcal{Y}_5) + 2n\kappa\mu g(\eta(\mathcal{Y}_1)h\mathcal{Y}_2 - \eta(\mathcal{Y}_2)h\mathcal{Y}_1,\mathcal{Y}_5) \\ &\qquad + 2n\kappa\nu g(\eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2 - \eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1,\mathcal{Y}_5) + \left(\kappa - \frac{\tau}{2n(2n+1)}\right) S(\eta(\mathcal{Y}_2)\mathcal{Y}_1 - \eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5) \\ &\qquad + \mu S(\eta(\mathcal{Y}_2)h\mathcal{Y}_1 - \eta(\mathcal{Y}_1)h\mathcal{Y}_2,\mathcal{Y}_5) + \nu S(\eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1 - \eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2,\mathcal{Y}_5). \end{aligned}$$

Consequently, taking into account of (1.7) we reach at

$$\begin{aligned} &g(\eta(\mathcal{Y}_1)\mathcal{Y}_2 - \eta(\mathcal{Y}_2)\mathcal{Y}_1,\mathcal{Y}_5)\left(\left(L_c - \kappa + \frac{\tau}{2n(2n+1)}\right)(2n\kappa - \lambda + \alpha) - \nu(\kappa + \alpha^2)\right) \\ &+ g(\eta(\mathcal{Y}_1)\mathcal{Y}_2 - \eta(\mathcal{Y}_2)\mathcal{Y}_1,\phi h\mathcal{Y}_5)\left(-L_c + \kappa - \frac{\tau}{2n(2n+1)} - \nu(\alpha - \lambda + 2n\kappa)\right) \end{aligned}$$

$$-g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,h\mathcal{Y}_5)\mu(2n\kappa-\lambda+\alpha)+g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\phi\mathcal{Y}_5)\mu(\kappa+\alpha^2)=0.$$

This implies that

$$\begin{aligned}&\left(\left(L_c-\kappa+\frac{\tau}{2n(2n+1)}\right)(2n\kappa-\lambda+\alpha)-\nu(\kappa+\alpha^2)\right)\mathcal{Y}_5\\&+\mu(\kappa+\alpha^2)\phi\mathcal{Y}_5+\left(-L_c+\kappa-\frac{\tau}{2n(2n+1)}-\nu(\alpha-\lambda+2n\kappa)\right)\phi h\mathcal{Y}_5\\&-\mu(2n\kappa-\lambda+\alpha)h\mathcal{Y}_5=0.\end{aligned}\tag{2.9}$$

Both sides of this equation are multiplied by ξ, we have

$$\left(L_c-\kappa+\frac{\tau}{2n(2n+1)}\right)(2n\kappa-\lambda+\alpha)=\nu(\kappa+\alpha^2).\tag{2.10}$$

Then (2.9) reduces

$$\mu(\kappa+\alpha^2)\phi\mathcal{Y}_5+\left(-L_c+\kappa-\frac{\tau}{2n(2n+1)}-\nu(\alpha-\lambda+2n\kappa)\right)\phi h\mathcal{Y}_5-\mu(2n\kappa-\lambda+\alpha)h\mathcal{Y}_5=0.\tag{2.11}$$

Substituting $h\mathcal{Y}_5$ for $\mathcal{Y}_5$ and using (2.11), we obtain

$$\begin{aligned}&\mu(\kappa+\alpha^2)\phi h\mathcal{Y}_5-(\kappa+\alpha^2)\left(-L_c+\kappa-\frac{\tau}{2n(2n+1)}-\nu(\alpha-\lambda+2n\kappa)\right)\phi\mathcal{Y}_5\\&-\mu(\kappa+\alpha^2)(2n\kappa-\lambda+\alpha)\phi^2\mathcal{Y}_5=0.\end{aligned}\tag{2.12}$$

Furthermore, applying h to (2.11) from left side and by means of (1.7), we reach at

$$\begin{aligned}&-\mu(\kappa+\alpha^2)\phi h\mathcal{Y}_5+(\kappa+\alpha^2)\left(-L_c+\kappa-\frac{\tau}{2n(2n+1)}-\nu(\alpha-\lambda+2n\kappa)\right)\phi\mathcal{Y}_5\\&-\mu(\kappa+\alpha^2)(2n\kappa-\lambda+\alpha)\phi^2\mathcal{Y}_5=0.\end{aligned}\tag{2.13}$$

From (2.12) and (2.13) we conclude that

$$\mu(\kappa+\alpha^2)(2n\kappa-\lambda+\alpha)=0.\tag{2.14}$$

Hence from (2.11) because of $\kappa+\alpha^2\neq0$, we have

$$\left(-L_c+\kappa-\frac{\tau}{2n(2n+1)}-\nu(\alpha-\lambda+2n\kappa)\right)\phi h\mathcal{Y}_5+\mu(\kappa+\alpha^2)\phi\mathcal{Y}_5=0.\tag{2.15}$$

In the last equality, substituting $h\mathcal{Y}_5$ for $\mathcal{Y}_5$ and making use of (1.7), we have

$$-(\kappa+\alpha^2)\left(-L_c+\kappa-\frac{\tau}{2n(2n+1)}-\nu(\alpha-\lambda+2n\kappa)\right)\phi\mathcal{Y}_5+\mu(\kappa+\alpha^2)\phi h\mathcal{Y}_5=0.\tag{2.16}$$

Since $\kappa+\alpha^2\neq0$, from (2.15) and (2.16) we can infer

$$\left(-L_c+\kappa-\frac{\tau}{2n(2n+1)}-\nu(\alpha-\lambda+2n\kappa)\right)^2+\mu^2(\kappa+\alpha^2)=0$$

and

$$\left(L_c-\kappa+\frac{\tau}{2n(2n+1)}\right)\left(2n\kappa+\alpha-\frac{\tau}{2}\right)=\nu(\kappa+\alpha^2).$$

From (2.15), (2.16) and (2.17), we observe

$$\mu=0\ \text{ or }\ 4n\kappa+\alpha-\frac{\tau}{2}=0\ \text{ and }\ \nu=0.\tag{2.17}$$

If $\mu=0$, $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ an almost α-cosymplectic (κ,μ,ν)-space reduces α-cosymplectic $(\kappa,0,\nu)$.

Otherwise, if $4n\kappa+\alpha-\frac{\tau}{2}=0$, then we conclude that $4\lambda=\tau+2\alpha$ and $\nu=0$. In this case, $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ an almost α-cosymplectic (κ,μ,ν)-space reduces α-cosymplectic $(\kappa,\mu,0)$.

Thus, we have following the theorem:

Theorem 2.1. *The concircular Ricci-pseudosymmetric almost α-cosymplectic (κ,μ,ν) space admits Einstein soliton. Then at least one of the following statements is true:*

(i) *An almost α-cosymplectic $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$-$(\kappa,\mu,\nu)$-space reduces α-cosymplectic $(\kappa,\mu,0)$.*

(ii) *An almost α-cosymplectic $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$-$(\kappa,\mu,\nu)$-space reduces α-cosymplectic $(\kappa,0,\nu)$.*

(iii) *Almost α-cosymplectic $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$-space is expanding (resp. shrinking, steady) for $\tau>-2\alpha$ (resp. $\tau<-2\alpha$, $\tau=-2\alpha$).*

Now, we assume that projective Ricci-pseudosymmetric almost α-cosymplectic (κ,μ,ν) space admits Einstein soliton. Then, we have

$$(P(\mathcal{Y}_1,\mathcal{Y}_2)\cdot S)(\mathcal{Y}_4,\mathcal{Y}_5)=L_pQ(g,S)(\mathcal{Y}_4,\mathcal{Y}_5;\mathcal{Y}_1,\mathcal{Y}_2),$$

for all $\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_4,\mathcal{Y}_5\in\Gamma(T\widetilde{M}^{2n+1})$. This implies that

$$S(P(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_4,\mathcal{Y}_5)+S(\mathcal{Y}_4,P(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_5)=L_p\{S((\mathcal{Y}_1\wedge_g\mathcal{Y}_2)\mathcal{Y}_4,\mathcal{Y}_5)+S(\mathcal{Y}_4,(\mathcal{Y}_1\wedge_g\mathcal{Y}_2)\mathcal{Y}_5)\},$$

which form for $\mathcal{Y}_4=\xi$

$$\begin{aligned}&S(P(\mathcal{Y}_1,\mathcal{Y}_2)\xi,\mathcal{Y}_5)+S(\xi,P(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_5)\\&\quad=L_p\{S(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+S(\xi,g(\mathcal{Y}_2,\mathcal{Y}_5)\mathcal{Y}_1-g(\mathcal{Y}_5,\mathcal{Y}_1)\mathcal{Y}_2)\}.\end{aligned}\tag{2.18}$$

On the hand, making use of (1.6) and (1.16), we have

$$P(\mathcal{Y}_1,\mathcal{Y}_2)\xi=\mu(\eta(\mathcal{Y}_2)h\mathcal{Y}_1-\eta(\mathcal{Y}_1)h\mathcal{Y}_2)+\nu(\eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1-\eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2)\tag{2.19}$$

and

$$\eta(P(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_3)=\mu g(\eta(\mathcal{Y}_1)h\mathcal{Y}_2-\eta(\mathcal{Y}_2)h\mathcal{Y}_1,\mathcal{Y}_3)+\nu g(\eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2-\eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1,\mathcal{Y}_3).\tag{2.20}$$

Thus (2.19) and (2.20) are set in (2.18),

$$\begin{aligned}&\mu S(\eta(\mathcal{Y}_2)h\mathcal{Y}_1-\eta(\mathcal{Y}_1)h\mathcal{Y}_2,\mathcal{Y}_5)+2n\kappa\eta(P(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_5)+\nu S(\eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1-\eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2,\mathcal{Y}_5)\\&\quad=L_p\{S(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+2n\kappa\eta(g(\mathcal{Y}_2,\mathcal{Y}_5)\mathcal{Y}_1-\eta(\mathcal{Y}_1,\mathcal{Y}_5)\mathcal{Y}_2)\}.\end{aligned}\tag{2.21}$$

Taking into account (2.2), we reach at

$$\begin{aligned}&L_p\Big\{g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\phi\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi\mathcal{Y}_5)\\&+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+2n\kappa g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\mathcal{Y}_5)\Big\}\\&\quad=\mu\Big\{-g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h^2\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)\\&\qquad+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)\Big\}\\&\qquad+\nu\Big\{g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h^2\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)\end{aligned}$$

$$+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)\Big\}$$
$$+2n\kappa\mu g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,h\mathcal{Y}_5)+2n\kappa\nu g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\phi h\mathcal{Y}_5),$$

that is,

$$L_p\Big\{g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\phi\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi\mathcal{Y}_5)$$
$$+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+2n\kappa g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\mathcal{Y}_5)\Big\}$$
$$=\mu\Big\{(\kappa+\alpha^2)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)$$
$$+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)\Big\}+\nu\Big\{-(\kappa+\alpha^2)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)$$
$$-\alpha g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)\Big\}$$
$$+2n\kappa\mu g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,h\mathcal{Y}_5)+2n\kappa\nu g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\phi h\mathcal{Y}_5).$$

This yields to

$$g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)\left[L_p+\nu\left(\frac{\tau}{2}-\lambda+\alpha+2n\kappa\right)\right]$$
$$+g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)\mu\left[\alpha-\frac{\tau}{2}+\lambda+2n\kappa\right]$$
$$-\mu(\kappa+\alpha^2)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi\mathcal{Y}_5)$$
$$+g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)\left[L_p\left(-\alpha-2n\kappa+\frac{\tau}{2}-\lambda\right)+\nu(\kappa+\alpha^2)\right]=0,$$

which implies that

$$\left[L_p+\nu\left(\alpha+\frac{\tau}{2}-\lambda\right)+2n\kappa\nu\right]\phi h\mathcal{Y}_5+\mu\left[\alpha-\frac{\tau}{2}+\lambda+2n\kappa\right]h\mathcal{Y}_5$$
$$-\mu(\kappa+\alpha^2)\phi\mathcal{Y}_5+\left[L_p\left(-\alpha+\frac{\tau}{2}-\lambda-2n\kappa\right)+\nu(\kappa+\alpha^2)\right]\mathcal{Y}_5=0. \tag{2.22}$$

By inner product by ξ both of sides (2.22), we get

$$L_p\left(\alpha-\frac{\tau}{2}+\lambda+2n\kappa\right)=\nu(\kappa+\alpha^2). \tag{2.23}$$

Thus (2.22) reduce

$$\left[L_p+\nu\left(\alpha+\frac{\tau}{2}-\lambda+2n\kappa\right)\right]\phi h\mathcal{Y}_5+\mu\left[\alpha-\frac{\tau}{2}+\lambda+2n\kappa\right]h\mathcal{Y}_5-\mu(\kappa+\alpha^2)\phi\mathcal{Y}_5=0. \tag{2.24}$$

Applying ϕ to (2.24) and by means of (1.7), we have

$$-\left[L_p+\nu\left(\alpha+\frac{\tau}{2}-\lambda+2n\kappa\right)\right]h\mathcal{Y}_5+\mu\left[\alpha-\frac{\tau}{2}+\lambda+2n\kappa\right]\phi h\mathcal{Y}_5-\mu(\kappa+\alpha^2)\phi^2\mathcal{Y}_5=0. \tag{2.25}$$

Furthermore, if $\phi \mathcal{Y}_5$ is put instead of $\mathcal{Y}_5$ in (2.24) and using the second of (1.7) we have

$$\left[L_p + \nu\left(\alpha + \frac{\tau}{2} - \lambda + 2n\kappa\right)\right] h\mathcal{Y}_5 - \mu\left[\alpha - \frac{\tau}{2} + \lambda + 2n\kappa\right]\phi h\mathcal{Y}_5 - \mu(\kappa + \alpha^2)\phi^2\mathcal{Y}_5 = 0. \tag{2.26}$$

From the last two inequalities, we conclude that

$$\mu(\kappa + \alpha^2)\phi^2\mathcal{Y}_5 = 0.$$

Since $\kappa + \alpha^2 \neq 0$ and $\phi^2\mathcal{Y}_5 \neq 0$,

$$\mu = 0. \tag{2.27}$$

Eq. (2.26) tell us

$$L_p + \nu\left(\alpha + \frac{\tau}{2} - \lambda + 2n\kappa\right) = 0. \tag{2.28}$$

Thus, from (2.23) and (2.5) we observe

$$L_p\alpha = \nu(\kappa + \alpha^2) \tag{2.29}$$

Also, by making use of (2.5) and (2.28),

$$L_p + \nu(\alpha + \tau - 2\lambda) = 0. \tag{2.30}$$

In view of (2.29) and (2.30), we have

$$\nu(2\lambda\alpha + \kappa - \alpha\tau) = 0. \tag{2.31}$$

Thus, we have the following theorem.

Theorem 2.2. *Let $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ an almost α-cosymplectic (κ,μ,ν) space be projective Ricci-pseudosymmetric admitting Einstein soliton. Then The ambient manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ either reduces an almost α-cosymplectic $(\kappa,0,0)$-space or $\alpha(2\lambda - \tau) + \kappa = 0$.*

Now, we assume that W_1-Ricci-pseudosymmetric almost α-cosymplectic (κ,μ,ν) space admits Einstein soliton. Then we have

$$(W_1(\mathcal{Y}_1,\mathcal{Y}_2)\cdot S)(\mathcal{Y}_4,\mathcal{Y}_5) = L_{W_1}Q(g,S)(\mathcal{Y}_4,\mathcal{Y}_5;\mathcal{Y}_1,\mathcal{Y}_2),$$

for all $\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_4,\mathcal{Y}_5 \in \Gamma(T\widetilde{M}^{2n+1})$. This implies that

$$S(W_1(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_4,\mathcal{Y}_5) + S(\mathcal{Y}_4,W_1(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_5) = L_{W_1}\{S((\mathcal{Y}_1 \wedge_g \mathcal{Y}_2)\mathcal{Y}_4,\mathcal{Y}_5) + S(\mathcal{Y}_4,(\mathcal{Y}_1 \wedge_g \mathcal{Y}_2)\mathcal{Y}_5)\},$$

which form for $\mathcal{Y}_4 = \xi$

$$\begin{aligned} &S(W_1(\mathcal{Y}_1,\mathcal{Y}_2)\xi,\mathcal{Y}_5) + S(\xi,W_1(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_5) \\ &\quad = L_{W_1}\{S(\eta(\mathcal{Y}_2)\mathcal{Y}_1 - \eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5) + S(\xi,g(\mathcal{Y}_2,\mathcal{Y}_5)\mathcal{Y}_1 - g(\mathcal{Y}_5,\mathcal{Y}_1)\mathcal{Y}_2)\}. \end{aligned} \tag{2.32}$$

On the hand, making use of (1.6) and (1.14), we have

$$\begin{aligned} W_1(\mathcal{Y}_1,\mathcal{Y}_2)\xi &= 2\kappa[\eta(\mathcal{Y}_2)\mathcal{Y}_1 - \eta(\mathcal{Y}_1)\mathcal{Y}_2] + \mu(\eta(\mathcal{Y}_2)h\mathcal{Y}_1 - \eta(\mathcal{Y}_1)h\mathcal{Y}_2) \\ &\quad + \nu(\eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1 - \eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2) \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} \eta(W_1(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_3) &= 2\kappa g(\eta(\mathcal{Y}_1)\mathcal{Y}_2 - \eta(\mathcal{Y}_2)\mathcal{Y}_1,\mathcal{Y}_3) + \mu g(\eta(\mathcal{Y}_1)h\mathcal{Y}_2 - \eta(\mathcal{Y}_2)h\mathcal{Y}_1,\mathcal{Y}_3) \\ &\quad + \nu g(\eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2 - \eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1,\mathcal{Y}_3). \end{aligned} \tag{2.34}$$

Thus (2.33) and (2.34) are set in (2.32),

$$\begin{aligned}
&2\kappa S(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+\mu S(\eta(\mathcal{Y}_2)h\mathcal{Y}_1-\eta(\mathcal{Y}_1)h\mathcal{Y}_2,\mathcal{Y}_5)\\
&+\nu S(\eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1-\eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2,\mathcal{Y}_5)+2n\kappa\eta(W_1(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_5)\\
&\quad=L_{W_1}\{S(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+2n\kappa\eta(g(\mathcal{Y}_2,\mathcal{Y}_5)\mathcal{Y}_1-\eta(\mathcal{Y}_1,\mathcal{Y}_5)\mathcal{Y}_2)\}.
\end{aligned}\tag{2.35}$$

Taking into account (2.2), we reach at

$$\begin{aligned}
&L_{W_1}\Big\{g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\phi\mathcal{Y}_1-\eta(\mathcal{Y}_1)\phi\mathcal{Y}_2,\phi\mathcal{Y}_5)\\
&+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+2n\kappa g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\mathcal{Y}_5)\Big\}\\
&\quad=2\kappa\Big\{g\Big(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5\Big)-\alpha g(\eta(\mathcal{Y}_2)\phi\mathcal{Y}_1-\eta(\mathcal{Y}_1)\phi\mathcal{Y}_2,\phi\mathcal{Y}_5)\\
&\qquad+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)\Big\}\\
&\qquad+\mu\Big\{-g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h^2\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)\\
&\qquad+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)\Big\}\\
&\qquad+\nu\Big\{g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h^2\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)\\
&\qquad+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)\Big\}+2n\kappa^2g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\mathcal{Y}_5)\\
&\qquad+2n\kappa\mu g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,h\mathcal{Y}_5)+2n\kappa\nu g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\phi h\mathcal{Y}_5),
\end{aligned}$$

that is,

$$\begin{aligned}
&L_{W_1}\Big\{g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\phi\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi\mathcal{Y}_5)\\
&\quad+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+2n\kappa g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\mathcal{Y}_5)\Big\}\\
&\quad=2\kappa\Big\{g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\phi\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi\mathcal{Y}_5)\\
&\qquad+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)\Big\}\\
&\qquad+\mu\Big\{(\kappa+\alpha^2)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)\\
&\qquad+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)\Big\}+\nu\Big\{-(\kappa+\alpha^2)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)\\
&\qquad-\alpha g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)\Big\}\\
&\qquad+2n\kappa^2g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\mathcal{Y}_5)+2n\kappa\mu g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,h\mathcal{Y}_5)\\
&\qquad+2n\kappa\nu g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\phi h\mathcal{Y}_5).
\end{aligned}$$

This yields to

$$g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)\left[L_{W_1}-2\kappa+\alpha\nu-\nu\left(\frac{\tau}{2}-\lambda\right)+2n\kappa\nu\right]$$
$$+g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)\mu\left[\alpha-\frac{\tau}{2}+\lambda+2n\kappa\right]-\mu(\kappa+\alpha^2)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi\mathcal{Y}_5)$$
$$+g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)\left[L_{W_1}\left(-\alpha-2n\kappa+\frac{\tau}{2}-\lambda\right)+2\kappa\left(\alpha-\frac{\tau}{2}+\lambda\right)+\nu(\kappa+\alpha^2)+2n\kappa^2\right]=0,$$

which implies that

$$\left[L_{W_1}-2\kappa+\nu\left(\alpha-\frac{\tau}{2}+\lambda\right)+2n\kappa\nu\right]\phi h\mathcal{Y}_5+\mu\left[\alpha-\frac{\tau}{2}+\lambda+2n\kappa\right]h\mathcal{Y}_5-\mu(\kappa+\alpha^2)\phi\mathcal{Y}_5$$
$$+L_{W_1}\left[\left(-\alpha+\frac{\tau}{2}-\lambda-2n\kappa\right)+2n\kappa\left(\alpha-\frac{\tau}{2}+\lambda\right)+\nu(\kappa+\alpha^2)+2n\kappa^2\mathcal{Y}_5\right]=0. \tag{2.36}$$

By inner product by ξ both of sides (2.36), we get

$$L_{W_1}\left(\alpha-\frac{\tau}{2}+\lambda+2n\kappa\right)=2n\kappa\left(\alpha-\frac{\tau}{2}+\lambda\right)+\nu(\kappa+\alpha^2)+2n\kappa^2. \tag{2.37}$$

Thus (2.36) reduce

$$\left[L_{W_1}-2\kappa+\nu\left(\alpha-\frac{\tau}{2}+\lambda\right)+2n\kappa\nu\right]\phi h\mathcal{Y}_5+\mu\left[\alpha-\frac{\tau}{2}+\lambda+2n\kappa\right]h\mathcal{Y}_5-\mu(\kappa+\alpha^2)\phi\mathcal{Y}_5=0. \tag{2.38}$$

Applying ϕ to (2.38) and by means of (1.7), we have

$$-\left[L_{W_1}-2\kappa+\nu\left(\alpha-\frac{\tau}{2}+\lambda\right)+2n\kappa\nu\right]h\mathcal{Y}_5+\mu\left[\alpha-\frac{\tau}{2}+\lambda+2n\kappa\right]\phi h\mathcal{Y}_5-\mu(\kappa+\alpha^2)\phi^2\mathcal{Y}_5=0. \tag{2.39}$$

Furthermore, if $\phi\mathcal{Y}_5$ is put instead of $\mathcal{Y}_5$ in (2.38) and using the second of (1.7), we have

$$\left[L_{W_1}-2\kappa+\nu\left(\alpha-\frac{\tau}{2}+\lambda\right)+2n\kappa\nu\right]h\mathcal{Y}_5-\mu\left[\alpha-\frac{\tau}{2}+\lambda+2n\kappa\right]\phi h\mathcal{Y}_5-\mu(\kappa+\alpha^2)\phi^2\mathcal{Y}_5=0. \tag{2.40}$$

From the last two inequalities, we conclude that

$$\mu(\kappa+\alpha^2)\phi^2\mathcal{Y}_5=0.$$

Since $\kappa+\alpha^2\neq 0$ and $\phi^2\mathcal{Y}_5\neq 0$,

$$\mu=0. \tag{2.41}$$

Eq. (2.40) tell us

$$L_{W_1}-2\kappa+\nu\left(\alpha-\frac{\tau}{2}+\lambda\right)+2n\kappa\nu=0. \tag{2.42}$$

Thus from (2.42) and (2.5) we observe

$$L_{W_1}\alpha=2\kappa(\alpha-2n\kappa)+\nu(\kappa+\alpha^2)+2n\kappa^2, \tag{2.43}$$

and

$$L_{W_1}-2\kappa+\nu\alpha=0. \tag{2.44}$$

Also, by making use of (2.43) and (2.44)

$$\nu(2\alpha^2+\kappa)=2n\kappa^2.$$

Thus, we have the following theorem.

Theorem 2.3. *Let $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ an almost α-cosymplectic (κ,μ,ν) space be W_1-Ricci-pseudosymmetric admitting Einstein soliton. Then, the ambient manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ reduces an almost α-cosymplectic $(\kappa,0,\frac{2n\kappa^2}{2\alpha^2+\kappa})$-space.*

Now, we assume that W_2-Ricci-pseudosymmetric almost α-cosymplectic (κ,μ,ν) space admits Einstein soliton. Then, we have

$$(W_2(\mathcal{Y}_1,\mathcal{Y}_2)\cdot S)(\mathcal{Y}_4,\mathcal{Y}_5)=L_{W_2}Q(g,S)(\mathcal{Y}_4,\mathcal{Y}_5;\mathcal{Y}_1,\mathcal{Y}_2),$$

for all $\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_4,\mathcal{Y}_5\in\Gamma(T\widetilde{M}^{2n+1})$, this implies that

$$S(W_2(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_4,\mathcal{Y}_5)+S(\mathcal{Y}_4,W_2(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_5)=L_{W_2}\{S((\mathcal{Y}_1\wedge_g\mathcal{Y}_2)\mathcal{Y}_4,\mathcal{Y}_5)+S(\mathcal{Y}_4,(\mathcal{Y}_1\wedge_g\mathcal{Y}_2)\mathcal{Y}_5)\},$$

which form for $\mathcal{Y}_4=\xi$,

$$\begin{aligned}&S(W_2(\mathcal{Y}_1,\mathcal{Y}_2)\xi,\mathcal{Y}_5)+S(\xi,W_2(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_5)\\&\quad=L_{W_2}\{S(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+S(\xi,g(\mathcal{Y}_2,\mathcal{Y}_5)\mathcal{Y}_1-g(\mathcal{Y}_5,\mathcal{Y}_1)\mathcal{Y}_2)\}.\end{aligned}\tag{2.45}$$

On the hand, making use of (1.6) and (1.16), we have

$$\begin{aligned}W_2(\mathcal{Y}_1,\mathcal{Y}_2)\xi&=\kappa[\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2]+\mu[(\eta(\mathcal{Y}_2)h\mathcal{Y}_1-\eta(\mathcal{Y}_1)h\mathcal{Y}_2]\\&\quad+\nu[(\eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1-\eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2]-\frac{1}{2n}[\eta(\mathcal{Y}_2)Q\mathcal{Y}_1-\eta(\mathcal{Y}_1)Q\mathcal{Y}_2]\end{aligned}\tag{2.46}$$

and

$$\begin{aligned}\eta(W_2(\mathcal{Y}_1,\mathcal{Y}_2)\mathcal{Y}_3)&=\kappa g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\mathcal{Y}_3)+\mu g(\eta(\mathcal{Y}_1)h\mathcal{Y}_2-\eta(\mathcal{Y}_2)h\mathcal{Y}_1,\mathcal{Y}_3)\\&\quad+\nu g(\eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2-\eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1,\mathcal{Y}_3)+\frac{1}{2n}S(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_3).\end{aligned}\tag{2.47}$$

Thus (2.46) and (2.47) are set in (2.45),

$$\begin{aligned}&\kappa S(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+\mu S(\eta(\mathcal{Y}_2)h\mathcal{Y}_1-\eta(\mathcal{Y}_1)h\mathcal{Y}_2,\mathcal{Y}_5)\\&+\nu S(\eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1-\eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2,\mathcal{Y}_5)-\frac{1}{2n}S(\eta(\mathcal{Y}_2)Q\mathcal{Y}_1-\eta(\mathcal{Y}_1)Q\mathcal{Y}_2,\mathcal{Y}_5)\\&-\left(\kappa-\frac{1}{2n}\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+\mu g(\eta(\mathcal{Y}_1)h\mathcal{Y}_2-\eta(\mathcal{Y}_2)h\mathcal{Y}_1,\mathcal{Y}_5)\\&+\nu g(\eta(\mathcal{Y}_1)\phi h\mathcal{Y}_2-\eta(\mathcal{Y}_2)\phi h\mathcal{Y}_1,\mathcal{Y}_5)\\&\quad=L_{W_2}\{S(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+2n\kappa\eta(g(\mathcal{Y}_2,\mathcal{Y}_5)\mathcal{Y}_1-\eta(\mathcal{Y}_1,\mathcal{Y}_5)\mathcal{Y}_2)\}.\end{aligned}\tag{2.48}$$

Taking into account (2.2), we reach at

$$\begin{aligned}&L_{W_2}\Big\{g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)\\&+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)-2n\kappa g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)\Big\}\\&\quad=\kappa\Big\{g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\phi\mathcal{Y}_1-\eta(\mathcal{Y}_1)\phi\mathcal{Y}_2,\phi\mathcal{Y}_5)\\&\qquad+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)\Big\}\\&\qquad+\mu\Big\{(\kappa+\alpha^2)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)\Big\}\\
&+\nu\Big\{-(\kappa+\alpha^2)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)-\alpha g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)\\
&+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)\Big\}-\frac{1}{2n}\Big\{S(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi\mathcal{Y}_5)\\
&-\alpha S(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+\left(\frac{\tau}{2}-\lambda\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)\Big\}\\
&-\left(\kappa-\frac{1}{2n}\right)g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)+\mu g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)\\
&-\nu g(\eta(\mathcal{Y}_1)\mathcal{Y}_2-\eta(\mathcal{Y}_2)\mathcal{Y}_1,\phi h\mathcal{Y}_5).
\end{aligned}
\tag{2.49}
$$

Taking into account (2.2) in (2.49), we reach at

$$
\begin{aligned}
&g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi h\mathcal{Y}_5)\left[L_{W_2}-\kappa+\alpha\nu-\nu\left(\frac{\tau}{2}-\lambda\right)-\frac{\alpha}{2n}+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)-\nu\right]\\
&\quad+g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,h\mathcal{Y}_5)\left[\mu\left(\alpha-\frac{\tau}{2}+\lambda\right)-\frac{1}{2n}\right]\\
&\quad+g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\phi\mathcal{Y}_5)\left[-\mu(\kappa+\alpha^2)+\frac{\alpha}{2n}+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)\right]\\
&\quad+g(\eta(\mathcal{Y}_2)\mathcal{Y}_1-\eta(\mathcal{Y}_1)\mathcal{Y}_2,\mathcal{Y}_5)\Big[L_{W_2}\left(-\alpha-2n\kappa+\frac{\tau}{2}-\lambda\right)+\alpha\kappa-\kappa\left(\frac{\tau}{2}-\lambda\right)\\
&\quad+\nu(\kappa+\alpha^2)+\frac{\alpha^2}{2n}-\frac{\alpha}{n}\left(\frac{\tau}{2}-\lambda\right)+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)^2+\left(\kappa-\frac{1}{2n}\right)-\mu\Big]=0
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\left[L_{W_2}-\kappa+\alpha\nu-\nu\left(\frac{\tau}{2}-\lambda\right)-\frac{\alpha}{2n}+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)-\nu\right]\phi h\mathcal{Y}_5\\
&\quad+\left[\mu\left(\alpha-\frac{\tau}{2}+\lambda\right)-\frac{1}{2n}\right]h\mathcal{Y}_5+\left[-\mu(\kappa+\alpha^2)+\frac{\alpha}{2n}+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)\right]\phi\mathcal{Y}_5\\
&\quad+\Big[L_{W_2}\left(-\alpha-2n\kappa+\frac{\tau}{2}-\lambda\right)+\alpha\kappa-\kappa\left(\frac{\tau}{2}-\lambda\right)+\nu(\kappa+\alpha^2)\\
&\quad+\frac{\alpha^2}{2n}-\frac{\alpha}{n}\left(\frac{\tau}{2}-\lambda\right)+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)^2+\left(\kappa-\frac{1}{2n}\right)-\mu\Big]\mathcal{Y}_5=0.
\end{aligned}
\tag{2.50}
$$

By inner product by ξ both of sides (2.50), we get

$$
\begin{aligned}
&L_{W_2}\left(-\alpha-2n\kappa+\frac{\tau}{2}-\lambda\right)+\alpha\kappa-\kappa\left(\frac{\tau}{2}-\lambda\right)+\nu(\kappa+\alpha^2)\\
&\quad+\frac{\alpha^2}{2n}-\frac{\alpha}{n}\left(\frac{\tau}{2}-\lambda\right)+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)^2+\left(\kappa-\frac{1}{2n}\right)-\mu=0.
\end{aligned}
\tag{2.51}
$$

Thus (2.50) reduce

$$
\left[L_{W_2}-\kappa+\alpha\nu-\nu\left(\frac{\tau}{2}-\lambda\right)-\frac{\alpha}{2n}+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)-\nu\right]\phi h\mathcal{Y}_5+\left[\mu\left(\alpha-\frac{\tau}{2}+\lambda\right)-\frac{1}{2n}\right]h\mathcal{Y}_5
$$

$$+\left[-\mu(\kappa+\alpha^2)+\frac{\alpha}{2n}+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)\right]\phi\mathcal{Y}_5=0. \tag{2.52}$$

Applying ϕ to (2.52), we have

$$-\left[L_{W_2}-\kappa+\alpha\nu-\nu\left(\frac{\tau}{2}-\lambda\right)-\frac{\alpha}{2n}+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)-\nu\right]h\mathcal{Y}_5+\left[\mu\left(\alpha-\frac{\tau}{2}+\lambda\right)-\frac{1}{2n}\right]\phi h\mathcal{Y}_5$$
$$+\left[-\mu(\kappa+\alpha^2)+\frac{\alpha}{2n}+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)\right]\phi^2\mathcal{Y}_5=0. \tag{2.53}$$

Furthermore, if $\phi\mathcal{Y}_5$ is put instead of $\mathcal{Y}_5$ in (2.53), we have

$$\left[L_{W_2}-\kappa+\alpha\nu-\nu\left(\frac{\tau}{2}-\lambda\right)-\frac{\alpha}{2n}+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)-\nu\right]h\mathcal{Y}_5-\left[\mu\left(\alpha-\frac{\tau}{2}+\lambda\right)-\frac{1}{2n}\right]\phi h\mathcal{Y}_5$$
$$+\left[-\mu(\kappa+\alpha^2)+\frac{\alpha}{2n}+\frac{1}{2n}\left(\frac{\tau}{2}-\lambda\right)\right]\phi^2\mathcal{Y}_5=0. \tag{2.54}$$

From the last two inequalities, we conclude that

$$\mu(\kappa+\alpha^2)=\frac{\alpha+2n\kappa}{2n}.$$

Thus, we have the following theorem.

Theorem 2.4. *Let $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ an almost α-cosymplectic (κ,μ,ν) space be W_2-Ricci-pseudosymmetric admitting Einstein soliton. Then, the ambient manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ reduces an almost α-cosymplectic $\left(\kappa,\frac{\alpha+2n\kappa}{2n},\nu\right)$-space.*

3. Conclusion

This paper attempts to characterize cases of an almost α-cosymplectic (κ,μ,ν)-space admitting Einstein sloitons to be concircular Ricci pseudosymmetry, projective Ricci pseudosymmetry, W_1-curvature and the W_2-curvature Ricci pseudo symmetric.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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