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Research Article

A Novel Numerical Scheme for Time-Fractional Partial Integro Differential Equation of Parabolic Type

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Abstract. This work is devoted to study numerical methods for time-fractional integro-differential equations. In order to compute the approximate solutions for highly non-linear or linear forms of various time-fractional integro-differential models, we apply the extended and more generalized finite difference methods. First order and second order spacial derivatives are approximated by the central difference. The integral terms and Capto fractional terms are approximated by the composite trapezoidal rule. Particularly we derive error estimation and stability analysis of the finite difference method for a Volterra type fractional differential equation. Illustrative examples are provided in support of the proposed methods with three distinct problems.

Keywords. Time-fractional ADE, Variable parameters, Finite difference methods, Convergence analysis, Integro-partial differential equations

Mathematics Subject Classification (2020). 65M06, 65M15

1. Introduction

Let $\beta \in (0,1)$, consider the following parabolic problem with a time-fractional *Partial Integro Differential Equation* (*PIDE*) of the Voltera type (Das *et al*. [7])

$$\begin{cases} \frac{\partial^\beta \mathcal{Z}}{\partial t^\beta} - p(x,t)\frac{\partial^2 \mathcal{Z}}{\partial x^2} - q(x,t)\frac{\partial \mathcal{Z}}{\partial x} + \lambda \int_0^t \mathcal{K}(x,t-\xi)\mathcal{Z}(x,\xi)d\xi = f(x,t), & \text{for all } (x,t) \in \bar{\mathcal{D}} = [0,1]\times[0,1], \\ & \mathcal{Z}(x,0) = \phi(x); \\ & \text{for all } x \in [0,1], \\ \mathcal{Z}(0,t) = \psi_1(t),\ \mathcal{Z}(1,t) = \psi_2(t), & \text{for all } t \in (0,1], \end{cases} \tag{1.1}$$

where $p(x,t)$ and $q(x,t)$ are continuous and bounded functions in the given domain $\bar{\mathcal{D}}$, $\mathcal{K}(x,t-\xi)$ is a position (x) and time (t)-dependent absorbent term which may be related to diffusion and reaction. Here, the reaction term $\int_0^t \mathcal{K}(x,t-\xi)\mathcal{Z}(x,\xi)d\xi$ involved in the above equation be different from that used by Das and Gupta [6], Das *et al.* [7], Schot *et al.* [28], and $f(x,t)$ is the source term which is sufficiently finite time continuously differentiable function over $\bar{\mathcal{D}}$, $\frac{\partial^\beta}{\partial t^\beta}$ represents the Caputo fractional differential operator of fractional order β. Also, $f(x,t)$, $\phi(x)$, $\psi_1(t)$ and $\psi_2(t)$ are continuously differentiable function over $[0,1]$ and the integrand term $\mathcal{K}(x,t-\xi)$ is sufficiently finite time continuously differentiable and bounded function over $\bar{\mathcal{D}}$. We look on finding the approximate solution of the PIDE as analytical solution does not have closed form always.

Because of its many applications in science and engineering, the diffusion equation has been extensively investigated; however, the study takes on a different dimension when it is non-linear and when the time dependent derivative in the standard diffusion equation is substituted with a fractional derivative of order β. Non-linearity is a good topic since it may be predicted to a significant extent with a thorough understanding of the corresponding equations. It is essential to conduct a thorough and in-depth study of non-linear PDE that is connected to classical mechanics. Non-linear diffusion equations are a significant class of parabolic equations that are used in numerous image processing and computer vision techniques as well as a lot of physical problems such as phase transition in electrical, electronic and mechanical engineering, biological science problems and biochemistry. Singularity in the problem makes them demanding and complex. The fractional differential equations are currently receiving a lot of attention due to the reason that the fractional order integro partial differential equation system response eventually converges to the integer order response of the system. The generation of fractional Brownian motion is a significant result of these evolution equations.

Since fractional integro differential equations can accurately enumerate numerous phenomena, physical processes and chemical reactions more so than traditional integer order differential equations, they are receiving increasing amounts of attention in a variety of fields, including finance (Mainardi *et al.* [20]), biological systems (Benson *et al.* [2]), and systems exhibiting Hamilton chaos. An exact Fox H-function solution to the generalized linear fractional reaction-diffusion equation has been derived by Zahran [38]. A non-linear diffusion equation containing fractionally ordered spatial derivatives was solved by Silva *et al.* [31], and Lenzi *et al.* [17]. Using VIM, in [5], Das have examined the fractional diffusion equation's analytical solution when an outside force is present. In the fractional-order quadratic auto-catalysis model with linear admire's approximate solutions, Saad *et al.* [26] used a numerical method based on Lagrange polynomial interpolation. To solve the fractional *Reaction-Diffusion Equation* (*RDE*), use the efficient and potent hybrid analytical technique known as q-HASTM.

To solve the fractional partial differential equations analytically and obtain closed-form analytical solutions, a variety of techniques have been developed, including imaging technique, the Fourier transform technique, the Mellin transform technique, the Laplace transform technique, and the technique of separation of variables (Kilbas *et al.* [15], and Podlubny [23]). Only a very small number of fractional partial differential equations, like integer-order differential equations, have closed-form analytical solutions available. As a result, numerical methods must be applied generally. It appears necessary and perhaps more crucial to create numerical techniques for resolving problems involving fractional differential and integrals. Two methods for numerically discretizing fractional derivatives using finite difference techniques make up the main outline of work.

On the definition of fractional derivatives, one is based by Grunwald and Letnikov [**?**]. Meerschaert and Tadjeran in [22] presented a shifted Grunwald formula to approximate spatial fractional derivatives of order $0 \le \beta \le 1$. The alternative involves using the concepts of interpolating polynomials, such as the L^1 approximation (Langlands and Henry [16]), L^2 approximations (Lynch *et al.* [19]), to substitute the derivatives under the integration with difference quotients.

According to Gorenflo [12], and Sousa [32], the fractional diffusion was discretized using the shifted Grunwald formula and the advection term was roughly estimated using a first order upwind finite difference. A discrete L^2 norm's stability and convergence were investigated and examined. It was suggested by Liu *et al.* [18] that a L^2 technique be used to discretize the fractional Fokker-Planck equation. Based on the L^1 approximation, Du *et al.* [10] developed a compact difference scheme for the fractional diffusion-wave equation. The explicit and weighted averaged difference schemes based on the Grunwald-Letnikov approximation were introduced by Yuste and Acedo [37] and these two schemes were examined using the Von Neumann technique. In order to demonstrate for the fractional reaction-sub-diffusion equation and sub-diffusion equation respectively, the convergence and stability of the difference scheme, Chen *et al.* [3, 4] developed the difference scheme also using the Grunwald-Letnikov formula and presented Fourier techninique. Shen and Liu [30] provided a study of errors and suggested the space fractional diffusion problem using an explicit finite difference technique. In their study of using the equation of temporal fractional diffusion, an implicit difference approximation, Zhuang and Liu [40] explored the method's convergence and stability. Langlands and Henry [16] also looked into this issue, offered a numerical method that was implicit (L^1-approximation) and talked on the precision and stability of this technique. For an explanation of sub-diffusion using the fractional diffusion equation, Chen *et al.* [3] developed a Fourier technique. They also provided analysis of the difference approximation method's stability and overall accuracy. The implicit numerical techniques for the anomalous sub diffusion equation also have a new solution and analytical techniques recommended by Zhuang *et al.* [41].

When working on the non-linear integral equation, Gordji *et al.* in [11], and Wongyat and Sintunavarat in [36] demonstrated the solution's existence and uniqueness characteristics. From a numerical perspective, Wang *et al.* [35] established an effective using a weakly singular convolution kernel, the second-order Volterra integral equations can be evaluated numerically. When a differential operator appears in an integral equation, the equation is referred to as a *Integro-Differential Equation* (IDE). The model problem was defined in Banach spaces over an unrestricted domain and it was demonstrated by Tari and Shahmorad [33] that the requirements for existence and uniqueness of a class of solutions of non-linear IDEs apply. The alternate method is to use numerical methodologies because it can be difficult to solve a problem analytically at times. By Zhang and Hao [39], the matrix Tau technique was developed to resolve two dimensional linear *Volterra Integro-Differential Equations* (VIDEs). In addition to studying the linear Fredholm-VIDEs, Shahmorad [29] calculated the error boundaries for the specified numerical approach.

Many researchers have recently worked on PIDE. Thorwe and Bhalekar [34], to name a few, took into consideration a linear PIDE with a convolution kernel. To solve the equation analytically, they employed the Laplace transform technique. In his work, Dehghan [8] developed a second order numerical strategy to determine a PIDE with a weakly unique kernel. An IDE's analytical solution gets more challenging when fractional order derivatives are included. The presence and originality requirements were examined in the works of Hamoud *et al*. [14], and Matar [21]. In addition, a number of numerical techniques were created to evaluate the fractional order *Partial Integro Differential Equations* (PIDEs). With the purpose of evaluating fractional order *Partial Integro Differential Equation* (PIDE) with a finite delay, Abbas *et al*. [1] examined several conclusions about existence, uniqueness, and global asymptotic stability. Santra and Mohapatra [27] studied a time fractional partial integro-differential equation with a numerical solution of the Volterra type, where the time derivative is specified in the Caputo sense. The approximation solution converges to the precise answer after the error analysis is completed.

In this study, the fractional diffusion equation is solved using the finite difference method in the presence of both a linear external force and an absorbent term. Analytical formulations for various Brownian movements are obtained using the initial condition. An objective of the study is to error analysis on the fractional diffusion equation with the presence of the linear external force. To the author's knowledge, no one has yet solved the non-linear time fractional diffusion equation in the presence of the kind of external force and the specified kind of reaction term consider in our case.

2. Preliminaries

This section contains definitions and characteristics of fractional derivatives and integrals that we will utilise in our subsequent analysis (see Diethelm [9], and Podlubny [23]).

Definition 2.1. Let V and W be normed linear spaces over the underling field $\mathbb{R}$ or $\mathbb{C}$ (Santra and Mohapatra [27]) then the linear operator $A : V \to W$ is said to be bounded, if there exist a positive constant C such that

$$\|A(v)\|_W \le C\|v\|_V, \quad \text{for all } v \in V. \tag{2.1}$$

Theorem 2.1. *Let V and W be normed linear spaces* (*Santra and Mohapatra* [27])*, if and only if the linear operator $A : V \to W$ is continuous throughout V, then it is bounded.*

Definition 2.2. Let us say that $V \subseteq \mathbb{R}^n$, the function $f : V \to V$ maps the contraction if the following inequality holds,

$$\|f(v_1) - f(v_2)\| \le c\|v_1 - v_2\|, \quad \text{for all } v_1, v_2 \in V, \tag{2.2}$$

where c denotes the contraction constant with range $0 \le c < 1$.

Definition 2.3. Let V and W represent two normed linear spaces over $\mathbb{R}$ or $\mathbb{C}$ and $D_v : V \to W$ is a linear operator. If there exists a positive constant C such that (Santra and Mohapatra [27]), the linear operator D_x is said to be bounded, if it satisfy the following inequality:

$$\left\| \frac{\partial}{\partial v}[\mathcal{E}(v)(v-w)] \right\| \le C\|\mathcal{E}(v)\|\,\|v-w\|, \quad \text{for all } v \in V,\ w \in W. \tag{2.3}$$

Theorem 2.2. *Assuming that $V \subseteq \mathbb{R}^n$ is complete and that $\zeta : V \to V$ is a contraction mapping then ζ has a unique fixed point p^* in V.*

Definition 2.4. For every $i = 0,1,\ldots,\mathcal{M}$, $j = 0,1,\ldots\mathcal{N}$, assuming that the mesh function $\mathcal{V}_i^j$ corresponding to $\mathcal{V} : \bar{\Omega} \subset \mathbb{R}^2 \to \mathbb{R}$ where $\mathcal{V}$ is a continuous function (Santra and Mohapatra [27]). Then, we can define

$$\|\mathcal{V}\| = \max_{(x,t)\in\bar{\Omega}} |\mathcal{V}(x,t)| \quad \text{and} \quad \|\mathcal{V}^j\| = \max_{0\le i\le M} |\mathcal{V}_i^j|. \tag{2.4}$$

Definition 2.5. A real valued function $\varphi(t) \in C_\mu$, $\mu \in \mathbb{R}$ (Prakash and Kaur [24]) if there exist a number $p \in \mathbb{R}$ $(p > \mu)$ such that $\varphi(t) = t^p \varphi_1(t)$, where $\varphi_1(t) \in C[0,\infty)$ and $\varphi(t) \in C_\mu^m$ if $\varphi^{(m)} \in C_\mu$, $m \in \mathbb{N} \cup \{0\}$.

Definition 2.6. The Riemann-Liouville fractional integral operator of order $\beta > 0$ for the function $\varphi(t) \in C_\mu$, $\mu \ge -1$ is defined as (Ray and Bera [25]):

$$J^\beta \varphi(t) = \begin{cases} \frac{1}{\Gamma\beta} \int_0^t \frac{\varphi(v)}{(t-v)^{1-\beta}} dv, & \beta > 0,\ t > 0, \\ \varphi(t), & \beta = 0. \end{cases} \tag{2.5}$$

Definition 2.7. The Caputo fractional derivative operator of order $\beta > 0$ for the function $\varphi(t) \in C_{-1}^m$, $m \in \mathbb{N} \cup \{0\}$ is defined as (Ray and Bera [25]):

$$D_t^\beta \varphi(t) = \begin{cases} \frac{1}{\Gamma n-\beta} \int_0^t \frac{\varphi^{(n)}(v)}{(t-v)^{1-n+\beta}} dv, & n-1 < \beta < n,\ n \in \mathbb{N}, \\ \frac{d^n}{dt^n}\varphi(t), & \beta = n \in \mathbb{N}. \end{cases} \tag{2.6}$$

Following are some of the properties of Caputo-fractional derivatives:

(a) $D_t^\beta J_t^\beta \zeta(t) = \zeta(t)$.

(b) $J_t^\beta D_t^\beta \zeta(t) = \zeta(t) - \sum\limits_{k=0}^{n-1} \zeta^{(k)}(0^+)\frac{(t-a)^k}{k!}$, $t > 0$.

(c) $J_t^\beta D_t^\beta \zeta(x,t) = \zeta(x,t) - \zeta(x,0)$.

3. Continuous Problems

Theorem 3.1. *Let $p(x,t)$ and $q(x,t)$ be continuous and bounded in the region $\bar{\mathcal{D}}$ such that $0 < \|p(x,t)\| \le K_1$, $0 < \|q(x,t)\| \le K_2$ and $0 < \|\mathcal{K}(x,t-\xi)\| \le M$ where K_1, K_2 and M are positive generic constants, respectively. If the quantity $\frac{(C_1+C_2)(\beta+1)+|\lambda|M}{\Gamma(\beta+2)} < 1$ then there exists a unique solution $\mathcal{Z}(x,t)$ of* (1.1) *where C_1 and C_2 be two positive constants.*

Proof. Operating J^α on both side of equation (1.1) we obtained,

$$J^\beta \frac{\partial^\beta \mathcal{Z}}{\partial t^\beta} - J^\beta \left[p(x,t)\frac{\partial^2 \mathcal{Z}}{\partial x^2} \right] - J^\beta \left[q(x,t)\frac{\partial \mathcal{Z}}{\partial x} \right] + \lambda J^\beta \int_0^t \mathcal{K}(x,t-\xi)\mathcal{Z}(x,\xi)d\xi = J^\beta f(x,t). \tag{3.1}$$

That is,

$$\mathcal{Z}(x,t)-\mathcal{Z}(x,0)-J^{\beta}\left[p(x,t)\frac{\partial^2\mathcal{Z}}{\partial x^2}\right]-J^{\beta}\left[q(x,t)\frac{\partial\mathcal{Z}}{\partial x}\right]+\lambda J^{\beta}\int_0^t\mathcal{K}(x,t-\xi)\mathcal{Z}(x,\xi)d\xi=J^{\beta}f(x,t), \tag{3.2}$$

$$\mathcal{Z}(x,t)=\phi(x)+J^{\beta}\left[p(x,t)\frac{\partial^2\mathcal{Z}}{\partial x^2}\right]+J^{\beta}\left[q(x,t)\frac{\partial\mathcal{Z}}{\partial x}\right]-\lambda J^{\beta}\int_0^t\mathcal{K}(x,t-\xi)\mathcal{Z}(x,\xi)d\xi+J^{\beta}f(x,t). \tag{3.3}$$

Let $\mathcal{Z}(x,t)=H\mathcal{Z}(x,t)$, for all $(x,t)\in\bar{\mathcal{D}}$ then

$$H\mathcal{Z}(x,t)=\phi(x)+J^{\beta}\left[p(x,t)\frac{\partial^2\mathcal{Z}}{\partial x^2}\right]+J^{\beta}\left[q(x,t)\frac{\partial\mathcal{Z}}{\partial x}\right]-\lambda J^{\beta}\int_0^t\mathcal{K}(x,t-\xi)\mathcal{Z}(x,\xi)d\xi+J^{\beta}f(x,t). \tag{3.4}$$

Let $\mathcal{Z}_1,\ \mathcal{Z}_2\in C(\bar{D})$, then we have,

$$H\mathcal{Z}_1(x,t)=\phi(x)+J^{\beta}\left[p(x,t)\frac{\partial^2\mathcal{Z}_1}{\partial x^2}\right]+J^{\beta}\left[q(x,t)\frac{\partial\mathcal{Z}_1}{\partial x}\right]-\lambda J^{\beta}\int_0^t\mathcal{K}(x,t-\xi)\mathcal{Z}_1(x,\xi)d\xi+J^{\beta}f(x,t) \tag{3.5}$$

and

$$H\mathcal{Z}_2(x,t)=\phi(x)+J^{\beta}\left[p(x,t)\frac{\partial^2\mathcal{Z}_2}{\partial x^2}\right]+J^{\beta}\left[q(x,t)\frac{\partial\mathcal{Z}_2}{\partial x}\right]-\lambda J^{\beta}\int_0^t\mathcal{K}(x,t-\xi)\mathcal{Z}_2(x,\xi)d\xi+J^{\beta}f(x,t). \tag{3.6}$$

Since

$$\begin{aligned}\|H\mathcal{Z}_1-H\mathcal{Z}_2\|&=\Bigg\|J^{\beta}\left[p(x,t)\frac{\partial^2}{\partial x^2}(\mathcal{Z}_1-\mathcal{Z}_2)\right]+J^{\beta}\left[q(x,t)\frac{\partial}{\partial x}\cdot(\mathcal{Z}_1-\mathcal{Z}_2)\right]\\&\quad-\lambda J^{\beta}\int_0^t\mathcal{K}(x,t-\xi)(\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi)d\xi\Bigg\|,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\|H\mathcal{Z}_1-H\mathcal{Z}_2\|&\le\left\|J^{\beta}\left[p(x,t)\frac{\partial^2}{\partial x^2}(\mathcal{Z}_1-\mathcal{Z}_2)\right]\right\|+\left\|J^{\beta}\left[q(x,t)\frac{\partial}{\partial x}\cdot(\mathcal{Z}_1-\mathcal{Z}_2)\right]\right\|\\&\quad+\left\|\lambda J^{\beta}\int_0^t\mathcal{K}(x,t-\xi)(\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi)d\xi\right\|\end{aligned} \tag{3.8}$$

$$\begin{aligned}\|H\mathcal{Z}_1-H\mathcal{Z}_2\|&\le\left\|\frac{1}{\Gamma(\beta)}\int_0^t(t-\xi)^{\beta-1}p(x,\xi)\frac{\partial^2}{\partial x^2}((\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi))d\xi\right\|\\&\quad+\left\|\frac{1}{\Gamma(\beta)}\int_0^t(t-\xi)^{\beta-1}q(x,\xi)\frac{\partial}{\partial x}((\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi))d\xi\right\|\\&\quad+\left\|\frac{\lambda}{\Gamma(\beta)}\int_0^t(t-\rho)^{\beta-1}\int_0^{\rho}\mathcal{K}(x,\rho-\xi)(\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi)d\rho d\xi\right\|\end{aligned} \tag{3.9}$$

$$\begin{aligned}\|H\mathcal{Z}_1-H\mathcal{Z}_2\|&\le\frac{1}{\Gamma(\beta)}\int_0^t(t-\xi)^{\beta-1}\|p(x,\xi)\|\left\|\frac{\partial^2}{\partial x^2}((\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi))\right\|d\xi\\&\quad+\frac{1}{\Gamma(\beta)}\int_0^t(t-\xi)^{\beta-1}\|q(x,\xi)\|\left\|\frac{\partial}{\partial x}((\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi))\right\|d\xi\\&\quad+\frac{|\lambda|}{\Gamma(\beta)}\int_0^t(t-\rho)^{\beta-1}\int_0^{\rho}\|\mathcal{K}(x,\rho-\xi)\|\,\|(\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi)\|d\rho d\xi.\end{aligned} \tag{3.10}$$

Now, by Definitions 2.1 and 2.3, and using Theorem 2.1, we get

$$\|H\mathcal{Z}_1-H\mathcal{Z}_2\|\le\frac{1}{\Gamma(\beta)}\left(\frac{t^{\beta}}{\beta}\right)\|p(x,\xi)\|\left\|\frac{\partial^2}{\partial x^2}(\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi)\right\|$$

$$+\frac{1}{\Gamma(\beta)}\left(\frac{t^\beta}{\beta}\right)\|q(x,\xi)\|\left\|\frac{\partial}{\partial x}(\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi)\right\|+\frac{|\lambda|M}{\Gamma(\beta+2)}\|(\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi)\|, \tag{3.11}$$

$$\|H\mathcal{Z}_1-H\mathcal{Z}_2\|\leq\frac{K_1\cdot K_4\cdot(\beta+1)\|(\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi)\|}{\Gamma(\beta+2)}+\frac{K_2\cdot K_3\cdot(\beta+1)\|(\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi)\|}{\Gamma(\beta+2)}$$
$$+\frac{|\lambda|M}{\Gamma(\beta+2)}\|(\mathcal{Z}_1-\mathcal{Z}_2)(x,\xi)\|, \tag{3.12}$$

where K_1, K_2, K_3, K_4 and M are positive constants. Therefore, we have

$$\|H\mathcal{Z}_1-H\mathcal{Z}_2\|\leq\frac{(K_1\cdot K_4+K_2\cdot K_3)(\beta+1)+|\lambda|M}{\Gamma(\beta+2)}\|\mathcal{Z}_1-\mathcal{Z}_2\|. \tag{3.13}$$

Since K_1, K_2, K_3, K_4, λ and M be chosen such a way that

$$\frac{(K_1K_4+K_2K_3)(\beta+1)+|\lambda|M}{\Gamma(\beta+2)}<1 \quad\text{and}\quad \frac{(C_1+C_2)(\beta+1)+|\lambda|M}{\Gamma(\beta+2)}<1,$$

where $C_1=K_1K_4$, $C_2=K_2K_3$, therefore we have

$$\|H\mathcal{Z}_1-H\mathcal{Z}_2\|<\|\mathcal{Z}_1-\mathcal{Z}_2\|. \tag{3.14}$$

This proves that H is a contraction function and we look that $(C(\bar{\mathcal{D}}),\|\cdot\|)$ is a Banach space. Hence by Theorem 2.2 one can conclude that equation (1.1) has a unique solution $\mathcal{Z}(x,t)$ in $\bar{\mathcal{D}}$. □

4. Numerical Approximations

Let us consider $\mathcal{M}$ and $\mathcal{N}$ be two fixed positive integers. Define the grid $x_i=ih$ for $i=0,1,2,3,\dots\mathcal{M}$ and $t_j=j\tau$ for $j=0,1,2,3,\dots\mathcal{N}$, where h is the step length for spatial direction and τ is the step length for time direction and are defined by $h=\frac{1}{\mathcal{M}}$ and $\tau=\frac{1}{\mathcal{N}}$. Then, the uniform mesh is defined as $\bar{\mathcal{D}}_1=\{(x_i,t_j):i=0,1,2,3,\dots\mathcal{M},\,j=0,1,2,3,\dots\mathcal{N}\}$. Let $\{\mathcal{Z}(x_i,t_j)\}_{i=0,\,j=0}^{\mathcal{M},\,\mathcal{N}}$ be the exact solution and denote $\{\mathcal{Z}_i^j\}_{i=0,\,j=0}^{\mathcal{M},\,\mathcal{N}}$ as the approximate solution at each mesh point (x_i,t_j) for the equation (1.1).

Standard approximations are used to discretize the first and second order spatial derivatives as follows, $\frac{\partial\mathcal{Z}}{\partial x}(x_i,t_j)\approx D_x^0\mathcal{Z}_i^j=\frac{\mathcal{Z}_{i+1}^j-\mathcal{Z}_i^j}{h}$, and $\frac{\partial^2\mathcal{Z}}{\partial x^2}(x_i,t_j)\approx\delta_x^2\mathcal{Z}_i^j=\frac{\mathcal{Z}_{i-1}^j-2\mathcal{Z}_i^j+\mathcal{Z}_{i+1}^j}{h^2}$, respectively. The Caputo-fractional derivative $D_t^\beta\mathcal{Z}$ which is discretized by the following

$$D_t^\beta\mathcal{Z}(x_i,t_j)=\frac{1}{\Gamma(1-\beta)}\sum_{k=0}^{j-1}\int_{s=t_k}^{t_{k+1}}(t_j-s)^{-\beta}\frac{\partial\mathcal{Z}}{\partial s}(x_i,s)ds.$$

Furthers approximated as,

$$D_t^\beta\mathcal{Z}(x_i,t_j)\approx D_{\mathcal{N}}^\beta\mathcal{Z}_i^j=\frac{1}{\Gamma(1-\beta)}\sum_{k=0}^{j-1}\frac{\mathcal{Z}_i^{k+1}-\mathcal{Z}_i^k}{\tau}\int_{s=t_k}^{t_{k+1}}(t_j-s)^{-\beta}ds$$
$$=\frac{1}{\tau^\beta\Gamma(2-\beta)}\sum_{k=0}^{j-1}(\mathcal{Z}_i^{k+1}-\mathcal{Z}_i^k)d_{j-k},$$

where $d_k=k^{1-\beta}-(k-1)^{1-\beta}$, $k\geq 1$.

Equation (1.1) reduces as

$$D_{\mathcal{N}}^\beta\mathcal{Z}(x_i,t_j)-p(x_i,t_j)\delta_x^2\mathcal{Z}(x_i,t_j)-q(x_i,t_j)D_x^0\mathcal{Z}(x_i,t_j)+\lambda\int_0^{t_j}\mathcal{K}(x_i,t_j-\xi)\mathcal{Z}(x_i,\xi)d\xi$$
$$={}^{(1)}R_i^j+{}^{(2)}R_i^j+{}^{(3)}R_i^j+f(x_i,t_j), \tag{4.1}$$

for $1 \le i \le \mathcal{M}-1$, $1 \le j \le \mathcal{N}$ where $\mathcal{Z}(x_0,t_j) = \psi_1(t_j)$, $\mathcal{Z}(x_M,t_j) = \psi_2(t_j)$ for $0 < j \le \mathcal{N}$ and $\mathcal{Z}(x_i,t_0) = \phi(x_i)$ for $0 \le i \le \mathcal{M}$. Here

$$\left.\begin{aligned} {}^{(1)}R_i^j &= (D_{\mathcal{N}}^{\alpha} - D_t^{\alpha})\mathcal{Z}(x_i,t_j), \\ {}^{(2)}R_i^j &= p(x_i,t_j)\Big(\tfrac{\partial^2}{\partial x^2} - \delta_x^2\Big)\mathcal{Z}(x_i,t_j), \\ {}^{(3)}R_i^j &= q(x_i,t_j)\Big(\tfrac{\partial}{\partial x} - D_x^0\Big)\mathcal{Z}(x_i,t_j). \end{aligned}\right\} \tag{4.2}$$

are the remainder terms.

The composite approximation of the trapezoidal shape, which is provided by, is used to approximate the integral term as follows,

$$\begin{aligned} &\int_0^{t_j} \mathcal{K}(x_i,t_j-\xi)\mathcal{Z}(x_i,\xi)d\xi \\ &= \sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}} \mathcal{K}(x_i,t_j-\xi)\mathcal{Z}(x_i,\xi)d\xi \\ &= \frac{\lambda\tau}{2}\sum_{k=0}^{j-1}[\mathcal{K}(x_i,t_j-t_{k+1})\mathcal{Z}(x_i,t_{k+1}) + \mathcal{K}(x_i,t_j-t_k)\mathcal{Z}(x_i,t_k)] + {}^{(4)}R_i^j, \end{aligned} \tag{4.3}$$

where

$${}^{(4)}R_i^j = \lambda\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}(t_{k+1/2}-\xi)\frac{d}{d\xi}[\mathcal{K}(x_i,t_j-\xi)\mathcal{Z}(x_i,\xi)]d\xi. \tag{4.4}$$

Therefore, using equation (4.3) in equation (4.1), we get

$$\begin{cases} D_{\mathcal{N}}^{\beta}\mathcal{Z}(x_i,t_j) - p(x_i,t_j)\cdot\delta_x^2\mathcal{Z}(x_i,t_j) - q(x_i,t_j)D_x^0\mathcal{Z}(x_i,t_j) & \\ \quad + \frac{\lambda\tau}{2}\sum\limits_{k=0}^{j-1}[\mathcal{K}(x_i,t_j-t_{k+1})\mathcal{Z}(x_i,t_{k+1}) & \\ \quad + \mathcal{K}(x_i,t_j-t_k)\mathcal{Z}(x_i,t_k)] = R_i^j + f(x_i,t_j), & \text{for } 1 \le i \le \mathcal{M}-1,\ 1 \le j \le \mathcal{N}, \\ \mathcal{Z}(x_0,t_j) = \psi_1(t_j) \text{ and } \mathcal{Z}(\mathcal{M},t_j) = \psi_2(t_j), & 0 < j \le \mathcal{N}, \\ \mathcal{Z}(x_i,t_0) = \phi(x_i), & 0 \le i \le \mathcal{M}. \end{cases} \tag{4.5}$$

The remainder term R_i^j is defined as below,

$$\begin{aligned} R_i^j &= {}^{(1)}R_i^j + {}^{(2)}R_i^j + {}^{(3)}R_i^j - {}^{(4)}R_i^j \\ &= (D_{\mathcal{N}}^{\beta} - D_t^{\beta})\mathcal{Z}(x_i,t_j) + p(x_i,t_j)\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)\mathcal{Z}(x_i,t_j) + q(x_i,t_j)\left(\frac{\partial}{\partial x} - D_x^0\right)\mathcal{Z}(x_i,t_j) \\ &\quad - \lambda\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}(t_{k+1/2}-\xi)\frac{d}{d\xi}[\mathcal{K}(x_i,t_j-\xi)\mathcal{Z}(x_i,\xi)]d\xi. \end{aligned} \tag{4.6}$$

Now neglecting R_i^j then we can write the equation (4.5) in discrete form as below,

$$\begin{cases} D_{\mathcal{N}}^{\alpha}\mathcal{Z}_i^j - p(x_i,t_j)\cdot\delta_x^2\mathcal{Z}_i^j - q(x_i,t_j)D_x^0\mathcal{Z}_i^j & \\ \quad + \frac{\lambda\tau}{2}\sum\limits_{k=0}^{j-1}[\mathcal{K}(x_i,t_j-t_{k+1})\mathcal{Z}_i^{k+1} + \mathcal{K}(x_i,t_j-t_k)\mathcal{Z}_i^k] = f(x_i,t_j), & \\ \mathcal{Z}_0^j = \psi_1(t_j) \ \text{ and } \ \mathcal{Z}_M^j = \psi_2(t_j), & 0 < j \le \mathcal{N}, \\ \mathcal{Z}_i^0 = \phi(x_i), & 0 \le i \le \mathcal{M}. \end{cases} \tag{4.7}$$

Now using the values of $D_x^0\mathcal{Z}(x_i,t_j)$, $\delta_x^2\mathcal{Z}(x_i,t_j)$ and $D_t^\beta\mathcal{Z}(x_i,t_j)$ in equation (4.7) we get,

$$\begin{cases} \frac{1}{\tau^\beta\Gamma(2-\beta)}\sum\limits_{k=0}^{j-1}(\mathcal{Z}_i^{k+1}-\mathcal{Z}_i^k)d_{i-k}-\frac{p(x,t)}{h^2}(\mathcal{Z}_{i-1}^j-2\mathcal{Z}_i^j+\mathcal{Z}_{i+1}^j) \\ -\frac{q(x,t)}{h}(\mathcal{Z}_{i+1}^j-\mathcal{Z}_i^j)+\frac{\lambda\tau}{2}\sum\limits_{k=0}^{j-1}[\mathcal{K}(x_i,t_j-t_{k+1})\mathcal{Z}_i^{k+1}+\mathcal{K}(x_i,t_j-t_k)\mathcal{Z}_i^k]=f(x_i,t_j). \end{cases} \tag{4.8}$$

Equation (4.8) reduces as

$$\begin{cases} \left(-\frac{p(x,t)}{h^2}\right)\mathcal{Z}_{i-1}^j+\left(\frac{d_1}{\tau^\beta\Gamma(2-\beta)}+\frac{2a}{h^2}+\frac{q(x,t)}{h}+\frac{\lambda\tau}{2}\mathcal{K}(x_i,0)\right)\mathcal{Z}_i^j+\left(-\frac{p(x,t)}{h^2}-\frac{q(x,t)}{h}\right)\mathcal{Z}_{i+1}^j \\ =f(x_i,t_j)+\frac{d_1}{\tau^\beta\Gamma(2-\beta)}\mathcal{Z}_i^{j-1}-\frac{1}{\tau^\beta\Gamma(2-\beta)}\sum\limits_{k=0}^{j-2}(\mathcal{Z}_i^{k+1}-\mathcal{Z}_i^k)d_{j-k} \\ -\frac{\lambda\tau}{2}(\mathcal{K}(x_i,t_j-t_{j-1}))\mathcal{Z}_i^{j-1}-\frac{\lambda\tau}{2}\sum\limits_{k=0}^{j-2}(\mathcal{K}(x_i,t_j-t_{k+1})\mathcal{Z}_i^{k+1}+\mathcal{K}(x_i,t_j-t_k)\mathcal{Z}_i^k). \end{cases} \tag{4.9}$$

Equation (4.9) reduces as

$$\begin{cases} A_i\mathcal{Z}_{i-1}^j+B_i\mathcal{Z}_i^j+C_i\mathcal{Z}_{i+1}^j=D(i;0,1,2,3,\dots j-1), & \text{for } 1\le i\le\mathcal{M}-1,\ 1\le j\le\mathcal{N}, \\ \mathcal{Z}_0^j=\psi_1(t_j) \text{ and } \mathcal{Z}_\mathcal{M}^j=\psi_2(t_j), & \text{for all } 0<j\le\mathcal{N}, \\ \mathcal{Z}_i^0=\phi(x_i), & \text{for all } 0\le i\le\mathcal{M}. \end{cases} \tag{4.10}$$

For each $j=1,2,3,\dots\mathcal{N}$ the coefficients A_i, B_i, C_i and D $(i;0,1,2,3,\dots j-1)$ are defined by

$$\begin{cases} A_i=\left(-\frac{p(x,t)}{h^2}\right), \\ B_i=\left(\frac{d_1}{\tau^\beta\Gamma(2-\beta)}+\frac{2p(x,t)}{h^2}+\frac{q(x,t)}{h}+\frac{\lambda\tau}{2}\mathcal{K}(x_i,0)\right), \\ C_i=\left(-\frac{p(x,t)}{h^2}-\frac{q(x,t)}{h}\right), \\ D(i;0,1,2,3,\dots j-1)=f(x_i,t_j)+\frac{d_1}{\tau^\beta\Gamma(2-\beta)}\mathcal{Z}_i^{j-1}-\frac{1}{\tau^\beta\Gamma(2-\beta)}\sum\limits_{k=0}^{j-2}(\mathcal{Z}_i^{k+1}-\mathcal{Z}_i^k)d_{j-k} \\ \qquad -\frac{\lambda\tau}{2}(\mathcal{K}(x_i,t_j-t_{j-1}))\mathcal{Z}_i^{j-1}-\frac{\lambda\tau}{2}\sum\limits_{k=0}^{j-2}(\mathcal{K}(x_i,t_j-t_{k+1})\mathcal{Z}_i^{k+1} \\ \qquad +\mathcal{K}(x_i,t_j-t_k)\mathcal{Z}_i^k), \end{cases} \tag{4.11}$$

for $i=0,1,2,3,\dots\mathcal{M}-1$.

At any time label, there are $\mathcal{M}-1$ unknowns, $\mathcal{Z}_1^j,\mathcal{Z}_2^j,\mathcal{Z}_3^j,\mathcal{Z}_4^j,\dots\mathcal{Z}_{\mathcal{M}-1}^j$ and $\mathcal{M}-1$ number of equations are written as

$$\begin{cases} 1\cdot\mathcal{Z}_0^j=\mathcal{Z}_0^j, \\ A_1\mathcal{Z}_0^j+B_1\mathcal{Z}_1^j+C_1\mathcal{Z}_2^j=D(1;0,1,2,3\dots j-1), \\ 0\cdot\mathcal{Z}_0^j+A_2\mathcal{Z}_1^j+B_2\mathcal{Z}_2^j+C_2\mathcal{Z}_3^j=D(2;0,1,2,3\dots j-1), \\ 0\cdot\mathcal{Z}_0^j+0.\mathcal{Z}_1^j+A_3\mathcal{Z}_2^j+B_3\mathcal{Z}_3^j+C_3\mathcal{Z}_4^j=D(3;0,1,2,3\dots j-1), \\ \vdots \\ A_{\mathcal{M}-1}\mathcal{Z}_{\mathcal{M}-2}^j+B_{\mathcal{M}-1}\mathcal{Z}_{\mathcal{M}-1}^j+C_{\mathcal{M}-1}\mathcal{Z}_\mathcal{M}^j=D(\mathcal{M}-1;0,1,2,3\dots j-1), \\ 0\cdot\mathcal{Z}_0^j+0\cdot\mathcal{Z}_1^j+0\cdot\mathcal{Z}_2^j+0\cdot\mathcal{Z}_3^j+\dots+\mathcal{Z}_\mathcal{M}^j=\mathcal{Z}_\mathcal{M}^j. \end{cases} \tag{4.12}$$

We can write the aforementioned linear system in matrix form as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0\ldots & 0 & 0 & 0 & 0 \\ 0 & A_1 & B_1 & C_1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & A_2 & B_2 & C_2 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & A_3 & B_3 & C_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & \ldots & \ldots & \ldots & \ldots & 0 & A_{M-1} & B_{M-1} & C_{M-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{Z}_0^j \\ \mathcal{Z}_1^j \\ \mathcal{Z}_2^j \\ \mathcal{Z}_3^j \\ \vdots \\ ]\mathcal{Z}_{M-1}^j \\ \mathcal{Z}_M^j \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_0^j \\ D(1;0,1,2,3\ldots j-1) \\ D(2;0,1,2,3\ldots j-1) \\ D(3;0,1,2,3\ldots j-1) \\ \vdots \\ D(M-1;0,1,2,3\ldots j-1) \\ \mathcal{Z}_M^j \end{bmatrix}.$$

For $\sigma = 2,3,\ldots \mathcal{M}-1$, we define the coefficient of the above system as follows:

$$\begin{cases} A(\sigma,\sigma-1) = \left(-\frac{p}{h^2}\right), \\ A(\sigma,\sigma) = \left(\frac{d_1}{\tau^\beta\Gamma(2-\beta)} + \frac{2p}{h^2} + \frac{q}{h} + \frac{\lambda\tau}{2}\mathcal{K}(x_i,0)\right), \\ A(\sigma,\sigma+1) = \left(-\frac{p}{h^2} - \frac{q}{h}\right), \\ b(\sigma,1) = D(i;0,1,2,3,\ldots j-1) \\ \qquad = f(x_i,t_j) + \frac{d_1}{\tau^\beta\Gamma(2-\beta)}\mathcal{Z}_i^{j-1} - \frac{1}{\tau^\beta\Gamma(2-\beta)}\sum\limits_{k=0}^{j-2}(\mathcal{Z}_i^{k+1} - \mathcal{Z}_i^k)d_{j-k} \\ \qquad -\frac{\lambda\tau}{2}(\mathcal{K}(x_i,t_j-t_{j-1}))\mathcal{Z}_i^{j-1} - \frac{\lambda\tau}{2}\sum\limits_{k=0}^{j-2}(\mathcal{K}(x_i,t_j-t_{k+1})\mathcal{Z}_i^{k+1} \\ \qquad +\mathcal{K}(x_i,t_j-t_k)\mathcal{Z}_i^k). \end{cases} \tag{4.13}$$

5. Convergence Analysis

In this section, we establish the truncation error estimations for temporal derivative approximation D_N^β, second order spatial derivative approximation δ_x^2 and the trapezoidal approximation for the integral component of the equation. The stability property has then been used to derive the error bounds for the computed solution $\mathcal{Z}_i^j$ at each uniform mesh point (x_i,t_j).

Lemma 5.1. *Suppose that the solution to* (1.1) *satisfies the condition* $\left|\frac{\partial^{k_1}\mathcal{Z}}{\partial x^{k_1}}\right| \le C$ *for* $k_1 = 0,1,2,3,4$ *and* $\left|\frac{\partial^{k_2}\mathcal{Z}}{\partial t^{k_2}}\right| \le C(1+t^{\beta-k_2})$ *respectively for* $k_2 = 0,1,2$. *Hence, we get the following truncation error bound for each* $(x_i,t_j) \in \bar{\mathcal{D}}_1$

$$\|{}^{(1)}R_i^j\| \le C\cdot N^{-\min(2-\beta,\beta+1)}. \tag{5.1}$$

Proof. The proof of this lemma can be found in [13]. □

Lemma 5.2. *The following truncation error bound is satisfied by the discrete operator* δx^2

$$\|{}^{(2)}R_i^j\| \le Ch^2. \tag{5.2}$$

Proof. Applying Taylor's series expansion, we can easily show that

$$\left\|p(x_i,t_j)\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)\mathcal{Z}(x_i,t_j)\right\| \le Ch^2, \quad \text{for all } (x_i,t_j) \in \bar{\mathcal{D}}_1. \tag{5.3}$$

Therefore, observing $p(x_i,t_j)$ is continuous in given domain, we get the required bound. □

Lemma 5.3. *The discrete operator D_x^0 satisfies the following truncation error bound*

$$\|^{(3)}R_i^j\| \le Ch. \tag{5.4}$$

Proof. Since $q(x_i,t_j)$ be continuous and bounded in the given domain. Applying Taylor's series expansion, we can easily show that

$$\left\| q(x_i,t_j)\left(\frac{\partial}{\partial x} - D_x^0\right)\mathcal{Z}(x_i,t_j)\right\| \le Ch, \quad \text{for all } (x_i,t_j) \in \bar{\mathcal{D}}_1. \tag{5.5}$$

□

Lemma 5.4. *For each $i = 0,1,2,3,4,\dots M$ and $j = 0,1,2,3,\dots N$ the remainder term $^{(4)}R_i^j$ satisfies the following inequality*

$$\|^{(4)}R_i^j\| \le CN^{-1}. \tag{5.6}$$

Proof.

$$\begin{aligned}
\|^{(4)}R_i^j\| &= \left\| \lambda \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} (t_{k+1/2} - \xi)\frac{d}{d\xi}[K(x_i,t_j-\xi)\mathcal{Z}(x_i,\xi)]d\xi \right\| \\
&\le \lambda \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} (t_{k+1/2} - \xi)\left\| \frac{d}{d\xi}[\mathcal{K}(x_i,t_j-\xi)\mathcal{Z}(x_i,\xi)]\right\| d\xi \\
&\le \lambda \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} (t_{k+1/2} - \xi)\left\| \left[-\frac{\partial}{\partial t}\mathcal{K}(x_i,t_j-\xi)\mathcal{Z}(x_i,\xi) + \mathcal{K}(x_i,t_j-\xi)\cdot\frac{\partial \mathcal{Z}}{\partial \xi}(x_i,\xi)\right]\right\| d\xi \\
&\le C\tau \int_0^{t_j}\left(1 + \frac{\partial \mathcal{Z}}{\partial \xi}(x_i,\xi)\right)d\xi \le CN^{-1}.
\end{aligned} \tag{5.7}$$

Let $\|e_i^j\| = \|\mathcal{Z}(x_i,t_j) - \mathcal{Z}_i^j\|$ be the point-wise error at the point $(x_i,t_j) \in \bar{\mathcal{D}}_1$. Now subtracting of equation (4.7) from equation (4.5), we have obtained the following error equation

$$\begin{cases}
D_{\mathcal{N}}^{\beta} e_i^j - p(x_i,t_j)\cdot\delta_x^2 e_i^j - q(x_i,t_j)D_x^0 e_i^j \\
\quad + \frac{\lambda\tau}{2}\sum\limits_{k=0}^{j-1}[\mathcal{K}(x_i,t_j-t_{k+1})e_i^{k+1} + \mathcal{K}(x_i,t_j-t_k)e_i^k] = R_i^j, \\
e_0^j = 0,\ e_M^j = 0, & \text{for all } 0 < j \le \mathcal{N}, \\
e_i^0 = 0, & \text{for all } 0 \le i \le \mathcal{M}.
\end{cases} \tag{5.8}$$

The term R_i^j is the remainder term which is defined in equation (4.6). □

Lemma 5.5. *For each uniform mesh $(x_i,t_j) \in \bar{\mathcal{D}}_1$ the solution of* (4.7) *satisfies*

$$\|\mathcal{Z}(x_i,t_j) - \mathcal{Z}_i^j\| \le \tau^{\beta}\Gamma(2-\beta)\sum_{k=1}^{j}\theta_{j-k}\|R_i^k\|, \tag{5.9}$$

where, the stability multipliers θ_m's are defined by $\theta_0 = 0$, $\theta_m = \sum_{k=1}^{m}(d_k - d_{k+1})\theta m - k$ for $m = 1,2,3,\dots$ and R_i^j stands for

$$\begin{aligned}
R_i^j &= {}^{(1)}R_i^j + {}^{(2)}R_i^j + {}^{(3)}R_i^j - {}^{(4)}R_i^j \\
&= (D_{\mathcal{N}}^{\beta} - D_t^{\beta})\mathcal{Z}(x_i,t_j) + p(x_i,t_j)\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)\mathcal{Z}(x_i,t_j) + q(x_i,t_j)\left(\frac{\partial}{\partial x} - D_x^0\right)\mathcal{Z}(x_i,t_j) \\
&\quad - \lambda\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}(t_{k+1/2}-\xi)\frac{d}{d\xi}[\mathcal{K}(x_i,t_j-\xi)\mathcal{Z}(x_i,\xi)]d\xi.
\end{aligned} \tag{5.10}$$

Proof. Proof can be found in [27]. □

Theorem 5.6. *If* $\{\mathcal{Z}(x_i,t_j)\}_{i=0,j=0}^{\mathcal{M},\mathcal{N}}$ *and* $\{\mathcal{Z}_i^j\}_{i=0,j=0}^{\mathcal{M},\mathcal{N}}$ *be exact and approximate solution at each uniform mesh* (x_i,t_j) *of equation* (1.1) *obtained using the scheme* (4.7) *then for each uniform mesh point* $(x_i,t_j)\in\bar{\mathcal{D}}_1$*, the error bounds are estimated as follows:*

$$\|e_i^j\| \le C[\tau t_n^{\beta-1} + \tau + h^2]. \tag{5.11}$$

Proof. Details of the proof can be found in [27]. □

6. Numerical Examples and Graphical Discussion

In this section, we consider three test problems in order to validate the theoretical estimations established in the previous sections.

Example 6.1. Consider the following test problem

$$\begin{cases} \frac{\partial^\beta \mathcal{Z}}{\partial t^\beta} - p(x,t)\frac{\partial^2 \mathcal{Z}}{\partial x^2} - q(x,t)\frac{\partial \mathcal{Z}}{\partial x} + \lambda\int_0^t \mathcal{K}(x,t-s)\mathcal{Z}(x,s)ds = f(x,t), & \text{for } (x,t)\in[0,1]\times[0,1], \\ \mathcal{Z}(x,0)=0, & \text{for all } x\in[0,1], \\ \mathcal{Z}(0,t)=t+t^\beta,\ \mathcal{Z}(1,t)=0, & \text{for all } t\in(0,1], \end{cases} \tag{6.1}$$

where the kernel function is defined as $\mathcal{K}(x,t-s)=x(t-s)$, $p(x,t)=(1+x^2+t^2)$, $q(x,t)=(1+x^2)$.

Considering the following is the source function $f(x,t)$,

$$\begin{aligned} f(x,t) &= (1-x^2)\left(\frac{t^{1-\beta}}{\Gamma(2-\beta)} + \Gamma(1+\beta)\right) + 2p(x,t)(t+t^\beta) \\ &\quad + 2q(x,t)x(t+t^\beta) + \lambda x(1-x^2)\left(\frac{t^6}{6} + \frac{t^{2+\beta}}{(\beta+1)(\beta+2)}\right), \end{aligned} \tag{6.2}$$

then the problem considered in Example 6.1 has the exact solution that satisfies,

$$\mathcal{Z}(x,t)=(1-x^2)(t+t^\beta).$$

Let $\Delta\mathcal{E}_{\mathcal{M},\mathcal{N}} = \max_{(x_i,t_j)\in\bar{\mathcal{D}}_1}|\mathcal{Z}(x_i,t_j)-\mathcal{Z}_i^j|$ be the calculated inaccuracy at each mesh points (x_i,t_j) and $\Delta\mathcal{P}_{\mathcal{M},\mathcal{N}} = \log_2\left(\frac{\Delta\mathcal{E}_{\mathcal{M},\mathcal{N}}}{\Delta\mathcal{E}_{2\mathcal{M},2\mathcal{N}}}\right)$ be the convergence rate.

Table 1. Error, $\Delta\mathcal{E}_{\mathcal{M},\mathcal{N}}$ and rate of convergence, $\Delta\mathcal{P}_{\mathcal{M},\mathcal{N}}$ for Example 6.1

β	$M(=N)=64$	$M(=N)=28$	$M(=N)=256$	$M(=N)=512$	$M(=N)=1024$
0.2	0.0102	0.0096	0.0091	0.0088	0.0085
	0.0873	0.0617	0.0531	0.5225	
0.4	0.0118	0.0106	0.0094	0.0082	0.0071
	0.1550	0.1705	0.1940	0.2200	
0.6	0.0083	0.0065	0.0049	0.0036	0.0026
	0.3468	0.3988	0.4469	0.4862	
0.8	0.0034	0.0023	0.0014	0.0009	0.0005
	0.6050	0.6670	0.6780	0.7068	

Table 2. Comparison of solutions between the exact and FDM for Example 6.1

(x,t)	$\beta=0.4$		$\beta=0.8$	
	Exact	FDM	Exact	FDM
$(0.9,0.1)$	0.0944	0.0935	0.0490	0.0483
$(0.7,0.3)$	0.4677	0.4652	0.3474	0.3455
$(0.5,0.5)$	0.9433	0.9395	0.8057	0.8025
$(0.3,0.7)$	1.4255	1.4220	1.3207	1.3173
$(0.1,0.9)$	1.8398	1.8382	1.8006	1.7991

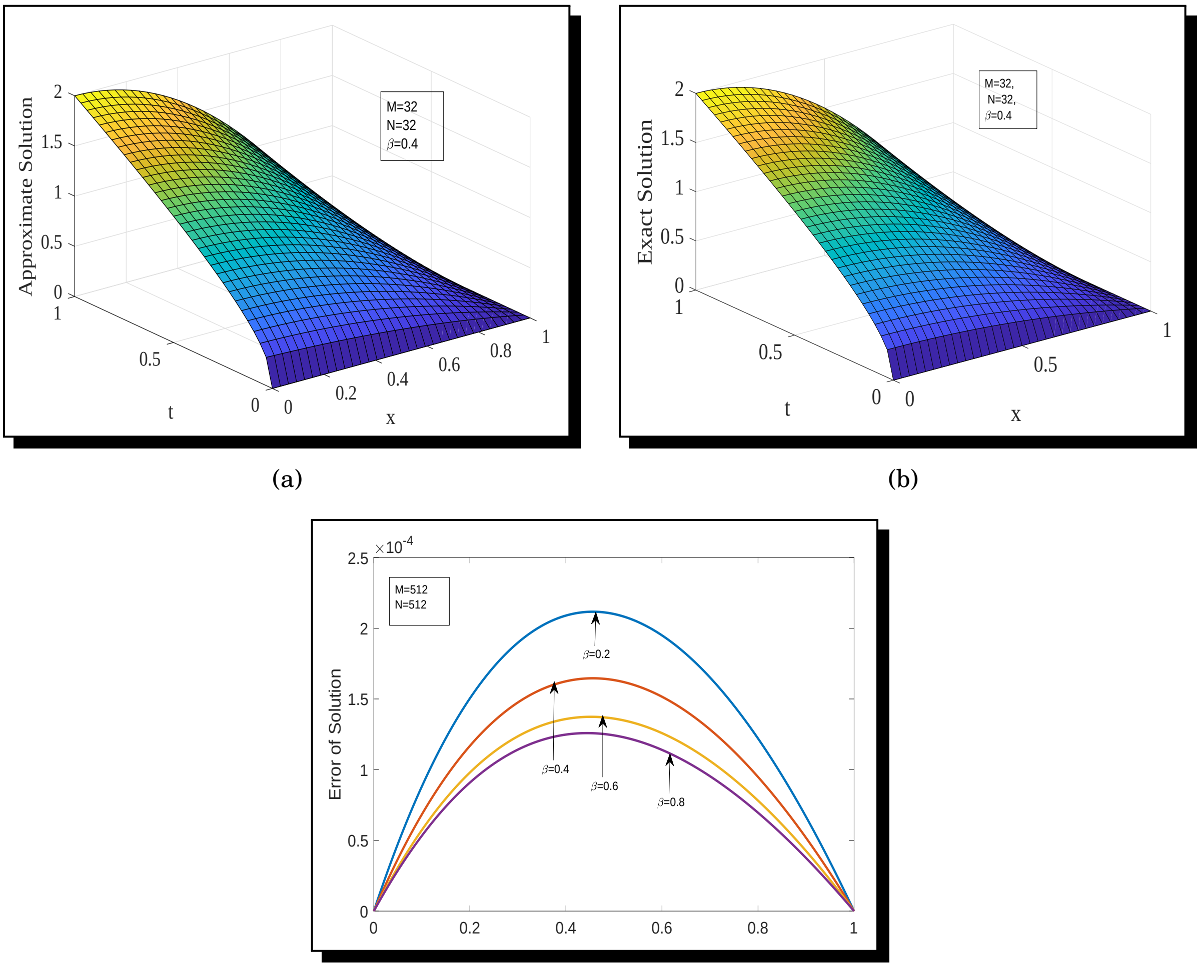


Figure 1. (a) Approximate solution for Example 6.1 with $M = N = 32$ and $\beta = 0.4$; (b) Exact solution for $M = N = 32$ and $\beta = 0.4$; (c) Error between the approximate and exact solution vs x for $M = N = 512$ and for different values of β for Example 6.1

The calculated error and order of convergence are presented in Table 1 and Table 2. Figure 1(a) represents surface of approximate solution in the $x-t$ plane with $M = N = 32$ and $\beta = 0.4$ for Example 6.1. Figure 1(b) represents the exact solution for various values of x and t with $M = N = 32$ and $\beta = 0.4$. Whereas Figure 1(c) plots error versus x for various values of $\beta = \{0.2, 0.4, 0.6, 0.8\}$ with $M = N = 512$. From the above plot we can see that initially error of solution is strictly increasing to attains maximum value and then strictly decreasing to attains minimum value. Finally, error at end points be zero. Also the associated error is decreasing as increasing values of β.

Example 6.2. Consider the following test problem

$$\begin{cases} \frac{\partial^\beta \mathcal{Z}}{\partial t^\beta} - p(x,t)\frac{\partial^2 \mathcal{Z}}{\partial x^2} - q(x,t)\frac{\partial \mathcal{Z}}{\partial x} + \lambda \int_0^t \mathcal{K}(x,t-s)\mathcal{Z}(x,s)ds = f(x,t), & \text{for all } (x,t) \in \bar{\mathcal{D}}, \\ \mathcal{Z}(x,0) = 0, & \text{for all } x \in [0,1], \\ \mathcal{Z}(0,t) = t^\beta,\ \mathcal{Z}(1,t) = et^\beta, & \text{for all } t \in (0,1], \end{cases} \tag{6.3}$$

where $\mathcal{K}(x,t-s) = e^{x(t-s)}$, $p(x,t) = (1+x^2+t^2)$, $q(x,t) = (1+x^2)$.

Here, we consider the source function f to be

$$f(x,t) = \Gamma(1+\beta)\cdot e^x - (p(x,t)+q(x,t))t^\beta e^x + \left[\frac{\lambda t^{1+\beta} \times e^{(1+t)x}}{(\beta+1)}\right]. \tag{6.4}$$

For this selecting of the source function $f(x,t)$, the exact solution of Example 6.2 is given by

$$\mathcal{Z}(x,t) = t^\beta e^x.$$

The error and order of convergence are calculated as given for Example 6.2 and are provided in Table 3 and Table 4.

Table 3. Error, $\Delta\mathcal{E}_{\mathcal{M},\mathcal{N}}$ and rate of convergence, $\Delta\mathcal{P}_{\mathcal{M},\mathcal{N}}$ for Example 6.2

β	$M(=N)=64$	$M(=N)=128$	$M(=N)=256$	$M(=N)=512$	$M(=N)=1024$
0.2	0.0336	0.0330	0.0328	0.0326	0.0326
	0.0232	0.0114	0.0055	0.0027	0
0.4	0.0302	0.0298	0.0296	0.0295	0.0294
	0.0208	0.0104	0.0051	0.0025	0
0.6	0.0275	0.0271	0.0269	0.0268	0.0267
	0.0212	0.0108	0.0054	0.0027	0
0.8	0.0251	0.0247	0.0245	0.0244	0.0243
	0.0226	0.0116	0.0059	0.0030	0

Table 4. Comparison of solutions between the exact value and FDM for Example 6.2

(x,t)	$\beta=0.4$		$\beta=0.8$	
	Exact	FDM	Exact	FDM
$(0.9,0.1)$	0.9776	0.9768	0.3894	0.3880
$(0.7,0.3)$	1.2439	1.2467	0.7685	0.7694
$(0.5,0.5)$	1.2495	1.2574	0.9469	0.9515
$(0.3,0.7)$	1.1705	1.1816	1.0148	1.0225
$(0.1,0.9)$	1.0596	1.0662	1.0159	1.0210

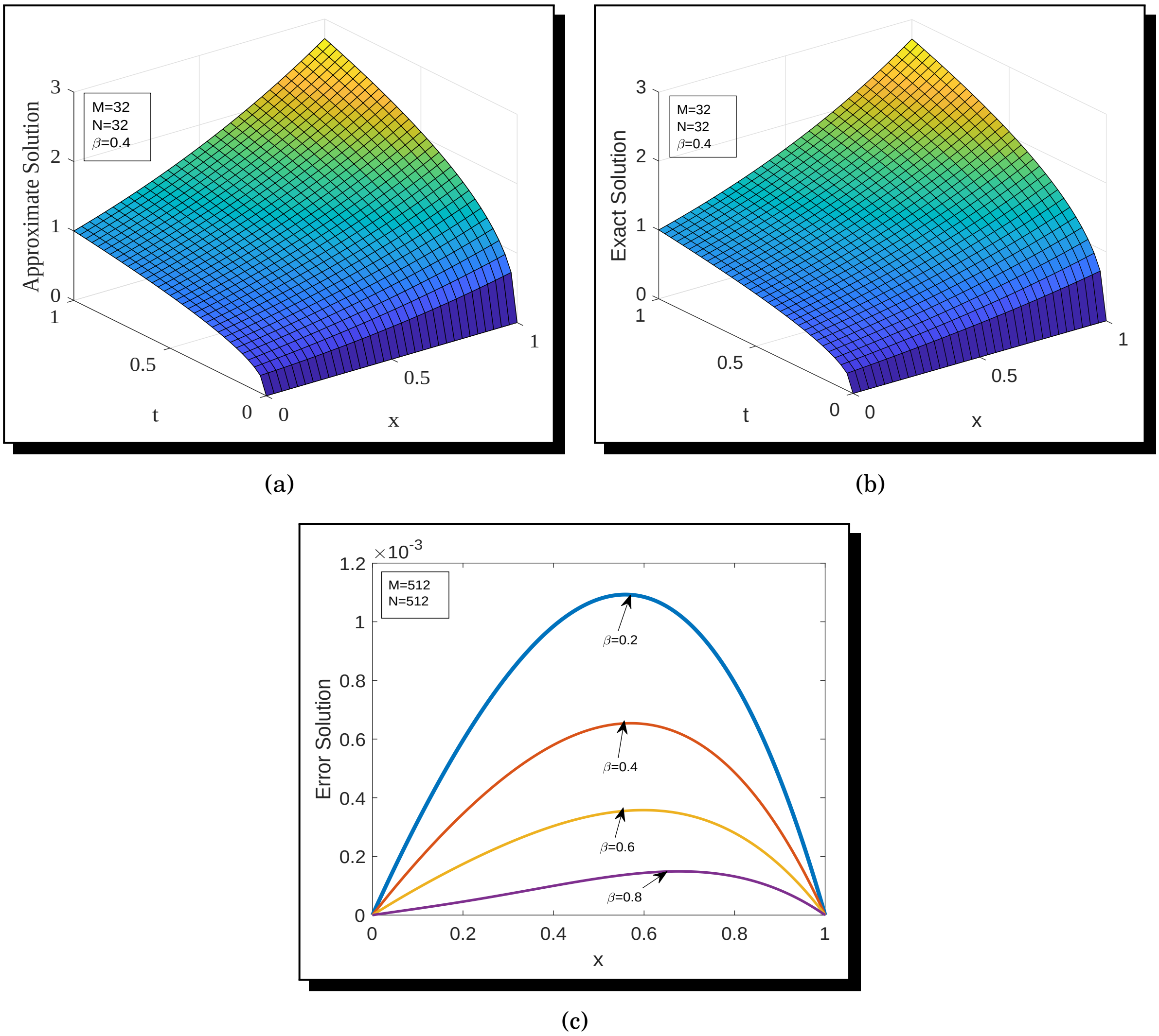


Figure 2. (a) Approximate solution vs t and x plot for $M = N = 32$ and $\beta = 0.4$; (b) Exact solution vs t and x plot for $M = N = 32$ and $\beta = 0.4$; (c) Error of solution vs x plot for $M = N = 512$ and for different values β

Figure 2(a) represents curved surface of approximate solution versus x, t of Example 6.2 with $M = N = 32$ and $\beta = 0.4$. Figure 2(b); for $M = N = 32$ and $\beta = 0.4$ represents curved surface of the exact solution versus x, t of Example 6.2. Figure 2(c); for $M = N = 512$ and $\beta = \{0.2, 0.4, 0.6, 0.8\}$ represents error versus x of Example 6.2.

Example 6.3. Consider the following test problem

$$\begin{cases} \frac{\partial^\beta \mathcal{Z}}{\partial t^\beta} - p(x,t)\frac{\partial^2 \mathcal{Z}}{\partial x^2} - q(x,t)\frac{\partial \mathcal{Z}}{\partial x} + \lambda \int_0^t \mathcal{K}(x,t-s)\mathcal{Z}(x,s)ds = f(x,t), & \text{for all } (x,t) \in [0,1]\times[0,1], \\ \mathcal{Z}(x,0) = 0, & \text{for all } x \in [0,1], \\ \mathcal{Z}(0,t) = 0,\ \mathcal{Z}(1,t) = t^\beta \cos(1), & \text{for all } t \in (0,1], \end{cases} \tag{6.5}$$

where $\mathcal{K}(x,t-s) = xt$, $p(x,t) = (1+x^2+t^2)$ and $q(x,t) = (1+x^2)$.

Here we consider,

$$f(x,t)=\Gamma(\beta+1)\sin\left(x+\frac{\pi x}{2}\right)+p(x,t)\left(1+\frac{\pi}{2}\right)^2 t^\beta \sin\left(x+\frac{\pi x}{2}\right)-q(x,t)t^\beta\left(1+\frac{\pi}{2}\right)\cos\left(x+\frac{\pi\cdot x}{2}\right)$$
$$+\lambda\left(xt\sin\left(x+\frac{\pi x}{2}\right)\frac{t^{\beta+1}}{(\beta+1)}\right). \tag{6.6}$$

For this selecting value of $f(x,t)$, the exact solution of Example 6.3 is provided by

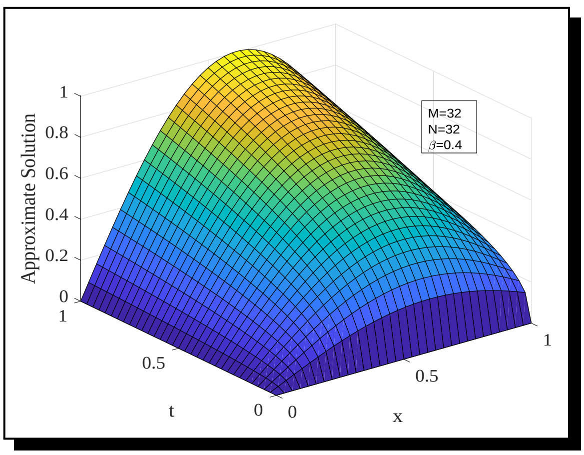
$$\mathcal{Z}(x,t)=t^\beta\sin\left(x+\frac{\pi x}{2}\right). \tag{6.7}$$

Table 5. Error, $\Delta\mathcal{E}_{\mathcal{M},\mathcal{N}}$ and rate of convergence, $\Delta\mathcal{P}_{\mathcal{M},\mathcal{N}}$ for Example 6.3

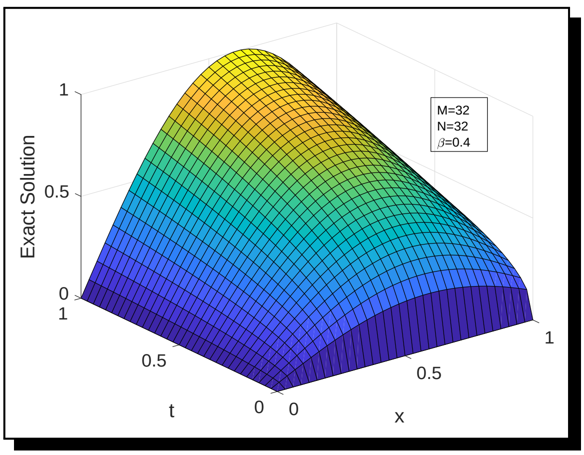
β	$M(=N)=64$	$M(=N)=128$	$M(=N)=256$	$M(=N)=512$	$M(=N)=1024$
0.2	0.0122	0.0109	0.0102	0.0097	0.0093
	0.1619	0.0999	0.0705	0.0603	0
0.4	0.0132	0.0116	0.0103	0.0089	0.0076
	0.1828	0.1828	0.1999	0.2246	0
0.6	0.0091	0.0071	0.0053	0.0039	0.0028
	0.3624	0.4074	0.4524	0.4923	0
0.8	0.0037	0.0024	0.0015	0.0010	0.0006
	0.6184	0.6785	0.6805	0.7203	0

Table 6. Comparison of solutions between the exact value and FDM for Example 6.3

(x,t)	$\beta=0.4$		$\beta=0.8$	
	Exact	FDM	Exact	FDM
$(0.9,0.1)$	0.2926	0.2908	0.1165	0.1155
$(0.7,0.3)$	0.6011	0.5964	0.3714	0.3685
$(0.5,0.5)$	0.7272	0.7210	0.5511	0.5464
$(0.3,0.7)$	0.6039	0.5989	0.5236	0.5192
$(0.1,0.9)$	0.2436	0.2418	0.2336	0.2318

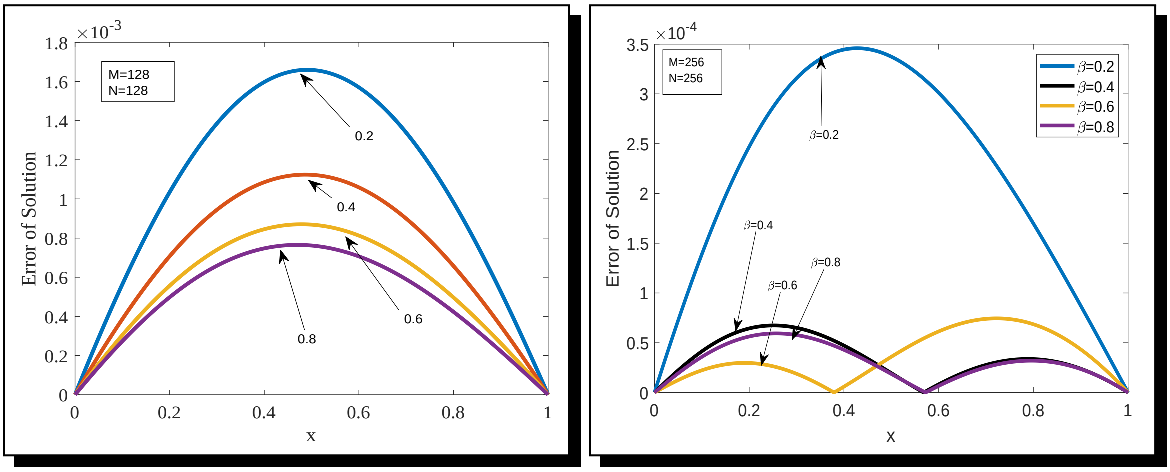


(a)



(b)

Figure *Contd*.



(c) (d)

Figure 3. (a) Approximate solution vs t and x plot for $M = N = 32$ and $\beta = 0.4$; (b) Exact solution vs t and x plot for $M = N = 32$ and $\beta = 0.4$; (c) Error of solution vs x plot for $M = N = 128$ and for different values β

Figure 3(a), for $M = N = 32$ and $\beta = 0.4$ represents curved surface of approximate solution versus x, t of Example 6.3. Figure 3(b), for $M = N = 32$ and $\beta = 0.4$ represents curved surface of the exact solution versus x, t, Example 6.3. Figure 3(c) for $M = N = 128$ and $\beta = \{0.2, 0.4, 0.6, 0.8\}$ represents error versus x of Example 6.3. From plot (c) initially error of solution is strictly increasing to attains maximum value and then strictly decreasing to attains minimum value. Finally, error at the end points be zero. Error is decreasing as increasing values of β. Figure 3(d), for $M = N = 256$ and $\beta = \{0.2, 0.4, 0.6, 0.8\}$ represents error versus x of Example 6.3. From plot (d) initially error of solution is strictly increasing to attains maximum value and then strictly decreasing to attains minimum value. Finally, error at the end points be zero.

7. Conclusions

In this work, we developed numerical methods for time-fractional integro-differential equations. In order to compute the approximate solutions for highly non-linear or linear forms of various time-fractional integro-differential models, we apply the extended and more generalized finite difference methods. This article expands three distinct examples of the linear time-fractional form of integro-differential models. Numerical experiments are carried out to confirm the theoretical estimations.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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