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Research Article

Travelling Wave Solutions to Fourth-Order Nonlinear Equation

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Abstract. In this paper, we study the soliton solutions of the fourth-order nonlinear partial differential equations (NPDE). The Riccati-Bernoulli (RB) sub-ODE method is applied to the fourth-order NPDE to investigate the exact and traveling wave solutions. We secure singular periodic wave solutions, kink-type soliton solution, dark soliton and singular soliton solution, which have unlimited application in mathematical physic, science and engineering. Some figures for the obtained solutions are demonstrated.

Keywords. Fourth-order nonlinear equation, Optical solitons, Traveling wave solutions, Riccati-Bernoulli sub-ODE method

Mathematics Subject Classification (2020). 35C08, 35A20, 35A09, 35L05, 35Qxx

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1. Introduction

In mathematics and physics, the study of NPDEs have lead to development of theories, application and concept that describe many different physical systems (Agrawal [2], and Whitham [27]). Mathematicians have been struggling to find the existence and uniqueness solutions of many complex problems that are used in various filed of study, such as applied science, engineering, chemistry, physics, astronomy and biological (Hasegawa *et al.* [8]). Various techniques for finding the solution to NPDEs have been studied by a number of researches, e.g., Abdou and Elhanbaly [1], Biswas [4], Chen and Yan [6], Ibrahim *et al.* [13], Inc *et al.* [14], Mirzazadeh *et al.* [18], Sulaiman *et al.* [21], Tchier *et al.* [24, 25], and the references cited therein. NPDEs techniques and its solutions are constantly used in continuum mechanics, fluid dynamics, chaos theory for dynamical systems, nonlinear optics, quantum theory and other related areas (Akinyemi *et al.* [1], Cangiani *et al.* [5], Fang *et al.* [7], Ibrahim [11], Khalaf *et al.* [16], Kudryashov [17], Sabawi [20], Sulaiman *et al.* [22], and Tao *et al.* [23]). The theories and concept of NPDEs can be extended to study the commutative of linear time varying systems (LTVSs) (Ibrahim *et al.* [9, 10, 12]).

The aim of this work is used the RB-sub equation technique to investigate the optical traveling wave solutions of the fourth-order NPDE. Some figures are used to demonstrate and support our results.

The fourth-order $(1+1)$-dimensional equations is given by,

$$\vartheta_{tt} = k\vartheta_{xx} + v\vartheta\vartheta_{xxxx} + \alpha\vartheta_x\vartheta_{xxx} + \beta\vartheta_{xx}^2 + \mu\vartheta_{xxtt}. \tag{1.1}$$

where k, v, α, β and μ are nonzero real parameters. Several methods have been adopted to acquire soliton solutions for various NLPDEs using the RB-sub equation method (Baleanu *et al.* [3], Karaman [15], Ozdemir *et al.* [19], Tchier *et al.* [26], and Yang *et al.* [28]).

We similarly apply the method to analyze traveling wave solutions for the fourth-order (NPDE). Notably, this method has not previously been employed for the proposed novel fourth-order NPDE. The paper is organized as follows: Section 2 outlines the methodology, Section 3 details its application and includes graphical results, and Section 4 concludes the study.

2. Overview of the RB Sub-ODE Technique

This section introduces the RB sub-equation methodology. Consider a nonlinear partial differential equation (NLPDE) in the form

$$P(\vartheta, \vartheta_t, \vartheta_x, \vartheta_{tt}, \vartheta_{xx}, \vartheta_{tx}, \dots) = 0, \tag{2.1}$$

where P is a polynomial. The RB sub-equation method is categories into three steps.

Step 1: We consider the following traveling wave transformation

$$\vartheta(\xi) = \vartheta(x,t), \quad \xi = K(x \pm vt), \tag{2.2}$$

that lead to the following ODE

$$P(\vartheta, \vartheta', \vartheta'', \dots) = 0, \tag{2.3}$$

where $\vartheta'(\xi) = \frac{d\vartheta}{d\xi}$.

Step 2: Let the solution to the RB eq. (2.2) be expressed as

$$\vartheta' = b\vartheta + a\vartheta^{2-m} + c\vartheta^m, \tag{2.4}$$

here a, b, c and m represent freely adjustable constants.

By differentiating eq. (2.3), the following relationships are derived

$$\vartheta'' = \vartheta^{-1-2m}(a\vartheta^2 + c\vartheta^{2m} + b\vartheta^{1+m})(-a(-2+m)U^2 + cm\vartheta^{2m} + b\vartheta^{1+m}), \tag{2.5}$$

$$\begin{aligned}\vartheta''' = \vartheta^{-2(1+m)}&(bu + a\vartheta^{2-m} + c\vartheta^m)(a^2(-2+m)(-3+2m)\vartheta^4 \\ &+ c^2m(-1+2m)\vartheta^{4m} + ab(-3+m)(-2+m)\vartheta^{3+m} \\ &+ (b^2+2ac)\vartheta^{2+2m} + bcm(1+m)\vartheta^{1+3m}),\end{aligned} \tag{2.6}$$

and so on.

[1] L. Akinyemi, U. Akpan, P. Veeresha, H. Rezazadeh and M. Inc, Computational techniques to study the dynamics of generalized unstable nonlinear Schrödinger equation, *Journal of Ocean Engineering and Science*, In Press (2022), DOI: 10.1016/j.joes.2022.02.011.

Observe that the solutions of eq. (2.3) leads to:

Case 1. As $m = 1$, the results of eq. (2.3) become

$$\vartheta(\xi) = Je^{(b+a+c)\xi}. \tag{2.7}$$

Case 2. As $m \neq 1$, $b = 0$ and $c = 0$, the results of eq. (2.3) become

$$\vartheta(\xi) = (a(m-1)(\xi + J))^{\frac{1}{m-1}}. \tag{2.8}$$

Case 3. As $m \neq 1$, $b \neq 0$ and $c = 0$, the results of eq. (2.3) become

$$\vartheta(\xi) = \left(Je^{(b(m-1)\xi)} - \frac{a}{b}\right)^{\frac{1}{m-1}}. \tag{2.9}$$

Case 4. As $m \neq 1$, $a \neq 0$ and $b^2 - 4ac < 0$, the results of eq. (2.3) becomes

$$\vartheta(\xi) = \left(-\frac{b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}\tan\left[\frac{(1-m)\sqrt{4ac-b^2}}{2}(\xi+J)\right]\right)^{\frac{1}{1-m}} \tag{2.10}$$

and

$$\vartheta(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{4ac-b^2}}{2a}\cot\left[\frac{(1-m)\sqrt{4ac-b^2}}{2}(\xi+J)\right]\right)^{\frac{1}{1-m}}. \tag{2.11}$$

Case 5. As $m \neq 1$, $a \neq 0$ and $b^2 - 4ac > 0$, the results of eq. (2.3) becomes

$$\vartheta(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\tanh\left[\frac{(1-m)\sqrt{b^2-4ac}}{2}(\xi+J)\right]\right)^{\frac{1}{1-m}} \tag{2.12}$$

and

$$\vartheta(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\coth\left[\frac{(1-m)\sqrt{b^2-4ac}}{2}(\xi+J)\right]\right)^{\frac{1}{1-m}}. \tag{2.13}$$

Case 6. As $m \neq 1$, $a \neq 0$ and $b^2 - 4ac = 0$, the results of eq. (2.3) become

$$\vartheta(\xi) = \left(\frac{1}{a(m-1)(\xi+J)} - \frac{b}{2a}\right)^{\frac{1}{1-m}} \tag{2.14}$$

where J is a constant.

Step 3. Substituting the derivatives of ϑ into eq. (2.2) transforms the equation into a function of ϑ. Grouping analogous terms and solving for the undetermined constants yields the solution to eq. (2.1), as detailed in reference [3].

2.1 Bäcklund Transformation

If $\vartheta_n(\xi)$ and $\vartheta_{n-1}(\xi)$ are solutions to eq. (2.1), the following holds:

$$\frac{d\vartheta_n(\xi)}{d\xi} = \frac{d\vartheta_n(\xi)}{d\vartheta_{n-1}(\xi)\xi}\frac{d\vartheta_{n-1}(\xi)}{d\xi} = \frac{d\vartheta_n(\xi)}{d\vartheta_{n-1}\xi}(a\vartheta_{n-1}^{2-m} + b\vartheta_{n-1} + c\vartheta_{n-1}^m), \tag{2.15}$$

namely,

$$\frac{d\vartheta_n(\xi)}{a\vartheta_n^{2-m} + b\vartheta_n + c\vartheta_n^m} = \frac{d\vartheta_{n-1}(\xi)}{a\vartheta_{n-1}^{2-m} + b\vartheta_{n-1} + c\vartheta_{n-1}^m}. \tag{2.16}$$

Integrating eq. (2.16) with respect to ξ leads

$$\vartheta_n(\xi)=\left(\frac{-cA_1+aA_2(\vartheta_{n-1}(\xi))^{1-m}}{bA_1+aA_2+aA_1(\vartheta_{n-1}(\xi))^{1-m}}\right)^{\frac{1}{1-m}}, \tag{2.17}$$

where A_1 and A_2 are arbitrary constants.

3. Applications

To address the fourth-order NLPDE in eq. (1.1), the traveling wave transformation is applied:

$$\vartheta(x,t)=\vartheta(\xi), \quad \xi=K(x+vt) \tag{3.1}$$

to eq. (1.1), and we obtained the following equation

$$K((k-v^2)\vartheta''+K^2(\beta+\gamma)\vartheta''^2+K^2(vU^{(4)}(\vartheta+\mu v)+\alpha\vartheta^{(3)}\vartheta'))=0. \tag{3.2}$$

Plugging eqs. (2.3), (2.5), (2.6) and its derivative into (3.2), setting $m=0$ and collecting all the coefficients of $U^i(\xi)$ (for $i=0,1,2,3,4,5,6$), and also equating each collection to zero, we have the following

$$\left.\begin{aligned}
\vartheta^0(\xi):\quad & bckK-bcKv^2+b^2c^2K^3\alpha+2ac^3K^3\alpha+b^2c^2K^3\beta+b^2c^2K^3\gamma+b^3cK^3v^2\mu\\
&+8abc^2K^3v^2\mu,\\
\vartheta^1(\xi):\quad & b^2kK+2ackK+b^3cK^3v+8abc^2K^3v-b^2Kv^2-2acKv^2+2b^3cK^3\alpha\\
&+10abc^2K^3\alpha+2b^3cK^3\beta+4abc^2K^3\beta+2b^3cK^3\gamma+4abc^2K^3\gamma+b^4K^3v^2\mu\\
&+22ab^2cK^3v^2\mu+16a^2c^2K^3v^2\mu,\\
\vartheta^2(\xi):\quad & 3abkK+b^4K^3v+22ab^2cK^3v+16a^2c^2K^3v-3abKv^2+b^4K^3\alpha+16ab^2cK^3\alpha\\
&+10a^2c^2K^3\alpha+b^4K^3\beta+10ab^2cK^3\beta+4a^2c^2K^3\beta+b^4K^3\gamma+10ab^2cK^3\gamma\\
&+4a^2c^2K^3\gamma+15ab^3K^3v^2\mu+60a^2bcK^3v^2\mu=0,\\
\vartheta^3(\xi):\quad & 2a^2kK+15ab^3K^3v+60a^2bcK^3v-2a^2Kv^2+8ab^3K^3\alpha+28a^2bcK^3\alpha\\
&+6ab^3K^3\beta+16a^2bcK^3\beta+6ab^3K^3\gamma+16a^2bcK^3\gamma+50a^2b^2K^3v^2\mu\\
&+40a^3cK^3v^2\mu=0,\\
\vartheta^4(\xi):\quad & 50a^2b^2K^3v+40a^3cK^3v+19a^2b^2K^3\alpha+14a^3cK^3\alpha+13a^2b^2K^3\beta+8a^3cK^3\beta\\
&+13a^2b^2K^3\gamma+8a^3cK^3\gamma+60a^3bK^3v^2\mu=0,\\
\vartheta^5(\xi):\quad & 60a^3bK^3v+18a^3bK^3\alpha+12a^3bK^3\beta+12a^3bK^3\gamma+24a^4K^3v^2\mu=0,\\
\vartheta^6(\xi):\quad & 24a^4K^3v+6a^4K^3\alpha+4a^4K^3\beta+4a^4K^3\gamma=0.
\end{aligned}\right\} \tag{3.3}$$

Solving the system of algebraic equations of eq. (3.3) lead to

$$\left.\begin{aligned}
a&=\frac{144c}{\mu^2(3\alpha+2\beta+2\gamma)^2},\\
b&=\frac{1}{6}(-3a\alpha\mu-2a\beta\mu-2a\gamma\mu),\\
v&=\frac{1}{12}(-3\alpha-2\beta-2\gamma),\\
K&=\frac{1}{144}(3\alpha+2\beta+2\gamma)^2.
\end{aligned}\right\} \tag{3.4}$$

With the solutions from eq. (3.4) with eqs. (2.7)-(2.14) and (3.1), we obtain the solutions of eq. (1.1) as:

The periodic singular can be reached as

$$\vartheta_1^{\pm}(x,t) = -\frac{(3\alpha+2\beta+2\gamma)^2\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)\mu^2}{1728c}+\frac{1}{288c}(3\alpha+2\beta+2\gamma)^2 \cdot\sqrt{\left(-\frac{1}{36}\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)^2+\frac{576c^2}{(3\alpha+2\beta+2\gamma)^2\mu^2}\right)\mu^2} \cdot\tan\left[\frac{1}{2}\left(J+\frac{1}{144}\left(x+\frac{1}{12}t(-3\alpha-2\beta-2\gamma)\right)(3\alpha+2\beta+2\gamma)^2\right) \cdot\sqrt{\left(-\frac{1}{36}\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)^2+\frac{576c^2}{(3\alpha+2\beta+2\gamma)^2\mu^2}\right)}\right], \tag{3.5}$$

$$\vartheta_2^{\pm}(x,t) = -\frac{(3\alpha+2\beta+2\gamma)^2\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)\mu^2}{1728c}-\frac{1}{288c}(3\alpha+2\beta+2\gamma)^2 \cdot\sqrt{\left(-\frac{1}{36}\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)^2+\frac{576c^2}{(3\alpha+2\beta+2\gamma)^2\mu^2}\right)\mu^2} \cdot\cot\left[\frac{1}{2}\left(J+\frac{1}{144}\left(x+\frac{1}{12}t(-3\alpha-2\beta-2\gamma)\right)(\beta+\gamma)(3\alpha+2\beta+2\gamma)\right) \cdot\sqrt{\left(-\frac{1}{36}\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)^2+\frac{576c^2}{(3\alpha+2\beta+2\gamma)^2\mu^2}\right)}\right]. \tag{3.6}$$

The dark optical soliton:

$$\vartheta_3^{\pm}(x,t) = -\frac{(3\alpha+2\beta+2\gamma)^2\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)\mu^2}{1728c}-\frac{1}{288c}(3\alpha+2\beta+2\gamma)^2 \cdot\sqrt{\left(\frac{1}{36}\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)^2-\frac{576c^2}{(3\alpha+2\beta+2\gamma)^2\mu^2}\right)\mu^2} \cdot\coth\left[\frac{1}{2}\left(J+\frac{1}{144}\left(x+\frac{1}{12}t(-3\alpha-2\beta-2\gamma)\right)(3\alpha+2\beta+2\gamma)^2\right) \cdot\sqrt{\left(\frac{1}{36}\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)^2-\frac{576c^2}{(3\alpha+2\beta+2\gamma)^2\mu^2}\right)}\right], \tag{3.7}$$

and the singular soliton:

$$\vartheta_4^{\pm}(x,t) = -\frac{(3\alpha+2\beta+2\gamma)^2\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)\mu^2}{1728c}-\frac{1}{288c}(3\alpha+2\beta+2\gamma)^2 \cdot\sqrt{\left(\frac{1}{36}\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)^2-\frac{576c^2}{(3\alpha+2\beta+2\gamma)^2\mu^2}\right)\mu^2}$$

$$\cdot\tanh\left[\frac{1}{2}\left(J+\frac{1}{144}\left(x+\frac{1}{12}t(-3\alpha-2\beta-2\gamma)\right)(3\alpha+2\beta+2\gamma)^2\right)\right.$$
$$\left.\cdot\sqrt{\left(\frac{1}{36}\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)^2-\frac{576c^2}{(3\alpha+2\beta+2\gamma)^2\mu^2}\right)}\,\right], \tag{3.8}$$

$$\vartheta_5^{\pm}(x,t)=\frac{1}{\left(\begin{array}{l} e^{-\frac{1}{864}\left(x+\frac{1}{12}t(-3\alpha-2\beta-2\gamma)\right)(3\alpha+2\beta+2\gamma)^2\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)} \\ \times J-\frac{864c}{(3\alpha+2\beta+2\gamma)^2\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)\mu^2}\end{array}\right)}, \tag{3.9}$$

$$\vartheta_6^{\pm}(x,t)=-\frac{(3\alpha+2\beta+2\gamma)^2\mu^2}{144c\left(J+\frac{1}{144}\left(x+\frac{1}{12}t(-3\alpha-2\beta-2\gamma)\right)(3\alpha+2\beta+2\gamma)^2\right)}$$
$$-\frac{(3\alpha+2\beta+2\gamma)^2\left(-\frac{432c\alpha}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\beta}{(3\alpha+2\beta+2\gamma)^2\mu}-\frac{288c\gamma}{(3\alpha+2\beta+2\gamma)^2\mu}\right)\mu^2}{1728c}. \tag{3.10}$$

Figure 1 present the periodic singular wave solution, that is $\vartheta_1(x,t)$ of eq. (3.5). We analogously considering the following parameters: $c=6$, $\alpha=8$, $\gamma=10$, $\beta=-0.5$, $\mu=7$, $J=-0.4$.

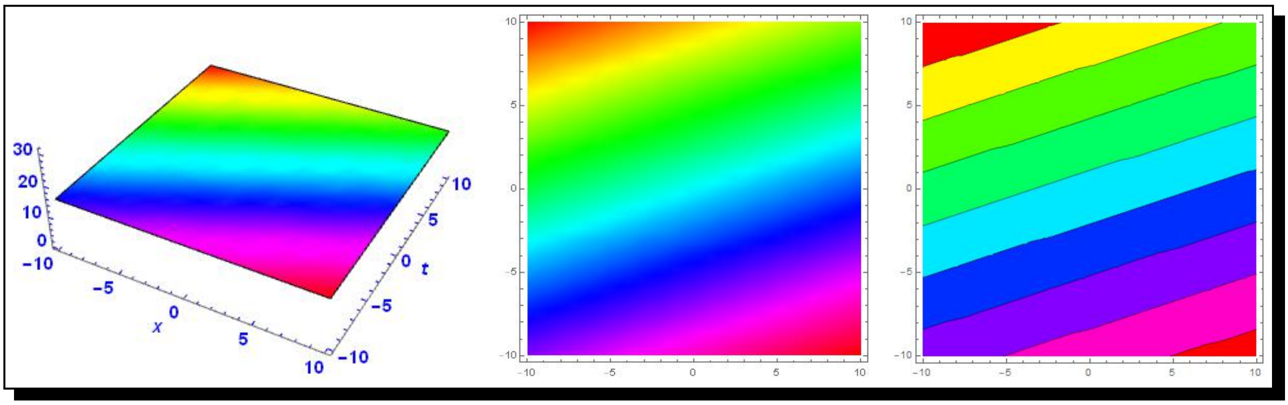

Figure 1. Plot of 3D, density and contour of (3.5)

Figure 2 present the periodic singular wave solution, that is $\vartheta_2(x,t)$ of eq. (3.4). We analogously considering the following parameters: $c=-6$, $\alpha=-5$, $\gamma=-10$, $\beta=0.5$, $\mu=0.25$, $J=1.5$.

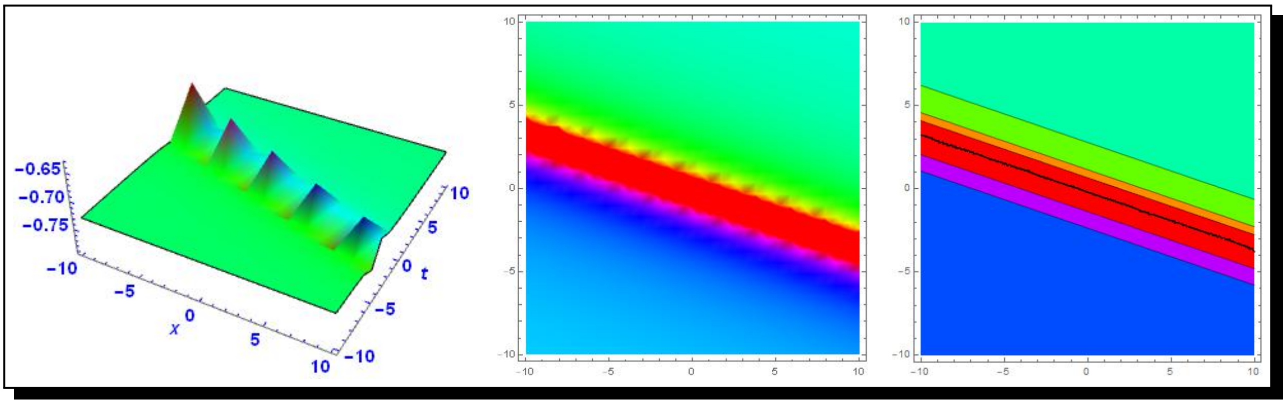

Figure 2. Plot of 3D, density and contour of (3.6)

Figure 3 present the dark soliton solution, that is $\vartheta_3(x,t)$ of eq. (3.7). We analogously considering the following parameters: $c=-6$, $\alpha=-10$, $\gamma=-10$, $\beta=2$, $\mu=0.25$, $J=1.5$.

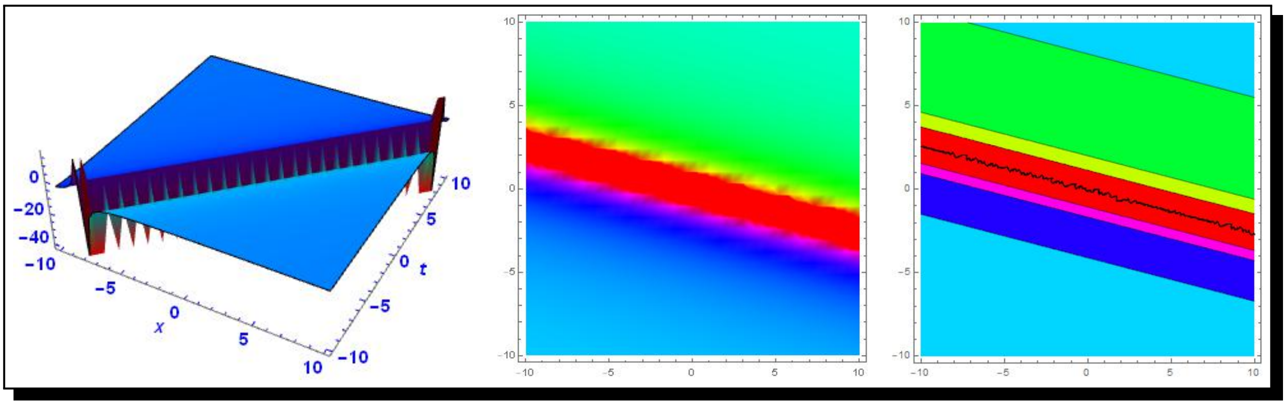

Figure 3. Plot of 3D, density and contour of (3.7)

Figure 4 present the singular soliton solution, that is $\vartheta_4(x,t)$ of eq. (3.8). We analogously considering the following parameters: $c=6$, $\alpha=-10$, $\gamma=-10$, $\beta=2$, $\mu=0.25$, $J=1.5$.

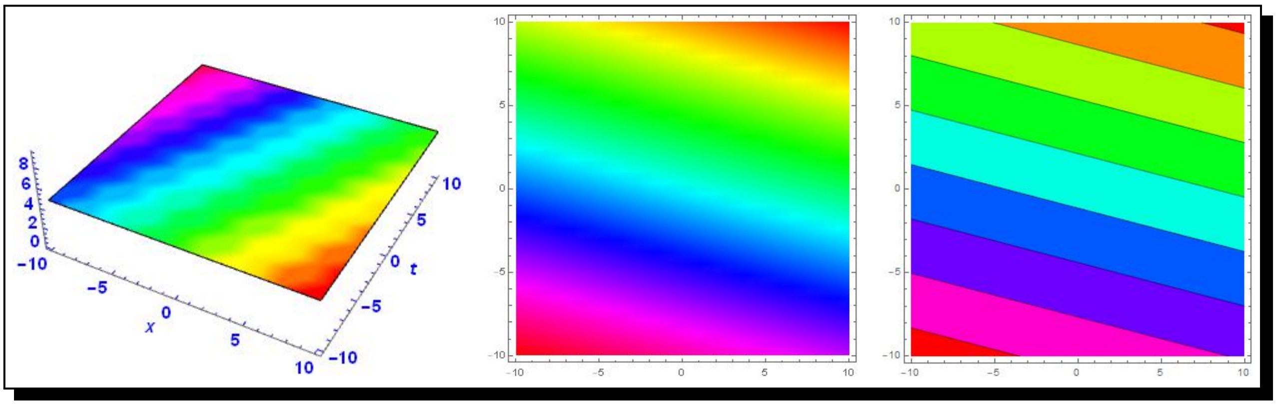

Figure 4. Plot of 3D, density and contour of (3.8)

Figure 5 present the periodic solution, that is $\vartheta_5(x,t)$ of eq. (3.9). We analogously considering the following parameters: $c=-6$, $\alpha=-5$, $\gamma=-10$, $\beta=-4$, $\mu=0.9$, $J=1$.

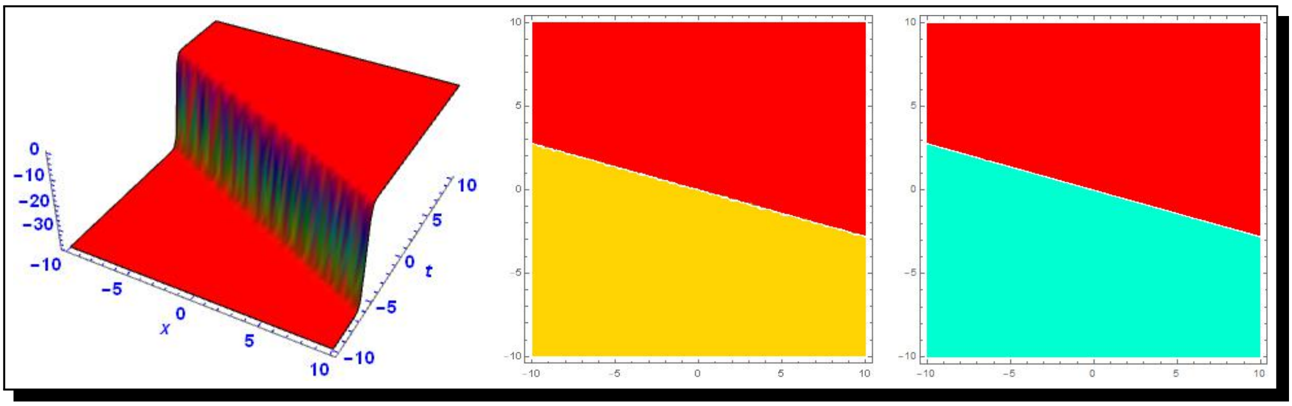

Figure 5. Plot of 3D, density and contour of (3.9)

Figure 6 present the periodic solution, that is $\vartheta_6(x,t)$ of eq. (3.10). We analogously considering the following parameters: $c=-6$, $\alpha=-5$, $\gamma=-10$, $\beta=-4$, $\mu=0.9$, $J=1$.

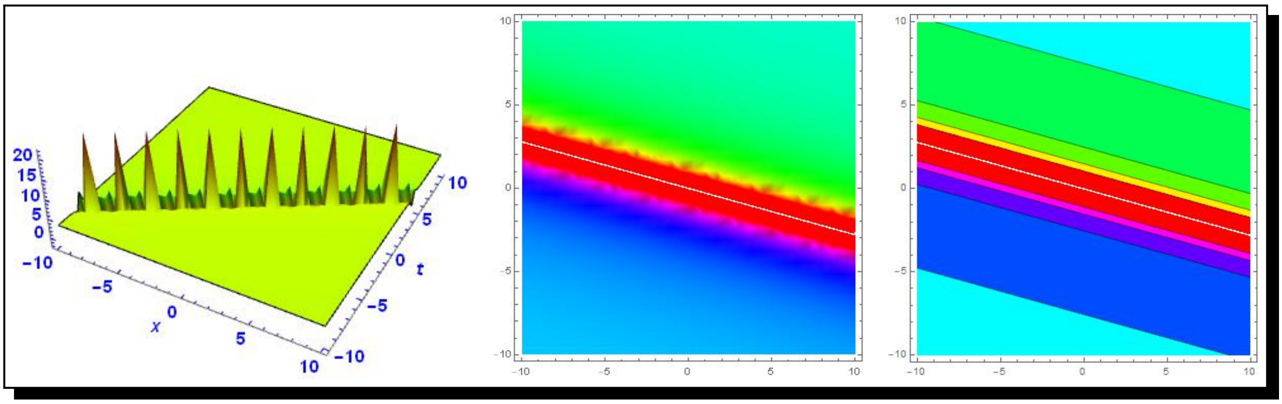

Figure 6. Plot of 3D, density and contour of (3.10)

4. Concluding Remarks

This paper investigate the solitary wave solution of the fourth-order NLPD using Riccati-Bernoulli sub-ODE equation method. We successfully secured some dark, singular solitons, and periodic wave solutions to the fourth-order NLPD. Moreover, the solutions obtained by the method is very important techniques in mathematics and have application in mathematical physics. We used Mathematica software to obtain the analytical solutions as well as plotting the figures.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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