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Research Article

Some Fixed Point Results on (α,β)-H-φ-Contraction Mappings in Partial Metric Spaces With Application

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Abstract. In this paper, we introduce the notion of H-φ-contraction, generalized H-φ-contraction, (α,β)-H-φ-contraction mappings and establish some fixed point results for such mappings in the context of partial metric spaces. An example is presented to illustrate the validity of the results. Further, the existence of the solution of nonlinear integral equation is discussed as an application of the result.

Keywords. H-φ-contraction mapping, Generalized H-φ-contraction mapping, (α,β)-H-φ-contraction mapping, Partial metric spaces

Mathematics Subject Classification (2020). Primary 47H10, Secondary 54H25

1. Introduction

Matthews [9] established the notion of *Partial Metric Space* (PMS) in which the self distance does not have to be equal to zero and proved the partial metric version of the Banach fixed point theorem during his research on the denotational semantics of data flow networks. In the framework of metric spaces, Boyd and Wong [3] introduced a class of mappings known as the φ-contraction mapping and proved several fixed point results for this class of mappings (for further findings, see Aydi and Karapinar [2], Kumar and Nziku [8], and Nziku and Kumar [12]). Further, Wardowski [17] introduced a new contraction called F-contraction and proved a fixed point theorem as a generalization of the Banach contraction principle. Recently, Piri and Kumam [13] extended the result of Wardowski. Following that, Singh and Sihag [16] defined a new contraction called η-contraction and established some fixed point results for this class of mappings. Samet *et al.* [15], on the other hand, proposed a new class of mappings known as α-ψ contractive type mappings and achieved some fixed point results for this class of mappings. Following that, Chandok [4] introduced the concept of (α,β)-admissible mappings and presented some outcomes. Mebawondu *et al.* [10, 11] recently introduced (α,β)-cyclic admissible mappings and established various fixed point results for this class of mappings in the framework of metric spaces.

Inspired by the works of Boyd and Wong [3], Singh and Sihag [16] and Mebawondu *et al.* [11], we introduce H-φ-contraction, generalized H-φ-contraction, (α,β)-H-φ-contraction mappings and obtain fixed point results for such mappings in partial metric spaces.

Matthews [9] presented the following generalization of metric space:

Definition 1.1 ([9]). Let $\mathcal{W}_\varrho$ be a set which is non-empty. A function $\varrho : \mathcal{W}_\varrho \times \mathcal{W}_\varrho \to [0,\infty)$ is said to be a partial metric on $\mathcal{W}_\varrho$ if the following conditions hold:
(PMS1) $\xi = \eta \Leftrightarrow \varrho(\xi,\xi) = \varrho(\eta,\eta) = \varrho(\xi,\eta)$;
(PMS2) $\varrho(\xi,\xi) \le \varrho(\xi,\eta)$;
(PMS3) $\varrho(\xi,\eta) = \varrho(\eta,\xi)$;
(PMS4) $\varrho(\xi,\eta) \le \varrho(\xi,\zeta) + \varrho(\zeta,\eta) - \varrho(\zeta,\zeta)$,
for all $\xi,\eta,\zeta \in \mathcal{W}_\varrho$.

Example 1.2 ([7]). Let $\mathcal{W}_\varrho = \{[a,b] : a,b \in \mathbb{R}, a \le b\}$ and define $\varrho([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then $(\mathcal{W}_\varrho,\varrho)$ is a partial metric space.

Example 1.3 ([7]). Let $\mathcal{W}_\varrho = [0,\infty)$ and define $\varrho(\xi,\eta) = \max\{\xi,\eta\}$. Then $(\mathcal{W}_\varrho,\varrho)$ is a partial metric space.

Lemma 1.4 ([9]). *Let $(\mathcal{W}_\varrho,\varrho)$ be a partial metric space.*

(a) *A sequence $\{\xi_i\}$ in $(\mathcal{W}_\varrho,\varrho)$ converges to a point $\xi \in \mathcal{W}_\varrho \Leftrightarrow \varrho(\xi,\xi) = \lim_{i\to\infty} \varrho(\xi_i,\xi)$.*

(b) *A sequence $\{\xi_i\}$ in $(\mathcal{W}_\varrho,\varrho)$ is a Cauchy sequence if $\lim_{j,i\to\infty} \varrho(\xi_i,\xi_j)$ exists and finite,*

(c) *$(\mathcal{W}_\varrho,\varrho)$ is complete if every Cauchy $\{\xi_i\}$ in $\mathcal{W}_\varrho$ converges to a point $\xi \in \mathcal{W}_\varrho$, such that*
$$\varrho(\xi,\xi) = \lim_{j,i\to\infty} \varrho(\xi_j,\xi_i) = \lim_{i\to\infty} \varrho(\xi_i,\xi) = \varrho(\xi,\xi).$$

Lemma 1.5 ([6, 9]). *Let ϱ be a partial metric on $\mathcal{W}_\varrho$, then the function $d^\varrho : \mathcal{W}_\varrho \times \mathcal{W}_\varrho \to \mathbb{R}^+$ such that*
$$d^\varrho(\xi,\eta) = 2\varrho(\xi,\eta) - \varrho(\xi,\xi) - \varrho(\eta,\eta)$$
is metric on $\mathcal{W}_\varrho$. Let $(\mathcal{W}_\varrho,\varrho)$ be a partial metric space. Then

(i) *a sequence $\{\xi_i\}$ in $(\mathcal{W}_\varrho,\varrho)$ is a Cauchy sequence $\Leftrightarrow$ it is a Cauchy sequence in the metric space $(\mathcal{W}_\varrho,d^\varrho)$,*

(ii) *$(\mathcal{W}_\varrho,\varrho)$ is complete $\Leftrightarrow$ $(\mathcal{W}_\varrho,d^\varrho)$ is complete. Moreover,*
$$\lim_{i\to\infty} d^\varrho(\xi_i,\xi) = 0 \Leftrightarrow \varrho(\xi,\xi) = \lim_{i\to\infty} \varrho(\xi_i,\xi) = \lim_{i,j\to\infty} \varrho(\xi_i,\xi_j).$$

Lemma 1.6 ([14]). *Assume that $\xi_n \to \zeta$ as $n \to \infty$ in a partial metric space $(\mathcal{W}_\varrho,\varrho)$ such that $\varrho(\zeta,\zeta)=0$. Then $\lim_{n\to\infty}\varrho(\xi_n,\eta)=\varrho(\zeta,\eta)$, for every $\eta\in\mathcal{W}_\varrho$.*

Lemma 1.7 ([14]). *If $\{\xi_i\}$ with $\lim_{i\to\infty}\varrho(\xi_i,\xi_{i+1})=0$ is not a Cauchy sequence in $(\mathcal{W}_\varrho,\varrho)$, and two sequences $\{i(p)\}$ and $\{j(p)\}$ of positive integers such that $i(p)>j(p)>p$, then following four sequences:*

$$\varrho(\xi_{j(p)},\xi_{i(p)+1}),\ \varrho(\xi_{j(p)},\xi_{i(p)}),\ \varrho(\xi_{j(p)-1},\xi_{i(p)+1}),\ \varrho(\xi_{j(p)-1},\xi_{i(p)})$$

tend to $\mu>0$ when $p\to\infty$.

Lemma 1.8 ([5]). *Let $(\mathcal{W}_\varrho,\varrho)$ be a partial metric space.*

(i) *If $\varrho(\xi,\zeta)=0$ then $\xi=\zeta$,*

(ii) *If $\xi\neq\zeta$ then $\varrho(\xi,\zeta)>0$.*

In 2012, Samet *et al.* [15] introduced α-admissible mapping as follows:

Definition 1.9 ([15]). Let $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ and $\alpha:\mathcal{W}_\varrho\times\mathcal{W}_\varrho\to[0,\infty)$. Γ is said to α-admissible if

$$\alpha(\xi,\eta)\geq 1\Rightarrow\alpha(\Gamma\xi,\Gamma\eta)\geq 1,$$

for all $\xi,\eta\in\mathcal{W}_\varrho$.

Further, in 2014, Alizadeh *et al.* [1] introduced cyclic (α,β)-admissible mappings as:

Definition 1.10 ([1]). Let Γ be a self mapping on $\mathcal{W}_\varrho$ and let $\alpha,\beta:\mathcal{W}_\varrho\to[0,\infty)$. We say Γ is cyclic (α,β)-admissible mapping if for any $\xi\in\mathcal{W}_\varrho$:

(i) $\alpha(\xi)\geq 1\Rightarrow\beta(\Gamma\xi)\geq 1$;

(ii) $\beta(\xi)\geq 1\Rightarrow\alpha(\Gamma\xi)\geq 1$.

Recently, Mebawondu and Mewomo [10] introduced a new class of mappings called (α,β)-cyclic admissible mappings as follows:

Definition 1.11 ([10]). Let Γ be a self mapping on $\mathcal{W}_\varrho$ and let $\alpha,\beta:\mathcal{W}_\varrho\times\mathcal{W}_\varrho\to[0,\infty)$. We say Γ is an (α,β)-cyclic admissible mapping if

(i) $\alpha(\xi,\eta)\geq 1\Rightarrow\beta(\Gamma\xi,\Gamma\eta)\geq 1$;

(ii) $\beta(\xi,\eta)\geq 1\Rightarrow\alpha(\Gamma\xi,\Gamma\eta)\geq 1$.

Lemma 1.12 ([10]). *Let $\mathcal{W}_\varrho$ be a non empty set and $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ be an (α,β)-cyclic admissible mapping. Suppose that there exists $\xi_0\in\mathcal{W}_\varrho$ such that $\alpha(\xi_0,\Gamma\xi_0)\geq 1$ and $\beta(\xi_0,\Gamma\xi_0)\geq 1$. Define the sequence $\{\xi_i\}$ by $\xi_{i+1}=\Gamma\xi_i$, then $\alpha(\xi_j,\xi_{j+1})\geq 1$ implies $\beta(\xi_i,\xi_{i+1})\geq 1$ and $\beta(\xi_j,\xi_{j+1})\geq 1$ implies $\alpha(\xi_i,\xi_{i+1})\geq 1$, for all $i,j\in\mathbb{N}\cup\{0\}$ with $j<i$.*

Boyd and Wong [3] introduced a class of mappings called the φ-contraction mapping and obtained following result:

Definition 1.13 ([3])**.** Let $(\mathcal{W}_\varrho,d)$ be a metric space and $\varphi:[0,\infty)\to[0,\infty)$ be a function such that $\varphi(t)<t$, for $t>0$. A self map $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ is called φ-contraction if

$$d(\Gamma\xi,\Gamma\eta)\le\varphi(d(\xi,\eta)),\quad \text{for all } \xi,\eta\in\mathcal{W}_\varrho.$$

Theorem 1.14 ([3])**.** *Let $(\mathcal{W}_\varrho,d)$ be a complete metric space and $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ a φ-contraction such that φ is upper semicontinuous from the right on $[0,\infty)$ and satisfies $\varphi(t)<t$, for all $t>0$. Then Γ has a unique fixed point.*

Definition 1.15 ([16])**.** Let $\mathcal{H}$ be the family of all functions $H:\mathbb{R}^+\to\mathbb{R}$ satisfying following conditions:

(i) H is strictly increasing, i.e. for all $\xi,\eta\in\mathbb{R}^+$ such that $\xi<\eta$, $H(\xi)<H(\eta)$;

(ii) for each sequence $\{\xi_i\}$ of positive numbers $\lim_{i\to\infty}(\xi_i)=0 \Leftrightarrow \lim_{i\to\infty}H(\xi_i)=0$;

(iii) H is continuous in $(0,\infty)$.

Singh and Sihag [16] defined a new contraction as follows:

Definition 1.16 ([16])**.** Let $(\mathcal{W}_\varrho,d)$ be a metric space and let $H\in\mathcal{H}$. A self map $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ is said to be an H-contraction if there exists $a\in(0,1)$ such that

$$d(\Gamma\xi,\Gamma\eta)>0\Rightarrow H(d(\Gamma\xi,\Gamma\eta))\le aH(d(\xi,\eta))$$

for all $\xi,\eta\in\mathcal{W}_\varrho$

Theorem 1.17 ([16])**.** *Let $(\mathcal{W}_\varrho,d)$ be a complete metric space and let self-map $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ be an H-contraction. Then Γ has a unique fixed point in $\mathcal{W}_\varrho$.*

2. Main Results

Definition 2.1. Let $(\mathcal{W}_\varrho,\varrho)$ be partial metric space and $\varphi:[0,\infty)\to[0,\infty)$ be a continuous function such that $\varphi(t)<t$, for $t>0$ and $\varphi(0)=0$. A self map $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ is called φ-contraction if

$$\varrho(\Gamma\xi,\Gamma\eta)\le\varphi(\varrho(\xi,\eta)),\quad \text{for all } \xi,\eta\in\mathcal{W}_\varrho.$$

Definition 2.2. Let $(\mathcal{W}_\varrho,\varrho)$ be partial metric space and $\varphi:[0,\infty)\to[0,\infty)$ be a continuous function such that $\varphi(t)<t$, for $t>0$ and $\varphi(0)=0$. A self map $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ is called generalized φ-contraction if

$$\varrho(\Gamma\xi,\Gamma\eta)\le\varphi(M(\xi,\eta)),\quad \text{for all } \xi,\eta\in\mathcal{W}_\varrho$$

and

$$M(\xi,\eta)=\max\left\{\varrho(\xi,\eta),\frac{\varrho(\xi,\Gamma\xi)+\varrho(\eta,\Gamma\eta)}{2},\frac{\varrho(\xi,\Gamma\eta)+\varrho(\eta,\Gamma\xi)}{2}\right\}.$$

Definition 2.3. Let $(\mathcal{W}_\varrho,\varrho)$ be partial metric space, $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ be φ-contraction and $H\in\mathcal{H}$. Then Γ is said to be H-φ-contraction if for $\Gamma\xi\neq\Gamma\eta$

$$H(\varrho(\Gamma\xi,\Gamma\eta))\le H(\varphi(\varrho(\xi,\eta))),\quad \text{for all } \xi,\eta\in\mathcal{W}_\varrho.$$

Definition 2.4. Let $(\mathcal{W}_\varrho,\varrho)$ be partial metric space, $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ be generalized φ-contraction and $H\in\mathcal{H}$. Then Γ is said to be generalized H-φ-contraction if for $\Gamma\xi\neq\Gamma\eta$:

$$H(\varrho(\Gamma\xi,\Gamma\eta))\le H(\varphi(M(\xi,\eta))),\quad \text{for all } \xi,\eta\in\mathcal{W}_\varrho$$

and

$$M(\xi,\eta)=\max\left\{\varrho(\xi,\eta),\frac{\varrho(\xi,\Gamma\xi)+\varrho(\eta,\Gamma\eta)}{2},\frac{\varrho(\xi,\Gamma\eta)+\varrho(\eta,\Gamma\xi)}{2}\right\}.$$

Definition 2.5. Let $(\mathcal{W}_\varrho,\varrho)$ be partial metric space, $\alpha,\beta:\mathcal{W}_\varrho\times\mathcal{W}_\varrho\to[0,\infty)$ be two functions and Γ be a generalized H-φ-contraction. The mapping Γ is said to be (α,β)-H-φ-contraction mapping if for all $\xi,\eta\in\mathcal{W}_\varrho$ with $\Gamma\xi\neq\Gamma\eta$, we have

$$\alpha(\xi,\Gamma\xi)\beta(\eta,\Gamma\eta)\ge 1\Rightarrow H(\varrho(\Gamma\xi,\Gamma\eta))\le H(\varphi(M(\xi,\eta))). \tag{2.1}$$

Theorem 2.6. *Let $(\mathcal{W}_\varrho,\varrho)$ be a complete partial metric space and $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ be self mapping. Suppose $\alpha,\beta:\mathcal{W}_\varrho\times\mathcal{W}_\varrho\to[0,\infty)$ be the mappings satisfying the conditions:*

(i) *Γ is (α,β)-cyclic admissible;*

(ii) *Γ is (α,β)-H-φ-contraction mapping;*

(iii) *there exists $\xi_0\in\mathcal{W}_\varrho$ such that $\alpha(\xi_0,\Gamma\xi_0)\ge 1$ and $\beta(\xi_0,\Gamma\xi_0)\ge 1$;*

(iv) *Γ is continuous.*

Then Γ has a fixed point in $\mathcal{W}_\varrho$.

Proof. Let ξ_0 be an arbitrary point such that $\alpha(\xi_0,\Gamma\xi_0)\ge 1$ and $\beta(\xi_0,\Gamma\xi_0)\ge 1$. Suppose, we have a sequence $\{\xi_n\}$ in $\mathcal{W}_\varrho$ such that $\xi_{n+1}=\Gamma\xi_n$, for all $n\in\mathbb{N}$.
If $\xi_n=\xi_{n+1}$ for some $n\in\mathbb{N}$, then ξ_n is a fixed point of Γ and the proof of existence is complete. Suppose $\xi_n\neq\xi_{n+1}$, for every $n\in\mathbb{N}$. Then $\varrho(\xi_n,\xi_{n+1})>0$ by Lemma 1.8. Now, since Γ is (α,β)-cyclic admissible, thus

$$\alpha(\xi_0,\Gamma\xi_0)=\alpha(\xi_0,\xi_1)\ge 1 \ \Rightarrow\ \beta(\Gamma\xi_0,\Gamma\xi_1)=\beta(\xi_1,\xi_2)\ge 1$$

and

$$\beta(\xi_1,\xi_2)\ge 1 \ \Rightarrow\ \alpha(\Gamma\xi_1,\Gamma\xi_2)=\alpha(\xi_2,\xi_3)\ge 1$$

and using induction, we have

$$\alpha(\xi_i,\xi_{i+1})\ge 1 \ \text{ and }\ \beta(\xi_i,\xi_{i+1})\ge 1,\quad \text{for all } i\in\mathbb{N}.$$

Now, since $\alpha(\xi_{i-1},\xi_i)\beta(\xi_i,\xi_{i+1})=\alpha(\xi_{i-1},\Gamma\xi_{i-1})\beta(\xi_i,\Gamma\xi_i)\ge 1$ from (2.1), we have

$$H(\varrho(\xi_i,\xi_{i+1}))=H(\varrho(\Gamma\xi_{i-1},\Gamma\xi_i))\le H(\varphi(M(\xi_{i-1},\xi_i))), \tag{2.2}$$

where

$$\begin{aligned}M(\xi_{i-1},\xi_i)&=\max\left\{\varrho(\xi_{i-1},\xi_i),\frac{\varrho(\xi_{i-1},\Gamma\xi_{i-1})+\varrho(\xi_i,\Gamma\xi_i)}{2},\frac{\varrho(\xi_{i-1},\Gamma\xi_i)+\varrho(\xi_i,\Gamma\xi_{i-1})}{2}\right\}\\&=\max\left\{\varrho(\xi_{i-1},\xi_i),\frac{\varrho(\xi_{i-1},\xi_i)+\varrho(\xi_i,\xi_{i+1})}{2},\frac{\varrho(\xi_{i-1},\xi_{i+1})+\varrho(\xi_i,\xi_i)}{2}\right\}.\end{aligned} \tag{2.3}$$

On the other side by (PMS4)

$$\varrho(\xi_{i-1},\xi_{i+1})\le\varrho(\xi_{i-1},\xi_i)+\varrho(\xi_i,\xi_{i+1})-\varrho(\xi_i,\xi_i). \tag{2.4}$$

Replacing (2.4) in (2.3), we get

$$M(\xi_{i-1},\xi_i) \le \max\{\varrho(\xi_{i-1},\xi_i),\varrho(\xi_i,\xi_{i+1})\}. \tag{2.5}$$

Now, using (2.5) in (2.2), we get

$$H(\varrho(\xi_i,\xi_{i+1})) \le H(\varphi(\max\xi\{\varrho(\xi_i,\xi_{i+1}),\varrho(\xi_{i-1},\xi_i)\})). \tag{2.6}$$

Now, if $\varrho(\xi_i,\xi_{i+1}) > \varrho(\xi_{i-1},\xi_i)$, then using the definitions of H and φ, we get

$$H(\varrho(\xi_i,\xi_{i+1})) \le H(\varphi(\varrho(\xi_i,\xi_{i+1}))) < H(\varrho(\xi_i,\xi_{i+1})).$$

Then

$$\varrho(\xi_i,\xi_{i+1}) < \varrho(\xi_i,\xi_{i+1})$$

which is a contradiction since $\varrho(\xi_i,\xi_{i+1}) > 0$. As a result, we get that $\{\varrho(\xi_i,\xi_{i+1}) : i \in \mathbb{N}\}$ is a decreasing sequence of non negative real numbers. Hence, it is convergent to a real number, therefore there exists $\delta \ge 0$ such that

$$\lim_{i\to\infty} \varrho(\xi_i,\xi_{i+1}) = \delta.$$

Let $\delta > 0$. Then from (2.6), we get

$$H(\varrho(\xi_i,\xi_{i+1})) \le H(\varphi(\varrho(\xi_{i-1},\xi_i))).$$

Now taking limit $i \to \infty$ and using the continuity of H and φ with fact that H is strictly increasing, $\varphi(t) < t$ for $t > 0$, we get

$$H(\delta) \le H(\varphi(\delta)) < H(\delta).$$

This is contradiction. Hence

$$\lim_{i\to\infty} \varrho(\xi_i,\xi_{i+1}) = 0. \tag{2.7}$$

Now, we show that $\{\xi_i\}$ is a Cauchy sequence in $\mathcal{W}_\varrho$ i.e. we prove that $\lim_{i,j\to\infty} \varrho(\xi_i,\xi_j) = 0$. We prove it by contradiction.

Let

$$\lim_{i,j\to\infty} \varrho(\xi_i,\xi_j) \ne 0.$$

Then sequences in Lemma 1.7 tends to $\mu > 0$, when $k \to \infty$.
Thus, we can see that

$$\lim_{k\to\infty} \varrho(\xi_{j(p)},\xi_{i(p)}) = \mu. \tag{2.8}$$

Additionally, corresponding to $j(p)$, we can choose $i(p)$ so that it is the smallest integer with with $i(p) > j(p) > p$. Then

$$\lim_{p\to\infty} \varrho(\xi_{i(p)-1},\xi_{j(p)}) = \mu. \tag{2.9}$$

Again,

$$\varrho(\xi_{j(p)-1},\xi_{i(p)-1}) \le \varrho(\xi_{j(p)-1},\xi_{i(p)}) + \varrho(\xi_{i(p)},\xi_{i(p)-1}) - \varrho(\xi_{i(p)},\xi_{i(p)}).$$

Letting $p \to \infty$ and using Lemma 1.7, we get

$$\lim_{p\to\infty} \varrho(\xi_{j(p)-1},\xi_{i(p)-1}) = \mu. \tag{2.10}$$

Now, using Lemma 1.12 we get

$$\alpha(\xi_{i(p)-1},\xi_{i(p)})\beta(\xi_{j(p)-1},\xi_{j(p)})=\alpha(\xi_{i(p)-1},\Gamma\xi_{i(p)-1})\beta(\xi_{j(p)-1},\quad \Gamma\xi_{j(p)-1})\geq 1. \tag{2.11}$$

In (2.1) replacing ξ by $\xi_{i(p)}$ and y by $\xi_{j(p)}$ respectively and using (2.11), we get

$$H(\varrho(\xi_{i(p)},\xi_{j(p)}))=H(\varrho(\Gamma\xi_{i(p)-1},\Gamma\xi_{j(p)-1}))\leq H(\varphi(M(\xi_{i(p)-1},\xi_{j(p)-1}))), \tag{2.12}$$

where

$$\begin{aligned} M&(\xi_{i(p)-1},\xi_{j(p)-1})\\ &=\max\left\{\varrho(\xi_{i(p)-1},\xi_{j(p)-1}),\frac{\varrho(\xi_{i(p)-1},\Gamma\xi_{i(p)-1})+\varrho(\xi_{j(p)-1},\Gamma\xi_{j(p)-1})}{2},\right.\\ &\qquad\left.\frac{\varrho(\xi_{i(p)-1},\Gamma\xi_{j(p)-1})+\varrho(\xi_{j(p)-1},\Gamma\xi_{i(p)-1})}{2}\right\}\\ &=\max\left\{\varrho(\xi_{i(p)-1},\xi_{j(p)-1}),\frac{\varrho(\xi_{i(p)-1},\xi_{i(p)})+\varrho(\xi_{j(p)-1},\xi_{j(p)})}{2},\frac{\varrho(\xi_{i(p)-1},\xi_{j(p)})+\varrho(\xi_{j(p)-1},\xi_{i(p)})}{2}\right\}. \end{aligned} \tag{2.13}$$

Letting $p\to\infty$ in (2.13) and using (2.7), (2.9), (2.10) and Lemma 1.7, we get

$$\lim_{p\to\infty}M(\xi_{i(p)-1},\xi_{j(p)-1})=\mu. \tag{2.14}$$

Now, letting $p\to\infty$ in (2.12) and using definitions of H and φ and (2.8), (2.14), we get

$$H(\mu)\leq H(\varphi(\mu))<H(\mu).$$

This is a contradiction. Therefore

$$\lim_{i,j\to\infty}\varrho(\xi_i,\xi_j)=0. \tag{2.15}$$

This implies that $\{\xi_i\}$ is a Cauchy sequence in $(\mathcal{W}_\varrho,\varrho)$ which is complete. Therefore, the sequence $\{\xi_i\}$ is convergent in the space $(\mathcal{W}_\varrho,\varrho)$, say $\lim_{i\to\infty}\varrho(\xi_i,\zeta)=0$. Again from Lemma 1.4, we get

$$\varrho(\zeta,\zeta)=\lim_{i\to\infty}\varrho(\xi_i,\zeta)=\lim_{j,i\to\infty}\varrho(\xi_i,\xi_j)=0.$$

Moreover, as Γ is continuous, we have

$$\zeta=\lim_{i\to\infty}\xi_{i+1}=\lim_{i\to\infty}\Gamma\xi_i=\Gamma\zeta. \qquad \square$$

We omit the continuity assumption of Γ in Theorem 2.6 in the following:

Theorem 2.7. *Let $(\mathcal{W}_\varrho,\varrho)$ be a complete partial metric space and $T:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ be self mapping. Suppose $\alpha,\beta:\mathcal{W}_\varrho\times\mathcal{W}_\varrho\to[0,\infty)$ be the mappings satisfying the conditions:*

(i) *Γ is (α,β)-cyclic admissible;*

(ii) *Γ is (α,β)-H-φ-contraction mapping;*

(iii) *there exists $\xi_0\in\mathcal{W}_\varrho$ such that $\alpha(\xi_0,\Gamma\xi_0)\geq 1$ and $\beta(\xi_0,\Gamma\xi_0)\geq 1$;*

(iv) *if for any sequence $\{\xi_i\}$ such that $\xi_i\to\zeta$ as $i\to\infty$, then $\alpha(\zeta,\Gamma\zeta)\geq 1$ and $\beta(\zeta,\Gamma\zeta)\geq 1$.*

Then Γ has a fixed point in $\mathcal{W}_\varrho$. Further, if ζ, ζ_1 are fixed points of Γ with $\alpha(\zeta,\Gamma\zeta)\geq 1$, $\alpha(\zeta_1,\Gamma\zeta_1)\geq 1$ and $\beta(\zeta,\Gamma\zeta)\geq 1$, $\beta(\zeta_1,\Gamma\zeta_1)\geq 1$, then $\zeta=\zeta_1$.

Proof. From the proof of Theorem 2.6, the sequence ξ_i defined by $\xi_{i+1} = \Gamma\xi_i$ is Cauchy in $\mathcal{W}_\varrho$. Now, suppose that (iv) holds. We have to show that $\Gamma\zeta = \zeta$. Assume that $\varrho(\Gamma\zeta,\zeta) > 0$. Since $\alpha(\xi_i,\xi_{i+1}) \geq 1$ and $\beta(\zeta,\Gamma\zeta) \geq 1$ we have $\alpha(\xi_i,\Gamma\xi_i)\beta(\zeta,\Gamma\zeta) \geq 1$. From (2.1), we have

$$H(\varrho(\xi_{i+1},\Gamma\zeta)) = H(\varrho(\Gamma\xi_i,\Gamma\zeta)) \leq H(\varphi(M(\xi_i,\zeta))), \tag{2.16}$$

where

$$\begin{aligned} M(\xi_i,\zeta) &= \max\left\{\varrho(\xi_i,\zeta), \frac{\varrho(\xi_i,\Gamma\xi_i)+\varrho(\zeta,\Gamma\zeta)}{2}, \frac{\varrho(\xi_i,\Gamma\zeta)+\varrho(\zeta,\Gamma\xi_i)}{2}\right\} \\ &= \max\left\{\varrho(\xi_i,\zeta), \frac{\varrho(\xi_i,\xi_{i+1})+\varrho(\zeta,\Gamma\zeta)}{2}, \frac{\varrho(\xi_i,\Gamma\zeta)+\varrho(\zeta,\xi_{i+1})}{2}\right\} \end{aligned} \tag{2.17}$$

Now, taking $i \to \infty$ in (2.17), we get

$$\lim_{i\to\infty} M(\xi_i,\zeta) \leq \varrho(\zeta,\Gamma\zeta). \tag{2.18}$$

Again, taking $n \to \infty$ in (2.16) and using (2.18) and definitions of H and φ, we get

$$H(\varrho(\zeta,\Gamma\zeta)) \leq H(\varphi(\varrho(\zeta,\Gamma\zeta))) < H(\varrho(\zeta,\Gamma\zeta))$$

which is a contradiction as $\varrho(\zeta,\Gamma\zeta) > 0$. Therefore, $\Gamma\zeta = \zeta$, i.e., ζ is a fixed point.
Further, suppose ζ and ζ_1 be two fixed point of Γ such that $\varrho(\zeta,\zeta_1) > 0$ and $\alpha(\zeta,\Gamma\zeta) \geq 1$, $\beta(\zeta_1,\Gamma\zeta_1) \geq 1$, then, we get $\alpha(\zeta,\Gamma\zeta)\beta(\zeta_1,\Gamma\zeta_1) \geq 1$. From (2.1), we have

$$H(\varrho(\zeta,\zeta_1)) = H(\varrho(\Gamma\zeta,\Gamma\zeta_1)) \leq H(\varphi(\varrho(M(\zeta,\zeta_1))), \tag{2.19}$$

where

$$\begin{aligned} M(\zeta,\zeta_1) &= \max\left\{\varrho(\zeta,\zeta_1), \frac{\varrho(\zeta,\Gamma\zeta)+\varrho(\zeta_1,\Gamma\zeta_1)}{2}, \frac{\varrho(\zeta,\Gamma\zeta_1)+\varrho(\zeta_1,\Gamma\zeta)}{2}\right\} \\ &= \max\left\{\varrho(\zeta,\zeta_1), \frac{\varrho(\zeta,\zeta)+\varrho(\zeta_1,\zeta_1)}{2}, \frac{\varrho(\zeta,\zeta_1)+\varrho(\zeta_1,\zeta)}{2}\right\} \\ &= \varrho(\zeta,\zeta_1). \end{aligned} \tag{2.20}$$

Putting (2.20) in (2.19) and using the definitions of H and φ, we get

$$H(\varrho(\zeta,\zeta_1)) \leq H(\varphi(\varrho(\zeta,\zeta_1))) < H(\varrho(\zeta,\zeta_1)).$$

which is a contradiction. Hence Γ has a unique fixed point.
This completes the proof. □

The theorem's consequences are shown below.

Corollary 2.8. *Let $(\mathcal{W}_\varrho,\varrho)$ be a complete partial metric space and $\Gamma : \mathcal{W}_\varrho \to \mathcal{W}_\varrho$ be self mapping. Suppose $\alpha,\beta : \mathcal{W}_\varrho \times \mathcal{W}_\varrho \to [0,\infty)$ be the mappings satisfying the conditions:*

(i) *Γ is (α,β)-cyclic admissible;*

(ii) *Γ is $(\alpha,\beta)-\varphi$-contraction mapping, i.e., $\alpha(\xi,\Gamma\xi)\beta(\eta,\Gamma\eta) \geq 1 \Rightarrow \varrho(\Gamma\xi,\Gamma\eta) \leq \varphi(\varrho(\xi,\eta))$;*

(iii) *there exists $\xi_0 \in \mathcal{W}_\varrho$ such that $\alpha(\xi_0,\Gamma\xi_0) \geq 1$ and $\beta(\xi_0,\Gamma\xi_0) \geq 1$;*

(iv) *Γ is continuous or if for any sequence $\{\xi_i\}$ such that $\xi_i \to \zeta$ as $i \to \infty$, then $\alpha(\zeta,\Gamma\zeta) \geq 1$ and $\beta(\zeta,\Gamma\zeta) \geq 1$.*

Then Γ has a fixed point in $\mathcal{W}_\varrho$.

Corollary 2.9. *Let $(\mathcal{W}_\varrho,\varrho)$ be partial metric space, $H\in\mathcal{H}$ and $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ be generalized H-φ-contraction mapping, i.e.,*

$$H(\varrho(\Gamma\xi,\Gamma\eta))\le H(\varphi(M(\xi,\eta))),\quad \text{for all } \xi,\eta\in\mathcal{W}_\varrho$$

and

$$M(\xi,\eta)=\max\left\{\varrho(\xi,\eta),\frac{\varrho(\xi,\Gamma\xi)+\varrho(\eta,\Gamma\eta)}{2},\frac{\varrho(\xi,\Gamma\eta)+\varrho(\eta,\Gamma\xi)}{2}\right\}.$$

Then Γ has a unique fixed point in $\mathcal{W}_\varrho$.

Corollary 2.10. *Let $(\mathcal{W}_\varrho,\varrho)$ be partial metric space and $\varphi:[0,\infty)\to[0,\infty)$ be a continuous function such that $\varphi(t)<t$, for $t>0$ and $\varphi(0)=0$. $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ be generalized φ-contraction mapping, i.e.,*

$$\varrho(\Gamma\xi,\Gamma\eta))\le\varphi(M(\xi,\eta)),\quad \text{for all } \xi,\eta\in\mathcal{W}_\varrho$$

and

$$M(\xi,\eta)=\max\left\{\varrho(\xi,\eta),\frac{\varrho(\xi,\Gamma\xi)+\varrho(\eta,\Gamma\eta)}{2},\frac{\varrho(\xi,\Gamma\eta)+\varrho(\eta,\Gamma\xi)}{2}\right\}.$$

Then Γ has a unique fixed point in $\mathcal{W}_\varrho$.

Example 2.11. Let $\mathcal{W}_\varrho=[0,\infty)$ and $\varrho(\xi,\eta)=\max\{\xi,\eta\}$. Then $(\mathcal{W}_\varrho,\varrho)$ is a complete partial metric space. Consider the mapping $\Gamma:\mathcal{W}_\varrho\to\mathcal{W}_\varrho$ defined by

$$\Gamma(\xi)=\begin{cases}\xi-\frac{4}{5}, & \xi\ge 1,\\ \frac{\xi}{5}, & 0\le\xi\le 1\end{cases}\tag{2.21}$$

and let $H,\varphi:[0,\infty)\to[0,\infty)$ be such that $H(t)=t$ and $\varphi(t)=\frac{t}{2}$, for all $t\ge 0$. If we define the functions $\alpha,\beta:\mathcal{W}_\varrho\times\mathcal{W}_\varrho\to[0,\infty)$ as

$$\alpha(\xi,\eta)=\begin{cases}\frac{5}{2}, & \xi,\eta\in[0,1],\\ 0, & \text{otherwise}\end{cases}\tag{2.22}$$

and

$$\beta(\xi,\eta)=\begin{cases}\frac{3}{2}, & \xi,\eta\in[0,1],\\ 0, & \text{otherwise.}\end{cases}\tag{2.23}$$

We show that contractive condition of Theorem 2.6 is satisfied. Without loss of generality we assume that $\xi\ge\eta$. We show that Γ is (α,β)-cyclic admissible. Let $\xi,\eta\in[0,1]$, then we have $\alpha(\xi,\eta)=\frac{5}{2}$, $\beta(\xi,\eta)=\frac{3}{2}$ and $\Gamma\xi=\frac{\xi}{5}$, $\Gamma\eta=\frac{\eta}{5}$. Now, as $\alpha(\xi,\eta)=\frac{5}{2}>1\Rightarrow\beta(\Gamma\xi,\Gamma\eta)=\beta\left(\frac{\xi}{5},\frac{\eta}{5}\right)=\frac{3}{2}>1$ and $\beta(\xi,\eta)=\frac{3}{2}>1\Rightarrow\alpha(\Gamma\xi,\Gamma\eta)=\alpha\left(\frac{\xi}{5},\frac{\eta}{5}\right)=\frac{5}{2}>1$. Therefore, Γ is (α,β)-cyclic admissible.
Again, for $\xi,\eta\in[0,1]$ $\alpha(\xi,\Gamma\xi)\beta(\eta,\Gamma\eta)=\alpha\left(\xi,\frac{\xi}{5}\right)\beta\left(\eta,\frac{\eta}{5}\right)=\frac{15}{4}>1$. Then, we get

$$\begin{aligned}H(\varrho(\Gamma\xi,\Gamma\eta))&=H\left(\varrho\left(\frac{\xi}{5},\frac{\eta}{5}\right)\right)=\frac{\xi}{5}\\&\le\frac{\xi}{2}=H\left(\frac{\xi}{2}\right)=H(\varphi(\xi))\\&=H\left(\varphi\left(\max\left\{\varrho(\xi,\eta),\frac{\varrho(\xi,\Gamma\xi)+\varrho(\eta,\Gamma\eta)}{2},\frac{\varrho(\xi,\Gamma\eta)+\varrho(\eta,\Gamma\xi)}{2}\right\}\right)\right)\end{aligned}$$

$$= H(\varphi(M(\xi,\eta))). \tag{2.24}$$

As a result, it meets the criterion of Theorem 2.6. Hence Γ has a fixed point, which in this case is 0.

3. Application

This section is influenced by the findings discussed in [8, 11], and the aim is to provide an application of Corollary 2.10 as a study of the existence of a unique solution to the nonlinear integral equation (3.1) presented below.

$$\xi(t) = h(t) + \int_a^b G(t,s)K(s,\xi(s))\,ds, \tag{3.1}$$

where $K : [a,b]\times\mathbb{R} \to \mathbb{R}$ and $G : [a,b]^2 \to [0,\infty)$ are continuous functions.

Let $\mathcal{W}_\varrho$ be the set of $C[a,b]$ of real continuous function on $[a,b]$ and let $\varrho : \mathcal{W}_\varrho \times \mathcal{W}_\varrho \to [0,\infty)$ be given by

$$\varrho(\xi,\eta) = \max_{a\le t\le b}\{\xi,\eta\}. \tag{3.2}$$

Then $(\mathcal{W}_\varrho,\varrho)$ is a complete partial metric space.

Theorem 3.1. *Let us consider the integral equation* (3.1) *as above. Also, suppose that it satisfies the following conditions:*

(i) *for $s\in[a,b]$ and $\xi,\eta\in\mathcal{W}_\varrho$ there exists a continuous function $\varphi:[0,\infty)\to[0,\infty)$ such that $\varphi(t)<t$, for $t>0$, $\varphi(0)=0$ and following inequality holds:*

$$|K(s,\xi(s)) - K(s,\eta(s))| \le \varphi|M(\xi,\eta)|,$$

where

$$M(\xi,\eta) = \max\left\{\varrho(\xi,\eta), \frac{\varrho(\xi,\Gamma\xi)+\varrho(\eta,\Gamma\eta)}{2}, \frac{\varrho(\xi,\Gamma\eta)+\varrho(\eta,\Gamma\xi)}{2}\right\}.$$

(ii) *there exists $\epsilon\in(0,1)$ such that $\max_{a\le t\le b}\int_a^b G(t,s)\,ds \le \epsilon$.*

Then, the integral equation (3.1) *has a unique solution.*

Proof. Let $\Gamma : \mathcal{W}_\varrho \to \mathcal{W}_\varrho$ be a mapping defined by

$$\Gamma\xi(t) = h(t) + \int_a^b G(t,s)K(s,\xi(s))\,ds \tag{3.3}$$

and $\varphi(t) = \epsilon t$. Then using the conditions of Theorem 3.1, we have

$$\begin{aligned}
\varrho(\Gamma\xi,\Gamma\eta) &= \max_{a\le t\le b}\{\Gamma\xi(t),\Gamma\eta(t)\} \\
&\le \max_{a\le t\le b}\left\{h(t)+\int_a^b G(t,s)K(s,\xi(s))\,ds, h(t)+\int_a^b G(t,s)K(s,\eta(s))\,ds\right\} \\
&\le \max_{a\le t\le b}\left\{\int_a^b G(t,s)\,ds[h(t)+K(s,\xi(s)), h(t)+K(s,\eta(s))]\right\} \\
&\le \int_a^b G(t,s)\,ds \max_{a\le t\le b}\left\{[h(t)+K(s,\xi(s)), h(t)+K(s,\eta(s))]\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le \epsilon \max_{a\le t\le b}\left\{[h(t)+K(s,\xi(s)),h(t)+K(s,\eta(s))]\right\}\\
&\le \epsilon \max\left\{\varrho(\xi,\eta),\frac{\varrho(\xi,\Gamma\xi)+\varrho(\eta,\Gamma\eta)}{2},\frac{\varrho(\xi,\Gamma\eta)+\varrho(\eta,\Gamma\xi)}{2}\right\}\\
&=\varphi(M(\xi,\eta)).
\end{aligned}
$$

Clearly, all of the conditions of Corollary 2.10 have been satisfied, showing that Γ has a unique fixed point. As a result, the integral equation (3.1) has a unique solution. □

4. Conclusion

We define H-φ-contraction, generalized H-φ-contraction, (α,β)-H-φ-contraction mappings in the frame work of partial metric spaces. We also established and obtained fixed point theorems in such spaces. Additionally, an example and an application to existence of the solution of nonlinear integral equation are provided to show the applicability of our obtained results.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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