Communications in Mathematics and Applications
Vol. 14, No. 3, pp. 1167–1180, 2023
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v14i3.2455

http://www.rgnpublications.com

Special Issue
Recent Trends in Mathematics and Applications
Proceedings of the International Conference of Gwalior Academy of Mathematical Sciences 2022
***Editors*: Vinod P. Saxena and Leena Sharma**

Research Article

Weakly Zero Divisor Graph of a Lattice

Vikas Kulal*[1], Anil Khairnar[2], Krishnat Masalkar[2] and Lata Kadam[2]

[1] *Department of Mathematics, School of Engineering and Sciences, MIT Art, Design and Technology University, Pune 412 201, Maharastra, India*
[2] *Department of Mathematics, Abasaheb Garware College, Pune 411 004, Maharastra, India*
***Corresponding author:** vikaskulal61@gmail.com

Received: February 16, 2023 **Accepted:** June 12, 2023

Abstract. For a lattice L, we associate a graph $WZG(L)$ called a weakly zero divisor graph of L. The vertex set of $WZG(L)$ is $Z^*(L)$, where $Z^*(L) = \{r \in L \mid r \neq 0, \exists\ s \neq 0$ such that $r \wedge s = 0\}$ and for any distinct u and v in $Z^*(L)$, $u - v$ is an edge in $WZG(L)$ if and only if there exists $p \in Ann(u) \setminus \{0\}$ and $q \in Ann(v) \setminus \{0\}$ such that $p \wedge q = 0$. In this paper, we determined the diameter, girth, independence number and domination number of $WZG(L)$. We characterized all lattices whose $WZG(L)$ is complete bipartite or planar. Also, we find a condition so that $WZG(L)$ is Eulerian or Hamiltonian. Finally, we study the affinity between the weakly zero divisor graph, the zero divisor graph and the annihilator-ideal graph of lattices.

Keywords. Zero divisor graph, Base of the element, Atom, Planar

Mathematics Subject Classification (2020). Primary: 06B10, Secondary: 06B99, 06D99

1. Introduction

Let $\mathbb{R}$ be a commutative ring. The zero divisor graph of $\mathbb{R}$ is introduced by Beck in [2]. It is denoted by $\Gamma(\mathbb{R})$. The $\Gamma(\mathbb{R})$ is an undirected graph with vertex set as $Z^*(\mathbb{R}) = Z(\mathbb{R}) \setminus \{0\}$, where $Z(\mathbb{R}) = \{r \in \mathbb{R} \mid \exists\ s \neq 0 \in \mathbb{R}$ such that $rs = 0\}$ and for any two distinct vertices u and v, $u - v$ is an edge in $\Gamma(\mathbb{R})$ if and only if $uv = 0$. In $\Gamma(\mathbb{R})$, authors mainly focused on its coloring. The weakly zero divisor graph $WG(\mathbb{R})$ is studied by Nikmehr *et al.* in [10]. It is an undirected (simple) graph with a vertex set as $Z(\mathbb{R})$ and for any two distinct vertices u and v, $u-v$ is an edge in $WG(\mathbb{R})$ if and only if there exists $p \in ann(u)$ and $q \in ann(v)$ such that $pq = 0$. Chelvam and Nithya [5], Anderson *et al.* [1], and Khairnar and Waphare [8] studied various graphs of algebraic structure.

Throughout this paper, $L = \langle L, \wedge, \vee \rangle$ is a lattice, and $K \neq \phi \subseteq L$ is a sublattice of L. A sublattice I of L is said to be ideal of L if $u \wedge i \in I$ for all $u \in L$ and $i \in I$. For any elements u and v in L with $v < u$, if there does not exists $z \in L$ such that $v < z < u$, then u covers v or v is covered by u (denoted by $u \succ v$ or $v \prec u$). For any $u \in L$, the set $(u] = \{v \in L : v \leq u\}$ is a principal ideal in L generated by u, if $\exists$ 0 and 1 such that $0 \leq u \leq 1$ then L is called bounded lattice, $Ann(u) = \{v \in L \mid u \wedge v = 0\}$, and if $0 \prec u$ or $u \prec 1$ then u is called atom or co-atom respectively. Let $A(L)$ be the set containing all atoms in L. Any elements u and v in L are said to be incomparable if and only if $u \nleq v$ and $v \nleq u$, we denote it by $u \parallel v$. A lattice L is called atomic if, for every element $v \in L$, $\exists$ an atom $a_v \in A(L)$ such that $a_v \leq v$. A lattice is called an atomistic, if it is atomic and each $u \in L$ is either an atom or a join of atoms in L. For concepts in lattice theory, we refer Birkhoff [3], and Grätzer [7]. For a lattice L with bottom element 0, the zero divisor graph is studied by Estaji and Khashyarmanesh [6]. It is denoted by $ZG(L)$. The $ZG(L)$ is a (undirected) graph with vertex set as $Z^*(L)$ and for any $u, v \in Z^*(L)$, $u - v$ is an edge in $ZG(L)$ if and only if $u \wedge v = 0$. Chelvam and Nithya [4] studied more properties of $ZG(L)$. Kulal *et al.* [9] studied the annihilator-ideal graph of an atomic lattice L with the smallest element 0. Let $N(L) = \{I \text{ is an ideal in } L \mid Ann(I) \neq \{0\}\}$. The annihilator-ideal graph $AnnIG(L)$ of a lattice L is a graph with vertex set as $N(L)$ and for any distinct vertices I and J, $I - J$ is an edge in $AnnIG(L)$ if and only if $J \cap Ann(I) \neq \{0\}$ or $I \cap Ann(J) \neq \{0\}$. Throughout this paper, L denotes a atomic lattice with the smallest element 0.

If V is a non-empty set (called vertices) and E is a set of 2 subsets of V (called edges), then $G = \langle V, E \rangle$ is called a graph on V with the edge set E. The 2-subset $e = \{u, v\} \in E$ is called an edge between u and v. In this case, we say that u and v are adjacent in G. If there exist subsets $V_1, V_2, \cdots, V_q$ of V such that $V = \bigcup_{i=1}^{q} V_i$, $V_i \cap V_j = \phi$, also for any $v_1, v_2 \in V_i$, $v_1 - v_2$ is not an edge and for any $v_1 \in V_i$, $v_2 \in V_j$, we have $v_1 - v_2$ is an edge for all $1 \leq i, j \leq q$, $i \neq j$, then the graph $G = \langle V, E \rangle$ is called a complete q-partite graph. It is denoted by H_q. If any two distinct vertices in graph G are adjacent, then G is called complete. A complete graph with p number of elements in V is denoted by K_p. If there exists a path joining any two vertices of graph G, then G is called a connected graph, otherwise, we say it is a disconnected graph. The distance between u and v is the length of shortest path from vertex u to vertex v, denoted by $d(u, v)$ and $\text{diam}(G) = \sup\{d(u, v) : u, v \in V\}$ is called a diameter of G. The length of shortest cycle in G is said to be the girth of G and it is denoted by $gr(G)$. An empty graph is a graph without vertices. The totally disconnected graph is a graph without edges. For the definitions of domination number ($\gamma(G)$), total domination number ($\gamma_t(G)$), independence number ($\alpha(G)$), chromatic number ($\chi(G)$) and clique number ($\omega(G)$), refer West [11]. We have, $A \bigvee B$ as a graph such that for any $u \in A$, $v \in B$, $u - v$ is an edge.

Definition 1.1. For a lattice L, we associate a graph $WZG(L)$ called the weakly zero divisor graph of L. It is a graph with the vertex set $Z^*(L) = \{a \in L \mid a \neq 0, \exists\ b \neq 0 \text{ such that } a \wedge b = 0\}$, and any two distinct vertices u and v are adjacent in $WZG(L)$ if and only if there exists a non zero $p \in Ann(u)$ and non zero $q \in Ann(v)$ such that $p \wedge q = 0$.

In the second section of this paper, we find the diameter, girth, independence number and domination number of $WZG(L)$. For any lattice L, we have shown that $WZG(L)$ is equal to $K_p \bigvee H_q$ for some p and q. We characterized all lattices whose $WZG(L)$ is complete bipartite or planar. Also, we find a condition so that $WZG(L)$ is Eulerian or Hamiltonian. In third section, we study the affinity between the weakly zero divisor graph, the zero divisor graph and the annihilator-ideal graph of a lattices.

2. Basic Properties of $WZG(L)$

In this section, the basic properties of $WZG(L)$ are studied. We have shown that $WZG(L)$ is connected with $\text{diam}(WZG(L)) \leq 2$ and $gr(WZG(L)) \in \{3,4,\infty\}$. We characterized all lattices whose $WZG(L)$ is complete bipartite or planar. Also, we find a condition so that $WZG(L)$ is Eulerian or Hamiltonian. Following is an immediate consequence from the definition of $WZG(L)$.

Lemma 2.1. *Let L be a lattice. Then subgraph induced by elements in $A(L)$ is complete in $WZG(L)$.*

Proof. Suppose $A(L) = \{a_1, a_2, \cdots, a_n\}$. Then for any distinct atoms a_i and a_j in $A(L)$, we have, $a_i \in Ann(a_j)$ and $a_j \in Ann(a_i)$. Since $a_i \wedge a_j = 0$, we have $a_i - a_j$ is an edge in $WZG(L)$. Thus subgraph induced by elements in $A(L)$ is complete in $WZG(L)$. □

In $WZG(L)$, the following definition is frequently useful.

Definition 2.2. If L is a lattice and $p \in L$, then the base of p is defined as the set of all atoms a_p of L with $a_p \leq p$. It is denoted by $\mathcal{B}(p)$. For every $p \in L$, we set $[\mathcal{B}(p)]^c = A(L) \setminus \mathcal{B}(p)$.

We have obtained the following results regarding the adjacency of vertices in $WZG(L)$.

Lemma 2.3. *Let L be a lattice. If u and v are distinct elements in $Z^*(L)$ with $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are distinct, then $u - v$ is an edge in $WZG(L)$.*

Proof. Suppose $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are not equal. If $\mathcal{B}(u) \nsubseteq \mathcal{B}(v)$ and $\mathcal{B}(v) \nsubseteq \mathcal{B}(u)$, then there exist atoms a_r and a_s in $A(L)$ with $a_r \in \mathcal{B}(u)$, $a_r \notin \mathcal{B}(v)$ and $a_s \in \mathcal{B}(v)$, $a_s \notin \mathcal{B}(u)$. Clearly, $a_r \in Ann(v)$ and $a_s \in Ann(u)$. Since $a_r \wedge a_s = 0$, we have, vertices u and v are adjacent in $WZG(L)$. Now, suppose $\mathcal{B}(u) \subset \mathcal{B}(v)$. Since $u, v \in Z^*(L)$, we have $|\mathcal{B}(v)| \leq n-1$ and $|\mathcal{B}(u)| \leq n-2$, where n be the number of atoms in L. Let b_s and b_r be the distinct atoms in $A(L)$ such that $b_s \notin \mathcal{B}(v)$ and $b_r \notin \mathcal{B}(u)$. Observe that $b_r \in Ann(u)$ and $b_s \in Ann(v)$. Since $b_r \wedge b_s = 0$, we have, vertex u is adjacent to vertex v in $WZG(L)$. Similarly, we prove the theorem if $\mathcal{B}(v) \subset \mathcal{B}(u)$. □

To check the converse of Lemma 2.3, consider a lattice L as shown in Figure 3. In $WZG(L)$, we have, $p - u$ is an edge, but $\mathcal{B}(p) = \{p\} = \mathcal{B}(u)$, therefore converse of Lemma 2.3 does not hold.

Theorem 2.4. *Let L be a lattice with the number of atoms in L is equal to n. If u and v are distinct elements in $Z^*(L)$, then u and v are not adjacent in $WZG(L)$ if and only if $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are equal with $|\mathcal{B}(u)| = |\mathcal{B}(v)| = n-1$.*

Proof. Let $A(L) = \{a_1, a_2, \cdots, a_n\}$. Suppose u and v are not adjacent in $WZG(L)$. Then by Lemma 2.3, $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are the same. To prove the theorem, we prove that the number of elements in $\mathcal{B}(u)$ and $\mathcal{B}(v)$ is equal to $n-1$. Since $u, v \in Z^*(L)$, we have $|\mathcal{B}(u)|, |\mathcal{B}(v)| \le n-1$. Suppose on the contrary $|\mathcal{B}(u)|, |\mathcal{B}(v)| < n-1$. Then there are distinct atoms a_r and a_s in $A(L)$ such that $a_r, a_s \notin \mathcal{B}(u) \cup \mathcal{B}(v)$. Therefore $a_r \wedge u = 0$ and $a_s \wedge v = 0$. Thus $a_r \in Ann(u)$ and $a_s \in Ann(v)$. Since $a_r \wedge a_s = 0$, we have, u and v are adjacent in $WZG(L)$, a contradiction. Thus number of elements in $\mathcal{B}(u)$ and $\mathcal{B}(v)$ is equal to $n-1$. Conversely, let $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are same with number of elements in $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are $n-1$. Suppose, on the contrary, u and v are adjacent in $WZG(L)$. By definition of $WZG(L)$, let p and q be the non-zero elements in L with $p \in Ann(u)$ and $q \in Ann(v)$ such that $p \wedge q = 0$. Thus $p \wedge u = 0 = q \wedge v$. Hence $p \wedge a_i = 0$ and $q \wedge a_j = 0$, $\forall\ a_i \in \mathcal{B}(u)$ and $\forall\ a_j \in \mathcal{B}(v)$. Since $|\mathcal{B}(u)| = n-1$, let a_t be the atom such that $a_t \notin \mathcal{B}(u)$. As L is atomic, we must have $a_t \in \mathcal{B}(p)$ and similarly $a_t \in \mathcal{B}(q)$. Clearly, $a_t \in \mathcal{B}(p \wedge q)$. Therefore, $p \wedge q \ne 0$ a contradiction. Thus $u - v$ is not an edge in $WZG(L)$. □

Let $A(L) = \{a_1, a_2, \cdots, a_n\}$ and for $1 \le k \le n$, denote $S_k = \{u \in Z^*(L) \mid a_i \in \mathcal{B}(u)$ for all i except $k\}$. Also, let $q = |\{S_k \mid S_k \ne \phi\}|$ and $p = |\{u \in Z^*(L) \mid u \notin S_k$, for any $k\}|$.

Example 2.5. Consider a lattice L_1 as shown in Figure 1. We have $A(L_1) = \{a_1, a_2, a_3\}$. Then S_1 is the set containing all $u \in Z^*(L_1)$ with $\mathcal{B}(u) = \{a_2, a_3\}$. Observe that, there does not exists $u \in Z^*(L_1)$ with $\mathcal{B}(u) = \{a_2, a_3\}$. Hence $S_1 = \phi$. Similarly, we can show that $S_2 = \phi = S_3$.

Example 2.6. Consider a lattice L_2 as shown in Figure 1. We have $A(L_2) = \{a_1, a_2, a_3\}$. Then $S_1 = \phi = S_2$ and $S_3 = \{e_3, e_3'\}$.

For a lattice L, define a relation $\sim$ on $Z^*(L)$ such that for all $u, v \in Z^*(L)$, $u \sim v$ if and only if $\mathcal{B}(u) = \mathcal{B}(v)$ with number of elements in $\mathcal{B}(u)$ are $n-1$, where n be the number of atoms in L. Note that $\sim$ is an equivalence relation. If L is any lattice, then in the following theorem, we show that $WZG(L) = K_p \bigvee H_q$ for some p and q.

Theorem 2.7. *If L is a lattice, then $WZG(L) = K_p \bigvee H_q$ for some p and q.*

Proof. Let $A(L) = \{a_1, \cdots, a_n\}$ and $u, v \in Z^*(L)$ be any distinct elements. By Theorem 2.4, there is no edge between u and v in $WZG(L)$ if and only if $\mathcal{B}(u) = \mathcal{B}(v)$ with $|\mathcal{B}(u)| = n-1$. That is $u \sim v$ if and only if $u - v$ is not an edge in $WZG(L)$. Now, for any $u \in Z^*(L)$, $[u] = \{v \in Z^*(L) \colon u \sim v\}$ is an equivalence class of u. Any two members in $[u]$ are not adjacent. If $[u]$ and $[v]$ are any two distinct equivalence classes, then for any $u_1 \in [u]$ and $v_1 \in [v]$, $u_1 \nsim v_1$. Therefore u_1 is adjacent to v_1. Let $A = \{u \in Z^*(L) \,\|\, \mathcal{B}(u)| = n-1\}$ and $B = \{v \in Z^*(L) \mid v \notin A\}$. Then $A \cup B = Z^*(L)$ and $A \cap B = \phi$. Let $u, v \in A$. If u and v both are in the same set S_m for some m, then $u \sim v$ and hence no edge between vertices u and v in $WZG(L)$. Now, if $u \in S_r$ and $v \in S_t$ for some $r \ne t$, then $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are distinct and hence $u \nsim v$. Therefore u and v are adjacent in $WZG(L)$. Hence $WZG(L)[A] = H_q$, where H_q is a complete q-partite graph with $q = |A|$. Let $z, w \in B$. Since $|\mathcal{B}(z)|$, $|\mathcal{B}(w)| \le n-2$, therefore we have $z \nsim w$. Hence $z - w$ is an edge in $WZG(L)$. Thus $WZG(L)[B] = K_p$, where $p = |B|$. Now, for all $x \in A$ and $y \in B$, we have $|\mathcal{B}(x)| = n-1$ and $|\mathcal{B}(y)| < n-1$. Thus $x \nsim y$ and hence $x - y$ is an edge in $WZG(L)[Z^*(L)]$. Thus $WZG(L) = K_p \bigvee H_q$ for some p and q. □

We illustrate the above theorem in the following examples.

Example 2.8. Let L_3 be a lattice shown in Figure 1. We have $A(L_3) = \{a_1, a_2, a_3\}$. Observe that $\mathcal{B}(e_1) = \{a_2, a_3\} = \mathcal{B}(e'_1)$ and there does not exist elements $x \in L$ other than e_1, e'_1 such that $\mathcal{B}(x) = \{a_2, a_3\}$. Hence $S_1 = \{e_1, e'_1\}$. Similarly, $S_2 = \{e_2, e'_2\}$, $S_3 = \{e_3, e'_3\}$. For a given lattice, we have $q = 3$ and $p = 3$. Therefore, $WZG(L_3) = K_3 \vee H_3$.

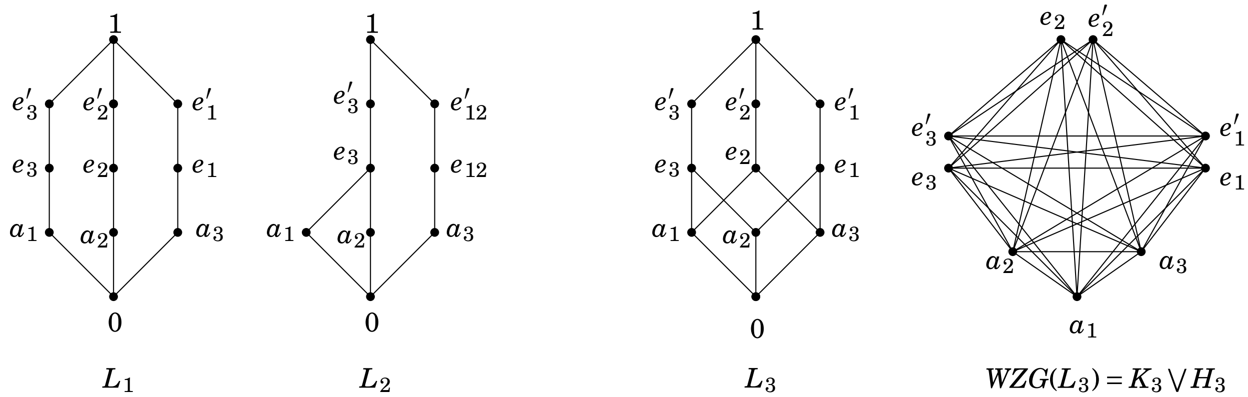

Figure 1

Example 2.9. Let $m_1, m_2, \cdots, m_n$ be n positive integers. Let L_1 be a lattice defined by relations, $0 \prec a_i \prec u_{i1} \prec u_{i2} \prec \cdots \prec u_{im_i} \prec 1$, for every $i = 1, 2, 3, \cdots n$. Then L_1 is a lattice with n number of atoms. Observe that $WZG(L_1) = K_p$, where $p = \left(\sum_{i=1}^{n} m_i\right) + n$. Let k be the least positive integer such that $n < 2^k - 2$ and L_2 be a lattice defined by the following relations:

(i) $0 \prec a_1 \prec u_{1,1} \prec u_{1,2} \prec \cdots \prec u_{1,m_1} \prec u_{1,m_1+1} \prec \cdots \prec u_{1,(m_1+m_2)} \prec u_{1,(m_1+m_2+1)}$ $\prec \cdots \prec u_{1,(m_1+m_2+m_3+\cdots+m_k)} \prec 1$.

(ii) $0 \prec a_i \prec u_{1,(m_1+m_2+\cdots+m_{i-1}+1)}$ for all $i = 2, 3, \cdots k$.

Then, lattice L_2 has k atoms and $WZG(L_2) = K_p \vee H_1$, where $p = (m_1 + m_2 + \cdots + m_{k-2}) + k - 2$ and H_1 is the totally disconnected graph with $m_{k-1} - 1$ number of vertices.

Following are the immediate consequences of Theorem 2.7.

Corollary 2.10. *If L is a lattice, then $WZG(L)$ is connected and* $\operatorname{diam}(WZG(L)) \leq 2$.

Proof. By Theorem 2.7, $WZG(L) = K_p \vee H_q$ for some p and q. Thus the statement is obvious. □

Definition 2.11. An element is said to be atomic in a lattice L if it is either an atom or a join of atoms. For any lattice L, the set of all atomic elements in $L \setminus \{1\}$ is denoted by $\mathcal{A}(L)$. An element is said to be nonatomic if it is not atomic.

Corollary 2.12. *If L is an atomistic lattice, then $WZG(L) = K_{|\mathcal{A}(L)|}$.*

Proof. Let $u, v \in Z^*(L)$ be any distinct elements. As L is atomistic, we have, $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are distinct and $Z^*(L) = \mathcal{A}(L)$. By Lemma 2.3, we have $WZG(L) = K_{|\mathcal{A}(L)|}$. □

Corollary 2.13. *If L is Boolean algebra $(P(\{1,2,3,\cdots,n\}, \cup, \cap))$, then $WZG(L) = K_{2^n-2}$.*

Proof. We know $L = (P(\{1,2,3,,n\},\cup,\cap))$ is atomistic and $\mathcal{A}(L) = P(\{1,2,3,\ldots,n\}) \setminus \{\phi,\{1,2,3,\cdots,n\}\}$. Observe that $|\mathcal{A}(L)| = 2^n - 2$. Thus, by Corollary 2.12, we have $WZG(L) = K_{2^n-2}$. □

Corollary 2.14. *If L is a lattice, then $WZG(L)$ is empty if and only if the number of atoms in L is one.*

Theorem 2.15. *If L is a lattice, then $WZG(L)$ is a complete bipartite or star if and only if the number of atoms in L is two.*

Proof. Let $WZG(L)$ be a complete bipartite or star. If $|A(L)| \geq 3$, then any three members from the set $A(L)$ form a cycle of length three, and hence $WZG(L)$ is not a bipartite, a contradiction. Also, if the number of atoms in L is one, then $WZG(L)$ is empty, a contradiction. Therefore $|A(L)| = 2$. Conversely, let $A(L) = \{a_1,a_2\}$. Note that, for any $u \in Z^*(L)$, if $\mathcal{B}(u) = \{a_1,a_2\}$, then u is not a member of $Z^*(L)$. Therefore, $Z^*(L) = S_1 \cup S_2$ and $S_1 \cap S_2 = \phi$. As $a_2 \in S_1$ and $a_1 \in S_2$, we have $S_1, S_2 \neq \phi$. Thus, $q = 2$. Since there does not exist a vertex in $Z^*(L)$, which is not a member of S_1 or S_2, we have $p = 0$. Hence $WZG(L) = K_0 \vee H_2 = H_2 = K_{|S_2|,|S_1|}$. Moreover, if $|S_2| = 1$ or $|S_1| = 1$, then $WZG(L)$ is star. □

Consider a lattice L with $|A(L)| = 2$. In the following result, we find the girth, domination number and independence number of $WZG(L)$. Also, we discuss the planarity of $WZG(L)$.

Corollary 2.16. *If L is a lattice such that $A(L) = \{a_1,a_2\}$, then*

(i) $gr(WZG(L)) = \{4,\infty\}$

(ii) $\gamma(WZG(L)) = \gamma_t(WZG(L)) \leq 2$.

(iii) $\alpha(WZG(L)) = \max\{|S_1|,|S_2|\}$.

(iv) *$WZG(L)$ is planar if and only if $|S_2| \leq 2$ or $|S_1| \leq 2$.*

Proof. Since the number of atoms in L are two, therefore by Theorem 2.15, we have $WZG(L) = H_2 = K_{|S_2|,|S_1|}$. Thus the statements (i), (ii), (iii) and (iv) are trivial. □

Consider a lattice L with the number of atoms in L three or more than three. We find the girth, domination number and independence number of $WZG(L)$ in the following theorem. Also, we discuss the planarity of $WZG(L)$.

Theorem 2.17. *If L is a lattice such that $|A(L)| \geq 3$, then the following statements hold.*

(i) *For any $a \in A(L)$ and $u \in Z^*(L)$ with $a \neq u$, $a - u$ is an edge in $WZG(L)$.*

(ii) $gr(WZG(L)) = 3$.

(iii) $\gamma(WZG(L)) = \gamma_t(WZG(L)) = 1$.

(iv) *If $A(L) = \{a_1,a_2,\cdots,a_n\}$, then $\alpha(WZG(L)) = \max\{|S_1|,|S_2|,\cdots,|S_n|\}$ or 1.*

(v) *If $|A(L)| = 3$, then $WZG(L)$ is planar if and only if $|Z^*(L)| = 3$ or $|Z^*(L)| = 4$ or $\mathcal{B}(u) = \mathcal{B}(v)$ with $|\mathcal{B}(u)| = 2$, $\forall\ u,v \in Z^*(L) \setminus A(L)$ when $|Z^*(L)| \geq 5$.*

(vi) *If $|A(L)| = 4$, then $WZG(L)$ is planar if and only if $|Z^*(L)| = 4$.*

(vii) *If $|A(L)| \geq 5$, then $WZG(L)$ is not a planar.*

Proof. (i): Let $a \in A(L)$ and $u \in Z^*(L)$ with $a \neq u$. Also, let a_i be the atom such that $a_i \notin \mathcal{B}(u)$. Then, $u \wedge a_i = 0$. Hence $a_i \in Ann(u)$. Since $|A(L)| \geq 3$, choose an atom a_k in $A(L)$ such that a_k is distinct from a and a_i. Then $a_k \wedge a = 0$. Therefore, $a_k \in Ann(a)$. Since $a_i \wedge a_k = 0$, we have vertex a is adjacent to vertex u.

(ii): By Lemma 2.1, any three distinct atoms form a cycle of length 3. Thus $gr(WZG(L)) = 3$.

(iii): By (i), for any $a \in A(L)$, $D = \{a\}$ is a dominating set. Hence $\gamma(WZG(L)) = \gamma_t(WZG(L)) = 1$.

(iv): If $WZG(L)$ is complete, then $\alpha(WZG(L)) = 1$. Suppose $WZG(L)$ is not complete. Then $|S_k| \geq 2$ for some k. Since S_k is an independent set for all values of k, therefore the statement follows easily.

(v): Let $|A(L)| = 3$. If $|Z^*(L)| = 3$, then $WZG(L) = K_3$. Also, if $|Z^*(L)| = 4$, then $WZG(L) = K_4$. Thus the statement is clear when $|Z^*(L)| = 3$ or $|Z^*(L)| = 4$. We prove the theorem when $|Z^*(L)| \geq 5$. Let $WZG(L)$ be a planar graph. Suppose $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are distinct for some $u, v \in Z^*(L) \setminus A(L)$. Then, by Lemma 2.3, $u - v$ is an edge in $WZG(L)$. Hence by (i), the elements in set $A(L)$ and vertices u, v form K_5 as a subgraph of $WZG(L)$, a contradiction. Thus $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are equal for every $u, v \in Z^*(L) \setminus A(L)$. Suppose there exists $u, v \in Z^*(L) \setminus A(L)$ such that $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are equal with number of elements in $\mathcal{B}(u)$ equal to one. Clearly, $|\mathcal{B}(v)| = 1$. Then by Theorem 2.4, we have, $u - v$ is an edge in $WZG(L)$. By (i), the elements in $A(L)$ and u, v forms K_5 as a subgraph in $WZG(L)$, a contradiction. Thus $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are equal with $|\mathcal{B}(u)| = 2, \forall u, v \in Z^*(L) \setminus A(L)$. Conversely, suppose $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are equal, with $|\mathcal{B}(u)| = 2$, $\forall\ u, v \in Z^*(L) \setminus A(L)$. Then by Theorem 2.4, for any $u, v \in Z^*(L) \setminus A(L)$, we have, vertex u and v not adjacent in $WZG(L)$. Moreover, $WZG(L) = K_3 \vee H_1$, where H_1 is the totally disconnected graph with $|Z^*(L) \setminus A(L)|$ number of vertices. Clearly, $WZG(L)$ has no subgraph isomorphic to $K_{3,3}$ or K_5. Thus $WZG(L)$ is planar.

(vi): Let $|A(L)| = 4$. Hence $|Z^*(L)| = 4$. Then $WZG(L) = K_4$, and thus the statement is clear.

(vii): Let $|A(L)| \geq 5$. Then members from $A(L)$ form K_5 as a subgraph of $WZG(L)$. Thus $WZG(L)$ is not planar. □

The converse of Theorem 2.17(i) is not true. Let L be a lattice shown in Figure 3(iii). In $WZG(L)$, the vertex u is adjacent to all vertices of $WZG(L)$, but u is not an atom.

The following corollary can be easily obtained from Theorem 2.17(ii) and Theorem 2.15.

Corollary 2.18. *If L is a lattice, then $WZG(L)$ is bipartite if and only if $WZG(L)$ is complete bipartite.*

Following corollary can be easily obtained from Theorem 2.17(i) and Lemma 2.1.

Corollary 2.19. *If L is a lattice such that the number of atoms in L is three or more than three, then there exists a vertex of $WZG(L)$ which is adjacent to every other vertex.*

In the following remark, we find a condition so that $WZG(L)$ is Eulerian.

Remark 2.20. For any lattice L, by Theorem 2.7, we have $WZG(L) = K_p \bigvee H_q$ for some p and q. Suppose $H_q = K_{m_1,m_2,\cdots,m_q}$ for some positive integers $m_1 \leq m_2 \leq \cdots \leq m_q$. Let $m = \sum_{i=1}^{q} m_i$ and p be a even number. Suppose $WZG(L)$ is Eulerian. Then m is odd. Hence $m - m_j$ is even for all $j = 1,2,\cdots,q$. Thus m_j is odd for all $j = 1,2,\cdots,q$. Then the degree of every vertex in H_q is odd in $WZG(L)$, a contradiction. Therefore, if p is even then $WZG(L)$ never be Eulerian. Let p be an odd number. Then $WZG(L)$ is Eulerian if and only if m is even if and only if $m - m_j$ is even for all $j = 1,2,\cdots,q$ if and only if all m_j have the same parity as that of m. That is, if $A(L) = \{a_1,a_2,\cdots,a_q\}$, then $WZG(L)$ is Eulerian if and only if p is odd, and $|S_k|$ is even for all k.

Remark 2.21. For any lattice L, by Theorem 2.7, we have $WZG(L) = K_p \bigvee H_q$ for some p and q. Suppose $H_q = K_{m_1,m_2,\cdots,m_q}$ for some positive integers $m_1 \leq m_2 \leq \cdots \leq m_q$. We know K_p is Hamiltonian. Therefore, $WZG(L)$ is Hamiltonian if and only if $m_q \leq \sum_{j \neq q} m_j$. That is, whenever $A(L) = \{a_1,a_2,\cdots,a_q\}$, then $WZG(L)$ is Hamiltonian if and only if $|S_q| \leq \sum_{j \neq q} |S_j|$.

3. Affinity Between $WZG(L)$, $ZG(L)$ and $AnnIG(L)$

In this section, we identify when can be $WZG(L)$ is identical to $ZG(L)$ and $AnnIG(L)$.

Remark 3.1. Let L be a lattice. Then L can be embedded in $I(L)$ (for $a \in L$, $a \to (a]$), where $I(L)$ denotes the set of all ideals in L. Moreover, if L is a finite lattice, then all ideals in $I(L)$ are principal. Observe that L and $I(L)$ are isomorphic. Then $(a] \in N(L)$ if and only if $a \in Z^*(L)$.

Lemma 3.2. *If L is a lattice, then the following statements hold.*

(i) *If $u - v$ is an edge in $ZG(L)$ for some distinct elements $u,v \in Z^*(L)$, then $u - v$ is an edge in $WZG(L)$.*

(ii) *If $(u]$ and $(v]$ are distinct ideals in $N(L)$ such that $(u]$ and $(v]$ are adjacent in $AnnIG(L)$, then u and v are adjacent in $WZG(L)$.*

Proof. (i): Let $u - v$ be an edge in $ZG(L)$. Therefore $u \wedge v = 0$. Clearly, $u \in Ann(v)$ and $v \in Ann(u)$. Therefore $u - v$ is an edge in $WZG(L)$. Thus, $ZG(L)$ is a subgraph of $WZG(L)$.

(ii): Suppose $(u]$ is adjacent to $(v]$ in $AnnIG(L)$. By [9, Lemma 2.2], $A((u])$ and $A((v])$ are distinct. That is $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are distinct. Hence, by Lemma 2.3, $u - v$ is an edge in $WZG(L)$. □

The converse of statements (i) and (ii) of Lemma 3.2 is not true. Let L be a lattice as shown in Figure 3. Observe that $p - u$ is an edge in $WZG(L)$, but $p - u$ is not an edge in $ZG(L)$ and $(p] - (u]$ is not an edge in $AnnIG(L)$.

Theorem 3.3. *If L is a lattice with $A(L) = \{a_1,a_2\}$, then $WZG(L)$, $ZG(L)$ and $AnnIG(L)$ are identical.*

Proof. Since $|A(L)| = 2$, by [6, Lemma 5.6], [9, Theorem 2.3] and Theorem 2.15, we have $WZG(L) = ZG(L) = AnnIG(L) = K_{|S_2|,|S_1|}$. □

The converse of Theorem 3.3 is not true. Observe that in Figure 2, we have, $WZG(L) = ZG(L) = AnnIG(L) = K_3$, but $|A(L)| = 3 \neq 2$.

If L is a lattice with only one atom, then $WZG(L)$, $ZG(L)$ and $AnnIG(L)$ are empty graphs. In the following theorem, we give a condition so that $WZG(L)$ and $ZG(L)$ are identical when L is a lattice with three or more than three atoms.

Theorem 3.4. *If L is a lattice with $|A(L)| \geq 3$, then $WZG(L)$ and $ZG(L)$ are identical if and only if $Z^*(L) = A(L)$.*

Proof. Suppose $WZG(L)$ and $ZG(L)$ are identical. Also, suppose $Z^*(L)$ and $A(L)$ are distinct. Let $u \in Z^*(L) \setminus A(L)$. Since L is atomic, let a be the atom in $A(L)$ such that a belongs to $\mathcal{B}(u)$. Therefore $a \wedge u \neq 0$. Thus u and a are not adjacent vertices in $ZG(L)$. But, by Theorem 2.17(i), $u - a$ is an edge in $WZG(L)$, a contradiction. Therefore $Z^*(L)$ and $A(L)$ are equal. The converse is clear by Lemma 2.1. □

Theorem 3.5 ([9])**.** *If L is a complete lattice, then $AnnIG(L)$ is complete if and only if L is atomistic. Moreover, if lattice L is atomistic, then $AnnIG(L) = K_m$, where $m = |L \setminus \{0,1\}|$.*

By Corollary 2.12 and Theorem 3.5, the following is an immediate consequence.

Proposition 3.6. *If L is a lattice and $AnnIG(L)$ is complete, then $WZG(L)$ and $AnnIG(L)$ are identical.*

To check the converse of Proposition 3.6, consider a lattice L shown in Figure 5. We have, $WZG(L)$ and $AnnIG(L)$ identical, but $AnnIG(L)$ is not complete.

In the following theorem, we gave a characterization for $WZG(L)$ and $AnnIG(L)$ to be identical when L is a lattice with a number of atoms in L are three or more than three.

Theorem 3.7. *If L is a lattice such that $A(L) = \{a_1, a_2, \cdots, a_n\}, n \geq 3$, then $WZG(L)$ and $AnnIG(L)$ are identical if and only if $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are distinct for each $u, v \in Z^*(L) \setminus \left(\bigcup_{i=1}^{n} S_i\right)$.*

Proof. Let $WZG(L)$ and $AnnIG(L)$ be identical. Suppose there are elements u and v in $Z^*(L) \setminus \left(\bigcup_{i=1}^{n} S_i\right)$ with $\mathcal{B}(u)$and $\mathcal{B}(v)$ are equal. Therefore number of elements in $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are less than $n-1$. Hence, by Theorem 2.4, there is an edge between u and v in $WZG(L)$. Since $A((u])$ and $A((v])$ are same, therefore by [9, Lemma 2.2], $(u] - (v]$ is not an edge in $AnnIG(L)$, a contradiction. Therefore $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are distinct for all $u, v \in Z^*(L) \setminus \left(\bigcup_{i=1}^{n} S_i\right)$. Conversely, let $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are distinct for all u and v in $Z^*(L) \setminus \left(\bigcup_{i=1}^{n} S_i\right)$. Therefore $|\mathcal{B}(u)|, |\mathcal{B}(v)| \leq n-2$. Then, by Lemma 2.3, u and v are adjacent in $WZG(L)$ if and only if $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are distinct if and only if $A((u])$ and $A((v)]$ are distinct if and only if ideal $(u]$ is adjacent to ideal $(v]$ in $AnnIG(L)$. Thus $WZG(L)$ and $AnnIG(L)$ are identical. □

In the following theorem, we discuss the properties of $WZG(D(n))$, where $D(n)$ is the lattice containing all divisors (positive) of a natural number n.

Theorem 3.8. *Let n be a natural number and $L = D(n)$ be the lattice containing all divisors of n. For $n > 1$, $n = p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}$ be the prime factorization, where $p_1, p_2, \cdots, p_k$, with $k \geq 2$ are* *distinct primes and $q_1 \le q_2 \le \cdots \le q_k$. Then*

(i) *$WZG(L)$ and $ZG(L)$ are identical if and only if $k=2$.*

(ii) *$WZG(L)$ and $AnnIG(L)$ are identical if and only if $k=2$ or $q_i=1$, $\forall\ i=1,\cdots,k$ when $k\neq 2$.*

(iii) *$WZG(L)$ is complete if and only if $AnnIG(L)$ is complete.*

(iv) *for $k=2$, $\gamma(WZG(L))=\gamma_t(WZG(L))\le 2$.*

(v) *for any $k\ge 3$, $\gamma(WZG(L))=\gamma_t(WZG(L))=1$.*

(vi) *for any $k\ge 2$, $\omega(WZG(L))=\chi(WZG(L))=\prod_{i=1}^{k}(q_i+1)-\left[\prod_{i=1}^{k}q_i+\left(\sum_{j=1}^{k}|S_j|\right)+2\right]+k$, where $|S_j|=\prod_{i=1,i\neq j}^{k}q_i$.*

(vii) *For any $k\ge 2$, $\alpha(WZG(L))=\prod_{i=2}^{k}q_i$*

(viii) *If $k=2$, then $WZG(L)$ is planar if and only if $q_1\le 2$.*

(ix) *If $k\ge 3$, then $WZG(L)$ is not planar.*

Proof. Let $U=\bigcup_{i=1}^{k}S_i$, where $S_k=\{u\in Z^*(L)|\ p_i\in\mathcal{B}(u)$ for all i except $k\}$. In this lattice $D(n)$, we have $A(L)=\{p_1,p_2,\cdots,p_k\}$. For any $1\le i,\ j\le k$ with $i\neq j$, we have $S_i\cap S_j=\phi$. Let $V=\{u\in Z^*(L)\,\|\,\mathcal{B}(u)|\le k-2\}$. Then $V(WZG(L))=U\cup V, U\cap V=\phi$, and so $\{U,V\}$ is a partition of $V(WZG(L))$.

(i): We have $|A(L)|=2$ if and only if $k=2$. Then by Theorem 3.3, the statement is trivial.

(ii): If $k=2$, then $|A(L)|=2$ and hence by Theorem 3.3, $WZG(L)=AnnIG(L)$. Let $k>2$ and $q_i=1$, $\forall\ i=1,\cdots,k$. Then L is a finite boolean lattice with $|L|=2^k$. By Corollary 2.13, we have $WZG(L)=K_{2^k-2}$. Hence by [9, Corollary 3.2], $WZG(L)=AnnIG(L)$. Conversely, let $WZG(L)$ and $AnnIG(L)$ be identical graphs. If $k=2$, then we are through. Let $k\neq 2$. Suppose $q_j\neq 1$ for some j. Then $A((p_j])=A((p_j^2])$. Hence by [9, Lemma 2.2], $(p_j]-(p_j^2]$ is not an edge in $AnnIG(L)$. But, by Theorem 2.17(i), we have $p_j-p_j^2$ is an edge in $WZG(L)$. Therefore $WZG(L)$ and $AnnIG(L)$ are not identical, a contradiction. Thus $q_i=1,\forall i=1,\cdots,k$.

(iii): Let $WZG(L)$ be complete. Suppose $q_j\neq 1$ for some j. Then for $u=p_1p_2\cdots p_j\cdots p_{k-1}$ and $v=p_1p_2\cdots p_j^2\cdots p_{k-1}$, we have $\mathcal{B}(u)=\{p_1,p_2,\cdots,p_{k-1}\}=\mathcal{B}(v)$ and since $|A(L)|=k$, by Theorem 2.4, $u-v$ is not an edge in $WZG(L)$, a contradiction. Therefore $q_i=1$, $\forall\ i=1,\cdots,k$. Then by the discussion in proof of (ii), $AnnIG(L)$ is complete. Conversely, suppose $AnnIG(L)$ is complete. If there exists at least one $q_j\neq 1$, then $A((p_j])=A((p_j^2])$ and hence $(p_j]-(p_j^2]$ is not an edge in $AnnIG(L)$, a contradiction. Thus by the discussion in proof of (ii), $WZG(L)$ is complete.

(iv): Let $k=2$. Then $WZG(L)$ is a complete bipartite graph.
Therefore $\gamma(WZG(L))=\gamma_t(WZG(L))\le 2$.

(v): Let $k\ge 3$. Then $|A(L)|\ge 3$ and hence by Theorem 2.17(iii), $\gamma(WZG(L))=\gamma_t(WZG(L))=1$.

(vi): Let $k\ge 2$. Then $|L|=\prod_{i=1}^{k}(q_i+1)-2$. Also, observe that $|U|=\sum_{j=1}^{k}|S_j|$, where $|S_j|=\prod_{i=1,i\neq j}^{k}q_i$. Therefore $WZG(L)=K_p\vee H_k$, where $p=\prod_{i=1}^{k}(q_i+1)-\left[\prod_{i=1}^{k}q_i+\left(\sum_{j=1}^{k}|S_j|\right)+2\right]$, with $|S_j|=\prod_{i=1,i\neq j}^{k}q_i$.

Therefore $\omega(WZG(L)) = \chi(WZG(L)) = \prod_{i=1}^{k}(q_i+1) - \left[\prod_{i=1}^{k} q_i + \left(\sum_{j=1}^{k}|S_j|\right)+2\right]+k$, where $|S_j| = \prod_{i=1,i\neq j}^{k} q_i$.

(vii): Let $k \geq 2$. Since $WZG(L) = K_p \vee H_k$, where $p = \prod_{i=1}^{k}(q_i+1) - 2 - \prod_{i=1}^{k} q_i - \sum_{j=1}^{k}|S_j|$ with $|S_j| = \prod_{i=1,i\neq j}^{k} q_i$ and for any $2 \leq l \leq k$, $|S_l| \leq |S_1|$. Therefore S_1 is the maximum independent set. Since $|S_1| = \prod_{i=2}^{k} q_i$, we have $\alpha(WZG(L)) = \prod_{i=2}^{k} q_i$.

(viii): Let $k = 2$ and $WZG(L)$ be planar. Then $WZG(L) = K_{q_1,q_2}$. Therefore the statement is trivial.

(ix): Let $k \geq 3$. Then $WZG(L) = K_p \vee H_q$ with $p \geq 3$ and $q \geq 3$. Clearly, $WZG(L)$ contains K_5 as a subgraph. Thus $WZG(L)$ is not planar. □

Figure 3(i,ii) illustrate parts (viii) and (ix) of Theorem 3.8.

In the following result, we discuss the relationship between the girth of $WZG(L)$, $ZG(L)$, and $AnnIG(L)$.

Theorem 3.9. *If L is a lattice, then $gr((WZG(L)) = gr(ZG(L)) = gr(AnnIG(L))$.*

Proof. If $A(L) = \{a_1, a_2\}$, then by Theorem 3.3, $WZG(L) = ZG(L) = AnnIG(L) = K_{|S_1|,|S_2|}$. If $|S_1| < 2$ or $|S_2| < 2$, then $gr((WZG(L)) = gr(ZG(L)) = gr(AnnIG(L)) = \infty$. If $|S_1|, |S_2| \geq 2$, then $gr((WZG(L)) = gr(ZG(L)) = gr(AnnIG(L)) = 4$. If $|A(L)| \geq 3$, then any three atoms make a cycle of length 3 in $WZG(L)$, $ZG(L)$ and $AnnIG(L)$. Hence $gr((WZG(L)) = gr(ZG(L)) = gr(AnnIG(L)) = 3$.

Thus, for any lattice L, we have $gr((WZG(L)) = gr(ZG(L)) = gr(AnnIG(L))$. □

Finally, in the following theorem, we discuss the relationship between the diameter of $WZG(L)$, $ZG(L)$, and $AnnIG(L)$.

Theorem 3.10. *If L is a lattice and $u, v \in Z^*(L)$ be distinct elements, then the following statements hold.*

(i) *If distance between u and v is three in $ZG(L)$, then $u - v$ is an edge in $WZG(L)$.*

(ii) *If* $\operatorname{diam}(WZG(L)) = 2$, *then* $\operatorname{diam}(ZG(L)) = 2$ *or* 3.

(iii) *If* $\operatorname{diam}(ZG(L)) = 1$, *then* $\operatorname{diam}(WZG(L)) = 1$.

(iv) *If* $\operatorname{diam}(WZG(L)) = 2$, *then* $\operatorname{diam}(AnnIG(L)) = 2$.

(v) *If* $\operatorname{diam}(AnnIG(L)) = 1$, *then* $\operatorname{diam}(WZG(L)) = 1$.

Proof. (i): Let $d(u,v) = 3$ in $ZG(L)$. Then $u \wedge p = 0$, $v \wedge q = 0$ and $u \wedge q \neq 0, v \wedge p \neq 0$ for some $p, q \in Z^*(L)$. This implies that $p \in Ann(u) \setminus Ann(v)$ and $q \in Ann(v) \setminus Ann(u)$. Therefore $Ann(u)$ and $Ann(v)$ are distinct. Hence $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are distinct. Thus by Lemma 2.3, $u - v$ is an edge in $WZG(L)$.

(ii): Suppose $|A(L)| = n$. Since $\operatorname{diam}(WZG(L)) = 2$, therefore, suppose u and v are the elements not adjacent in $WZG(L)$. Then $\mathcal{B}(u) = \mathcal{B}(v)$ with $|\mathcal{B}(u)| = n - 1$. Therefore $u \wedge v \neq 0$. Hence $u - v$ is not an edge in $ZG(L)$. Therefore $\operatorname{diam}(ZG(L)) = 2$ or 3.

(iii): Statement follows from Lemma 3.2(i).

(iv): Suppose $d(u,v) = 2$ in $WZG(L)$ for some u and v in $Z^*(L)$. Then by Lemma 2.3, $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are equal. Therefore $A((u])$ and $A((v])$ are distinct. Hence by [9, Lemma 2.2], $(u]$ and $(v]$ are not adjacent in $AnnIG(L)$. By [9, Corollary 2.4], we have $\text{diam}(AnnIG(L)) = 2$.

(v): Statement follows from Corollary 2.12 and Proposition 3.6. □

Example 3.11. If diam($WZG(L)$) = 1, then diam($ZG(L)$) = 1 or 2 or 3. It can be observe in Figure 2, Figure 3(iii) and Figure 4.

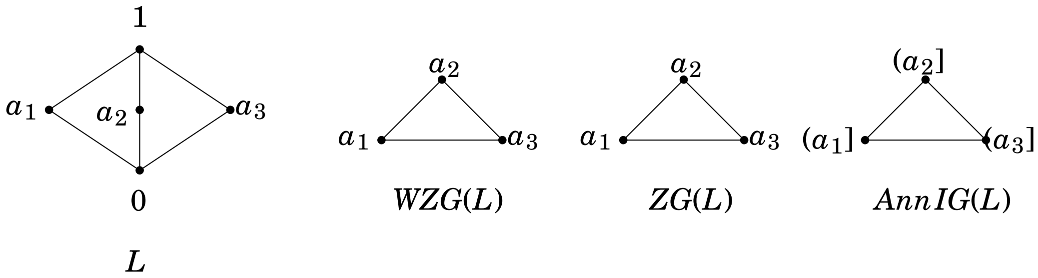

Figure 2

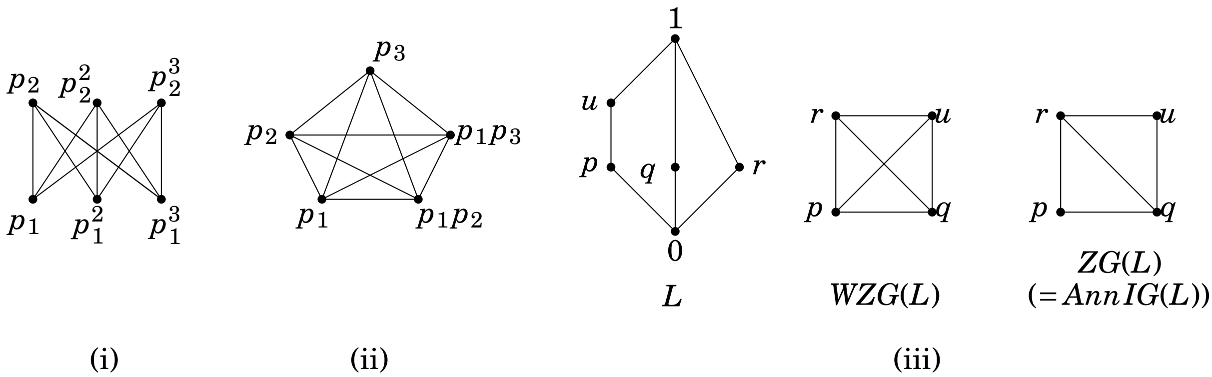

Figure 3

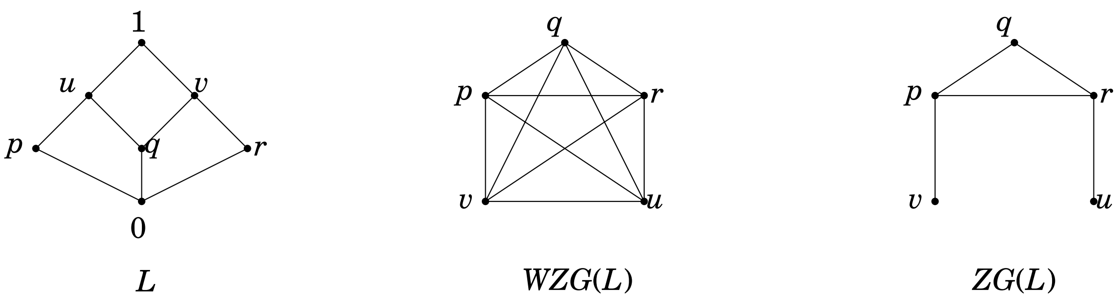

Figure 4

Example 3.12. If diam($WZG(L)$) = 2, then diam($ZG(L)$) = 2 or 3. To observe this, see Figure 5 and Figure 6.

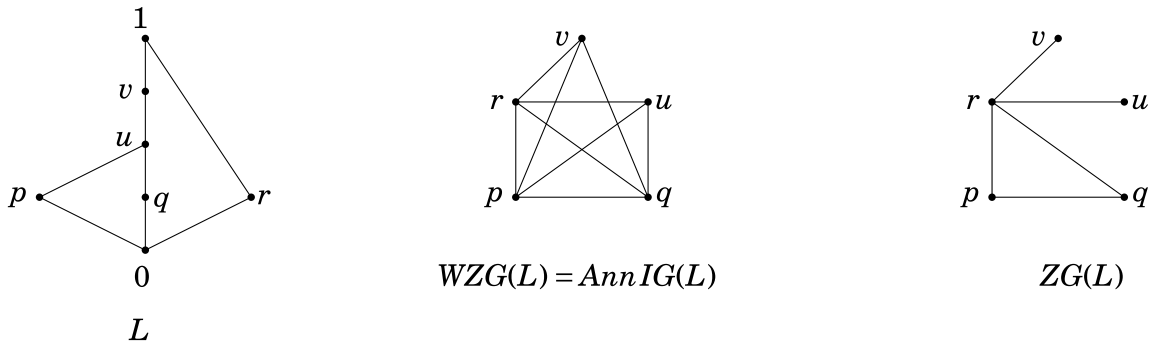

Figure 5

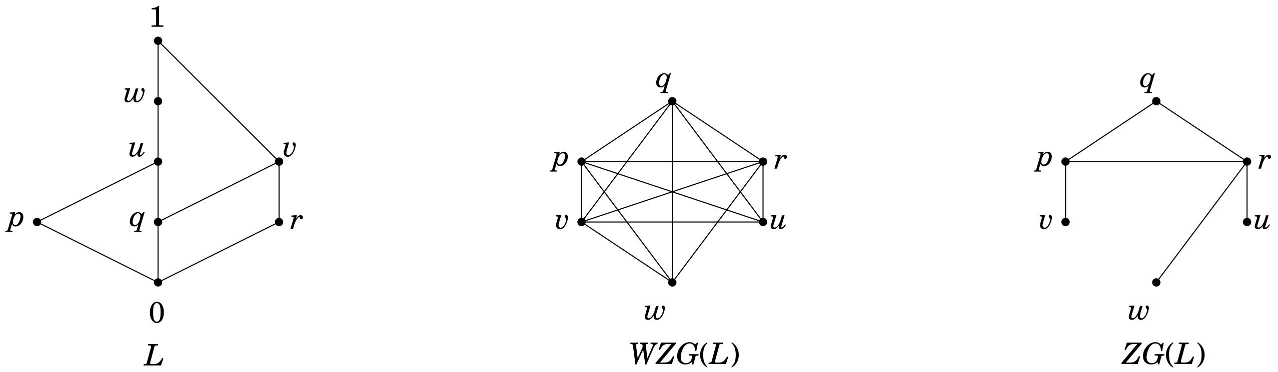

Figure 6

Example 3.13. If diam($WZG(L)$) = 1, then diam($AnnIG(L)$) = 1 or 2. To observe this, see Figure 2 and Figure 3(iii).

4. Conclusion

We have defined the weakly zero divisor graph of a lattice and determined its diameter, girth, domination number, and independence number. It is shown that the graph is a complete bipartite if and only if the number of atoms contained in the lattice is two. We have shown that the graph is not planar if the number of atoms contained in the lattice is five or more and also, characterized all lattices for which the graph is planar. We study the affinity between the weakly zero divisor graph, the zero divisor graph, and the annihilator-ideal graph of the lattices.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] D. F. Anderson, T. Asir, A. Badawi and T. T. Chelvam, *Graphs from Rings*, 1st edition, Springer International Publishing (2021).

[2] I. Beck, Coloring of commutative rings, *Journal of Algebra* **116**(1) (1988), 208 – 226, DOI: 10.1016/0021-8693(88)90202-5.

[3] G. Birkhoff, *Lattice Theory*, Colloquium Publications, Vol. **25**, American Mathematical Society, (1940), DOI: 10.2307/2268183.

[4] T. T. Chelvam and S. Nithya, A note on the zero divisor graph of a lattice, *Transactions on Combinatorics* **3**(3) (2014), 51 – 59, DOI: 10.22108/TOC.2014.5626.

[5] T. T. Chelvam and S. Nithya, Zero-divisor graph of an ideal of a near-ring, *Discrete Mathematics, Algorithms and Applications* **5**(1) (2013), 1350007, DOI: 10.1142/S1793830913500079

[6] E. Estaji and K. Khashyarmanesh, The zero divisor graph of a lattice, *Results in Mathematics* **61**(1) (2012), 1 – 11, DOI: 10.1007/s00025-010-0067-8.

[7] G. Grätzer, *Lattice Theory: Foundation*, 1st edition, Birkhäuser, Basel, xxx + 614 pages (2011), DOI: 10.1007/978-3-0348-0018-1.

[8] A. Khairnar and B. N. Waphare, Zero-divisor graphs of laurent polynomials and laurent power series, in: *Algebra and its Applications*, S. Rizvi, A. Ali and V. Filippis (editors), *Springer Proceedings in Mathematics & Statistics*, Vol. **174**, Springer, Singapore (2016), DOI: 10.1007/978-981-10-1651-6_21.

[9] V. Kulal, A. Khairnar and K. Masalkar, Annihilator ideal graph of a lattice, *Palestine Journal of Mathematics* **11**(iv) (2022), 195 – 204, URL: https://pjm.ppu.edu/paper/1214-annihilator-ideal-graph-lattice.

[10] M. J. Nikmehr, A. Azadi and R. Nikandish, The weakly zero-divisor graph of a commutative ring, *Revista de la Unión Matemática Argentina* **62**(1) (2021), 105 – 116, DOI: 10.33044/revuma.1677.

[11] D. B. West, *Introduction to Graph Theory*, 2nd edition, Pearson (2000).