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Research Article

Survival Signature Approach for Reliability Evaluation of Linear Consecutive *k*-out-of-*n* : *G* System

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Abstract. The present study concentrates on computing the reliability function for linear consecutive k-out-of-$n:G$ system with independent and identically distributed components. For systems where $2k \geq n$, we establish a formulation for the system survival signature. This is subsequently utilized to find a non-recursive representation of system reliability. The attained closed-form representation of system reliability empowers us to easily evaluate the performance of higher-order consecutive systems. The system signature is also evaluated with the assistance of the survival signature. Additionally, a method for calculating the system hazard rate function in light of its components' hazard rate functions is suggested. Both exponential and pareto distributions are considered in assessing the reliability function for such systems. A numerical example related to a quality control system provides a concrete illustration of the results achieved through the proposed method.

Keywords. Reliability, System signature, Survival signature, Linear consecutive k-out-of-$n:G$ system, Hazard rate function

Mathematics Subject Classification (2020). 62N05, 60K10

1. Introduction

Oil pipeline systems, telecommunication systems, quality control systems, microwave stations, and many other critical systems often use consecutive kind structures in their models and prime designs. A comprehensive review of existing literature reveals that the linear consecutive k-out-of-$n:F$ ($LC/k/n:F$) configuration is frequently examined. In this arrangement, n components are linked in sequence, and the system halts its operation upon the failure of k consecutive components. In relation to the $LC/k/n:F$ system, there exists a dual system referred to as the linear consecutive k-out-of-$n:G$ ($LC/k/n:G$) system. This system functions when there are not less than k successive functioning units in the system. Due to excessive applications in various advanced systems and industries, extensive research is in progress for studying the reliability of consecutive structures.

Numerous authors, including Bollinger and Salvia [1], Chiang and Niu [3], Cluzeau *et al.* [5], Kontoleaon [18], Lambiris and Papastavridis [21] and Lin [23] have given recursive equations or closed formulas in their studies for evaluating the reliability of $LC/k/n:F$ system. Using generating function technique, Lambiris and Papastavridis [21] developed a closed-form expression for $LC/k/n:F$ system reliability. Tong [33] first examined $LC/k/n:G$ system. Kuo *et al.* [19] further investigated the $LC/k/n:G$ system, identifying it as a mirror image of the $LC/k/n:F$ system. Zuo [37] again explored these systems through duality theory and developed a connection between the reliability bounds of both systems, by considering independent and identically distributed (iid) components. The literature survey reveals that multiple approaches such as combinatorial methods, recursive algorithms, and Markov chain imbedding (MCI) technology are employed in past to study consecutive systems. Fu and Hu [14] and Fu *et al.* [15] have estimated the reliabilities of $LC/k/n:F$ systems via the MCI technique. Gera [16] investigated consecutive-G systems having dependent components through matrix formulation using state space methods. A comprehensive survey on consecutive structures involving their properties and applications can be found in the review papers of Eryilmaz [9] and Triantafyllou [34], as well as in the documented monographs of Chiang and Hwang [2] and Kuo and Zuo [20].

The signature methodology given by Samaniego [31] is one of the significant tools in reliability engineering for analyzing various reliability characteristics of the coherent system. The researcher considered iid components and provided a mixed representation of the reliability function, in accordance with the common absolutely continuous distribution function. Navarro *et al.* [25] investigated coherent systems when the components are exchangeable and they defined minimal and maximal signatures. Navarro and Rychlik [27] extended Samaniego's mixture representation to systems having exchangeable components and obtained bounds for system reliability and expected lifetime. Through joint signature, Navarro *et al.* [29] explored the joint reliability for two systems when components are being used in sharing. Various approaches have been devised by Da *et al.* [7] as well as Jia *et al.* [17] for determining the system signature of different coherent systems. Coolen and Coolen-Maturi [6] broadened the concept of system signature to survival signature to encompass multiple types of components. The challenges faced in the reliability analysis of large-sized coherent systems having *iid* components or multiple components can be reduced by employing survival signature. Feng *et al.* [13] employed survival signature in the reliability analysis of real-world system involved in hydro power plant. Samaniego and Navarro [32] derived some theorems using survival signature and compared different coherent systems involving heterogeneous components. By computing survival signatures for coherent systems having multiple types of dependent components, Eryilmaz *et al.* [11] determined marginal and joint reliability importance measures. Li *et al.* [22] investigated the reliability of the hydraulic system through survival signature by considering load sharing. More recent overviews of survival signature and its applications to reliability evaluation and stochastic comparison can be found in the literature (Eryilmaz and Tuncel [12], Ding *et al.* [8], Yi *et al.* [35], Chopra and Kumar [4] and Qin and Coolen [30]).

The hazard rate is an essential concept in reliability analysis that can be utilized to assess a system's aging process. In addition, the failure rate ordering is extremely important since it compares system lifetimes based on hazard rate functions. The mixture representation of the coherent system lifetime provides a powerful tool for studying hazard rate function ordering properties. Samaniego [31], Navarro and Rychlik [27], Navarro [24] and Navarro and Rychlik [28] have exploited this concept in their research works.

There are some reliability studies on consecutive systems based on signature and survival signature methodology. Navarro and Eryilmaz [26], Eryilmaz [10] have adopted the signature approach in their works for studying consecutive structures. Navarro and Eryilmaz [26] considered exchangeable components in the $LC/k/n:G$ system and showed that when $2k \geq n$, the reliability function can be represented by the negative mixtures of two series (or parallel) systems. The authors also established the monotonicity and asymptotic properties of the system's mean residual life function. Eryilmaz [10] again considered consecutive systems with exchangeable components and obtained a representation for system reliability as a combination of the reliability of the order statistics. Eryilmaz and Tuncel [12] utilized a survival signature approach to probe the reliability of complex multi-state systems that cannot be repaired and contain multiple state components. They defined the j^{th} level survival signature for the $LC/k/n:G$ multi-state system, delineating the likelihood of the system surviving in a specific state or beyond. In connection with multi-state systems featuring diverse state components, Yi *et al.* [35] introduced the survival signature using a matrix form, applying the finite MCI technique. They put this method into practice to analyze the survival signature of particular $LC/k/n:G$ systems.

The aforementioned studies indicate that among consecutive-type structures mainly, $LC/k/n:F$ systems have been taken up. There are some reliability studies on $LC/k/n:G$ system that employ the concept of dual system Kuo *et al.* [19], Zuo [37] and Navarro and Eryilmaz [26]. In cases where $2k \geq n$, this article presents a novel approach, based on survival signature, for assessing the reliability characteristics of the $LC/k/n:G$ system with iid components. The present method, which is not based on duality, provides a closed-form representation for system reliability. As a result, it can be applied directly and effectively to analyze the performance of higher-order $LC/k/n:G$ systems. This is an attempt to employ the survival signature in assessing the reliability and hazard rate function specific to the $LC/k/n:G$ system. The results obtained are also compared with those from previous studies, offering a new perspective on this area of research.

The present article is divided into 5 sections. Section 2 includes notations, definitions, and basic formulas that are used in the current study. Section 3 presents the derivation of the main results for the survival signature, system signature, reliability, and hazard rate function concerning the $LC/k/n:G$ system with iid components. In Section 4, an illustrative example related to a quality control system is provided, along with the survival signature and reliability function for specific $LC/k/n:G$ systems. Finally, Section 5 discusses the conclusions, advantages, and future scope of the present study.

2. Definitions and Methodology

We have considered $LC/k/n:G$ system with iid components and presumed that the system's lifetime is T, while individual components have lifetimes $T_1, T_2, \dots, T_n$, all adhering to the common distribution function, $F(t)$. For a coherent system having n components, the system signature represented by $s = (s_1, s_2, \dots, s_i, \dots, s_n)$ is a probability vector, where s_i represents the probability of system breakdown on the ith component failure, i.e.,

$$s_i = P(T = T_{i:n}), \quad i = 1, 2, \dots, n,$$

where $T_{i:n}$ is the ith order statistic of the failure times of n components. In accordance with Samaniego [31], the reliability pertaining to the considered system as denoted by $R_G(k,n;t)$, is

$$R_G(k,n;t) = \sum_{i=1}^{n} s_i P(T_{i:n} > t).$$

Further, for the present $LC/k/n:G$ system, the survival signature represented by $\Phi_{k,n:G}(l)$, defines the probability that the system will operate if it has precisely l operational components in it. For exactly l functioning units, system state vector $x = (x_1, x_2, \dots, x_i, \dots, x_n)$ has exactly l number of ones and all others as zeros. There are $\binom{n}{l}$ such possible state vectors, and let S_l be the collection of all these state vectors. Mathematically, it will be

$$\Phi_{k,n:G}(l) = \binom{n}{l}^{-1} \sum_{x \epsilon S_l} \phi_{k,n:G}(x),$$

where $\Phi_{k,n:G}(x)$ is the binary valued structure function that is non-zero if the system is functioning. Using survival signature, $R_G(k,n;t)$ can be represented by the following formula

$$R_G(k,n;t) = \sum_{l=0}^{n} \Phi_{k,n:G}(l) \binom{n}{l} (F(t))^{n-l} (1 - F(t))^l. \tag{2.1}$$

Also, we have

$$\Phi_{k,n:G}(l) = \sum_{i=n-l+1}^{n} s_i. \tag{2.2}$$

Let $r_G(k,n;l)$ is the number of path sets concerning the present considered system having precisely l functioning units. The assumed function $r_G(k,n;l)$ can be computed by considering the sum of structure function values under the set S_l as given below:

$$r_G(k,n;l) = \sum_{x \epsilon S_l} \phi_{k,n:G}(x),$$

where $l = 0,1,\dots,n$.

Consequently, we have

$$r_G(k,n;l) = \binom{n}{l} \Phi_{k,n:G}(l), \tag{2.3}$$

where $l = 0,1,\dots,n$.

In the following section, eqns. (2.1) and (2.3) will be essential for examining the system's hazard rate function. This particular function refers to the likelihood that the system may fail in an upcoming time unit, even if it has been operating correctly in the time unit before. For $LC/k/n:G$ system, we have represented the hazard rate function as $h_G(k,n;t)$, and it is given by $h_G(k,n;t) = \lim_{\Delta t \to 0} \Pr(T \le t + \Delta t | T > t) = \frac{f_G(k,n;t)}{R_G(k,n;t)}$ for t such that $R_G(k,n;t)$ is positive, where $f_G(k,n;t) = -R_G{}'(k,n;t)$ is the probability density function (*pdf*) associated with the system's lifespan.

3. Main Results

3.1 Reliability Function of $LC/k/n:G$ System

The present study exploits the concept of survival signature in evaluating the system reliability. In the lemma below, we explicitly represent the number of path sets, $r_G(k,n;k+i)$, in the $LC/k/n:G$ system for a given fixed number of working components.

Lemma 3.1. *The number of path sets of $LC/k/n:G$ system consisting of iid components given that exactly $l = k+i$ components are working is*

$$r_G(k,n;k+i) = \binom{n-k}{i}(n-(k+i)+1), \tag{3.1}$$

where $i \in \{0,1,\dots,n-k\}$.

Proof. The derivation is based on the idea of examining the consecutive 1's in the state vectors present in the set S_l. The current assumed system will possibly work if there are not less than k functioning components in the system. Thus, we have considered $r_G(k,n;k+i)$ for $i \ge 0$. For $i = 0$, the system has precisely k working components, and the set S_k has total $\binom{n}{k}$ vectors. By combinatorial approach, we can find that the k functioning components can occur consecutively in n positions in $n-k+1$ ways. Therefore, the number of state vectors for which the system functions is $n-k+1$. Thus, we get

$$r_G(k,n;k) = n-k+1.$$

Hence, equation (3.1) holds for $i = 0$. Let us define these $n-k+1$ state vectors as:

$$w_{j,j+1,\dots,j+k-1} = (\underbrace{0,0,\dots,0}_{j-1},\underbrace{1,1,\dots,1}_{k},\underbrace{0,0,\dots,0}_{n-j-k+1}), \tag{3.2}$$

where $j = 1,2,\dots,n-k+1$.

Corresponding to $i = 1$, the $LC/k/n:G$ system has $k+1$ operational units and total $\binom{n}{k+1}$ state vectors in the set S_{k+1}. For finding path sets in this case, in each state vector given in equation (3.2), we can have the $(k+1)$th working component in $n-k$ positions. However, in these $(n-k+1)(n-k)$ cases some state vectors will repeat. Let us define

$$w^{j_1+k}_{j_1,j_1+1,\dots j_1+k-1} = (\underbrace{0,0,\dots,0}_{j_1-1},\underbrace{1,1,\dots,1}_{k+1},\underbrace{0,0,\dots,0}_{n-j_1-k}), \quad j_1 = 1,2,\dots,n-k \tag{3.3}$$

and

$$w^{j_2-1}_{j_2,j_2+1,\dots,j_2+k-1} = (\underbrace{0,0,\dots,0}_{j_2-2},\underbrace{1,1,\dots,1}_{k+1},\underbrace{0,0,\dots,0}_{n-j_2-k+1}), \quad j_2 = 2,3,\dots,n-k+1. \tag{3.4}$$

The $n-k$ state vectors given in equations (3.3) and (3.4) have $k+1$ consecutive 1's and form identical sets, so in the above consideration, $(n-k)$ state vectors are common. Therefore, we have

$$r_G(k,n;k+1) = (n-k+1)(n-k) - (n-k)$$

i.e.,

$$r_G(k,n;k+1) = (n-k)^2.$$

Hence equation (3.1) is true for $i = 1$.

Considering $i = m$, where $m \le n-k$, we can have m more 1's in $\binom{n-k}{m}$ ways. Among the $\binom{n-k}{m}(n-k+1)$ state vectors, repeated vectors are $\binom{n-k-1}{m-1}(n-k)$ as in each state vector given in equations (3.3) and (3.4), $m-1$ more 1's can be included in $\binom{n-k-1}{m-1}$ ways. Thus,

$$r_G(k,n;k+m) = (n-k+1)\binom{n-k}{m} - \binom{n-k-1}{m-1}(n-k)$$

or

$$r_G(k,n;k+m) = \binom{n-k}{m}(n-(k+m)+1).$$

Hence the result is true for all $i \in \{0,1,\dots,n-k\}$. □

Theorem 3.1. *If $2k \ge n$, the survival signature for the $LC/k/n:G$ system consisting of iid components is given by*

$$\Phi_{k,n:G}(l) = \begin{cases} \binom{n}{l}^{-1}\binom{n-k}{l-k}(n-l+1), & \text{for } l \ge k, \\ 0, & \text{otherwise}, \end{cases}$$

where $l \in \{0,1,\dots,n\}$.

Proof. Clearly, for less than k functioning components, the present studied system will not work, i.e., if $l < k$, then

$$\Phi_{k,n:G}(l) = 0. \tag{3.5}$$

However, for $l \ge k$ and $2k \ge n$, the survival signature is non-zero as we have chances of having k consecutive functioning components in the system. Using Lemma 3.1 and replacing $k+i$ with

l, the expression will be

$$r_G(k,n;l) = \binom{n-k}{l-k}(n-l+1). \tag{3.6}$$

Based on equations (2.3), (3.5), and (3.6), the obtained closed formula for system survival signature is

$$\Phi_{k,n:G}(l) = \begin{cases} \binom{n}{l}^{-1}\binom{n-k}{l-k}(n-l+1), & \text{for } l \geq k, \\ 0, & \text{otherwise,} \end{cases} \tag{3.7}$$

where $l \in \{0,1,\dots,n\}$. □

Theorem 3.2. *For $2k \geq n$, the non-recursive formula for the system reliability $R_G(k,n;t)$ for the LC/k/n : G system consisting of iid components having reliability function $R(t)$ is*

$$R_G(k,n;t) = \sum_{l=k}^{n} \binom{n-k}{l-k}(n-l+1)(R(t))^l(1-R(t))^{n-l}.$$

Proof. Using equation (2.1), the reliability concerning the considered present system possessing *iid* components with reliability $R(t)$ is

$$R_G(k,n;t) = \sum_{l=0}^{n} \Phi_{k,n:G}(l)\binom{n}{l}(R(t))^l(1-R(t))^{n-l}. \tag{3.8}$$

Further,

$$R_G(k,n;t) = \sum_{l=0}^{k-1} \Phi_{k,n:G}(l)\binom{n}{l}(R(t))^l(1-R(t))^{n-l} + \sum_{l=k}^{n} \Phi_{k,n:G}(l)\binom{n}{l}(R(t))^l(1-R(t))^{n-l}. \tag{3.9}$$

Using equations (3.7) and (3.9), we attain

$$R_G(k,n;t) = \sum_{l=k}^{n} \binom{n-k}{l-k}(n-l+1)(R(t))^l(1-R(t))^{n-l}.$$ □

3.2 Samaniego Signature of $LC/k/n : G$ System

Theorem 3.3. *For $2k \geq n$, the closed formula for ith element of Samaniego system signature of the LC/k/n : G system is represented as*

$$s_i = \begin{cases} \binom{n}{k}^{-1}\binom{n-i}{k}\left(\frac{ki}{n-i-k+1}-1\right), & \textit{for } i < n-k+1, \\ \binom{n}{k}^{-1}(n-k+1), & \textit{for } i = n-k+1, \\ 0, & \textit{otherwise,} \end{cases} \tag{3.10}$$

where $i \in \{1,2,\dots,n\}$.

Proof. Using equation (3.7), the another mixture representation of the survival signature will be

$$\Phi_{k,n:G}(l) = \begin{cases} \binom{n}{k}^{-1}\binom{l}{k}(n-l+1), & \text{for } l \geq k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

Using equation (2.2), the Samaniego signature is described as the recurrence relation of the survival signature, i.e.,

$$s_i = \Phi_{k,n:G}(n-i+1) - \Phi_{k,n:G}(n-i). \tag{3.12}$$

Using eqns. (3.11) and (3.12), for $i < n-k+1$, the closed formula for the system signature becomes

$$s_i = \binom{n}{k}^{-1}\binom{n-i}{k}\left(\frac{ki}{n-i-k+1}-1\right).$$

For $i = n-k+1$, the survival signature $\Phi_{k,n:G}(n-i)$ in equation (3.12) is zero. Therefore,

$$s_{n-k+1} = \binom{n}{k}^{-1}(n-k+1).$$

Since the survival signature involved in equation (3.11) vanishes for all $l < k$, thus the system signature also vanishes for all $i > n-k+1$. □

We evaluated the system signature of some $LC/k/n:G$ systems in Table 1, using the direct formula given in equation (3.10). In their article, Navarro and Eryilmaz [26] employed a different approach to study the signature of some $LC/k/n:G$ systems. Our findings align with the results of their study.

Table 1. System signature of $LC/k/n:G$ systems

System	System signature
$LC/2/3:G$	$\left(\frac{1}{3},\frac{2}{3},0\right)$
$LC/2/4:G$	$\left(0,\frac{1}{2},\frac{1}{2},0\right)$
$LC/3/5:G$	$\left(\frac{2}{10},\frac{5}{10},\frac{3}{10},0,0\right)$
$LC/3/6:G$	$\left(0,\frac{4}{10},\frac{4}{10},\frac{2}{10},0,0\right)$
$LC/4/5:G$	$\left(\frac{3}{5},\frac{2}{5},0,0,0\right)$
$LC/4/7:G$	$\left(\frac{1}{7},\frac{3}{7},\frac{11}{35},\frac{4}{35},0,0,0\right)$

3.3 Hazard Rate Function of the $LC/k/n:G$ System

Lemma 3.2. *Let $\Phi_{k,n:G}(l)$ represent the survival signature for $LC/k/n:G$ system possessing iid components having reliability function $R(t)$. The pdf for the $LC/k/n:G$ system lifetime, $f_G(k,n;t)$ is*

$$f_G(k,n;t) = \sum_{l=k}^{n}\Phi_{k,n:G}(l)\eta(l)\binom{n}{l}(R(t))^{l-1}(1-R(t))^{n-l}f(t),$$

where $\eta(l) = l - \frac{(l-k)(n-l+2)}{n-l+1}$ is a positive real-valued function and $f(t)$ be the pdf of the iid components lifetimes.

Proof. Let $f_G(k,n;t)$ be the *pdf* of the system lifetime T. The same can be derived by considering the derivative of $R_G(k,n;t)$ with a negative sign, i.e.,

$$f_G(k,n;t) = -\frac{d}{dt}R_G(k,n;t).$$

Using equations (3.8) and (3.7), we obtained $f_G(k,n;t)$ as

$$f_G(k,n;t) = -\frac{d}{dt}\left\{\sum_{l=k}^{n}\Phi_{k,n:G}(l)\binom{n}{l}(R(t))^{l}(1-R(t))^{n-l}\right\}. \tag{3.13}$$

To make above computations easy, we have replaced $\Phi_{k,n:G}(l)\binom{n}{l}$ with a_l, components' reliability and distribution function by 'p' and 'q', respectively. Accordingly, equation (3.13) reduces to

$$\begin{aligned} f_G(k,n;t) &= -\frac{d}{dt}\sum_{l=k}^{n} a_l p^l q^{n-l} \\ &= -\left\{ k a_k p^{k-1} q^{n-k} p' + \sum_{l=k}^{n-1} ((l+1)a_{l+1} - (n-l)a_l) p^l q^{n-l-1} p' \right\}, \end{aligned} \tag{3.14}$$

where p' is the derivative of 'p'.

For the present system, the recurrence relation for a_l can be written as

$$a_{l+1} = \frac{(n-l)^2}{(n-l+1)(l-k+1)} a_l .$$

Therefore, equation (3.14) becomes

$$f_G(k,n;t) = -\left\{ k a_k p^{k-1} q^{n-k} p' + \sum_{l=k}^{n-1} a_{l+1}\left((l+1) - \frac{(l-k+1)(n-l+1)}{n-l}\right) p^l q^{n-l-1} p' \right\},$$

i.e.,

$$f_G(k,n;t) = -\left\{ k a_k p^{k-1} q^{n-k} p' + \sum_{l=k+1}^{n} a_l \left(l - \frac{(l-k)(n-(l-1)+1)}{n-l+1}\right) p^{l-1} q^{n-(l-1)-1} p' \right\},$$

which gives

$$f_G(k,n;t) = \sum_{l=k}^{n} a_l \left(l - \frac{(l-k)(n-l+2)}{n-l+1}\right) p^{l-1} q^{n-l} (-p').$$

Thus, we obtain

$$f_G(k,n;t) = \sum_{l=k}^{n} \Phi_{k,n:G}(l)\eta(l)\binom{n}{l}(R(t))^{l-1}(1-R(t))^{n-l}(-R'(t)),$$

where $\eta(l) = l - \frac{(l-k)(n-l+2)}{n-l+1}$ is a positive real valued function since $k \le l \le n$ and $2k \ge n$. The function $\eta(l) = k$ for $l = k$ and $\eta(l) = 2k - n$ for $l = n$. For $LC/n/n : G$ system, the function $\eta(n) = n$.

So, *pdf* of the system lifetime is

$$f_G(k,n;t) = \sum_{l=k}^{n} \Phi_{k,n:G}(l)\eta(l)\binom{n}{l}(R(t))^{l-1}(1-R(t))^{n-l} f(t), \tag{3.15}$$

where $f(t)$ is *pdf* of *iid* components lifetime. □

This equation (3.15) gives the *pdf* for the system lifetime as the combination of *pdf* for the components' lifetime and reliability. Based on the lifetimes of the *iid* components, the subsequent theorem provides a closed formula for the hazard rate function of the system.

Theorem 3.4. *Let T be the lifetime for the $LC/k/n : G$ system. If the components lifetimes be iid with common reliability function $R(t)$ then the generalized mixture representation of the system hazard rate function $h_G(k,n;t)$ is*

$$h_G(k,n;t) = \frac{\sum_{l=k}^{n} \Phi_{k,n:G}(l)\eta(l)\binom{n}{l}(R(t))^l(1-R(t))^{n-l} h(t) \sum_{l=k}^{n} \Phi_{k,n:G}(l)\binom{n}{l}(R(t))^l(1-R(t))^{n-l}}{},$$

wherein the function $h(t)$ is components' hazard rate function and $\eta(l) = l - \frac{(l-k)(n-l+2)}{n-l+1}$ be the positive real valued function.

Proof. For the $LC/k/n:G$ system, the hazard rate function $h_G(k,n;t)$ according to equations (2.1) and (3.15) is represented as

$$h_G(k,n;t)=\frac{\sum_{l=k}^{n}\Phi_{k,n:G}(l)\eta(l)\binom{n}{l}(R(t))^{l-1}(1-R(t))^{n-l}f(t)}{\sum_{l=k}^{n}\Phi_{k,n:G}(l)\binom{n}{l}(R(t))^{l}(1-R(t))^{n-l}}.$$

Since $h(t)=\frac{f(t)}{R(t)}$, thus the system hazard rate function is

$$h_G(k,n;t)=\frac{\sum_{l=k}^{n}\Phi_{k,n:G}(l)\eta(l)\binom{n}{l}(R(t))^{l}(1-R(t))^{n-l}h(t)}{\sum_{l=k}^{n}\Phi_{k,n:G}(l)\binom{n}{l}(R(t))^{l}(1-R(t))^{n-l}}. \tag{3.16}$$

Equation (3.16) represents the hazard rate function of the considered system as the combination of components' hazard rate function. This is the generalized mixture representation for the hazard rate function of the $LC/k/n:G$ system. For exponentially distributed components, the hazard rate function $h_G(k,n;t)$ of the system reduces to

$$h_G(k,n;t)=\frac{\lambda\sum_{l=k}^{n}\Phi_{k,n:G}(l)\eta(l)\binom{n}{l}(e^{-\lambda t})^{l}(1-e^{-\lambda t})^{n-l}}{\sum_{l=k}^{n}\Phi_{k,n:G}(l)\binom{n}{l}(e^{-\lambda t})^{l}(1-e^{-\lambda t})^{n-l}},$$

where $\eta(l)=l-\frac{(l-k)(n-l+2)}{n-l+1}$ and λ is a parameter. □

4. Numerical Example

The important results discussed in the preceding section can be applied to a variety of practical problems. Consider a quality control system that inspects a sample of n produced units. This can be modeled as the $LC/k/n:G$ system. During the inspection, if any k consecutive units are found to be good, the sample is accepted. In our example, we consider a sample of 6 units and accept the sample if 4 consecutive units are found good during the inspection. This situation can be visualized as the $LC/4/6:G$ system. In this example, all components follow an exponential distribution with a constant failure rate λ, described by the distribution function $F(t)$. The components' reliability is expressed as $R(t)=e^{-\lambda t}$, while $f(t)$ designates the *pdf* and $h(t)$ signifies the hazard rate function. Using equation (3.7), we can determine the survival signature pertaining to the system as follows:

$$\Phi_{4,6:G}(l)=\begin{cases}\binom{6}{l}^{-1}\binom{2}{l-4}(6-l+1), & \text{for } l\geq 4,\\ 0, & \text{otherwise,}\end{cases}$$

Here, l varies from 0 to 6, the survival signature $\Phi_{4,6:G}(l)$ vanishes for $l\leq 3$. However $\Phi_{4,6:G}(4)$ is 1/5, $\Phi_{4,6:G}(5)$ is 2/3 and becomes 1 when all components are working.

Mathematically,

$$\Phi_{4,6:G}(l)=\begin{cases}0, & \text{for } l\leq 3,\\ \frac{1}{5}, & \text{for } l=4,\\ \frac{2}{3}, & \text{for } l=5,\\ 1, & \text{for } l=6.\end{cases} \tag{4.1}$$

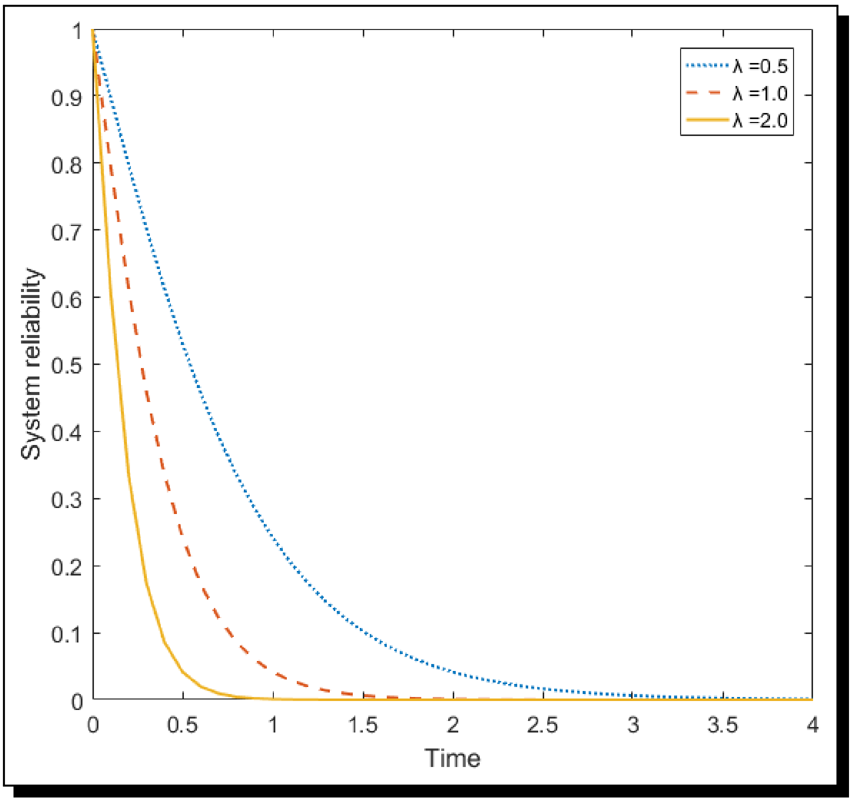

Figure 1. Reliability vs. Time

Using Theorem 3.2 and equation (4.1), we get

$$R_G(4,6;t) = 3(R(t))^4 - 2(R(t))^5, \tag{4.2}$$

i.e.,

$$R_G(4,6;t) = 3e^{-4\lambda t} - 2e^{-5\lambda t}.$$

Figure 1 demonstrates a reduction in the reliability of the $LC/4/6:G$ system as time progresses, with the increased hazard rate of its components. The closed formula for the $LC/4/6:G$ system's hazard rate function in respect of the components' hazard rate function as derived from equation (3.16) is as follows:

$$h_G(4,6;t) = \frac{\sum_{l=4}^{6} \Phi_{4,6:G}(l)\eta(l)\binom{6}{l}(R(t))^l(1-R(t))^{6-l}h(t)}{R_G(4,6;t)},$$

where $\eta(l) = l - \frac{(l-4)(8-l)}{7-l}$. Using equation (4.2), we obtain

$$h_G(4,6;t) = \frac{12-10R(t)}{3-2R(t)}h(t).$$

Since the components lifetime is exponentially distributed, therefore,

$$h_G(4,6;t) = \left(\frac{12-10e^{-\lambda t}}{3-2e^{-\lambda t}}\right)\lambda.$$

Figure 2 illustrates that the $LC/4/6:G$ system's hazard rate increases as the components' hazard rate increases.

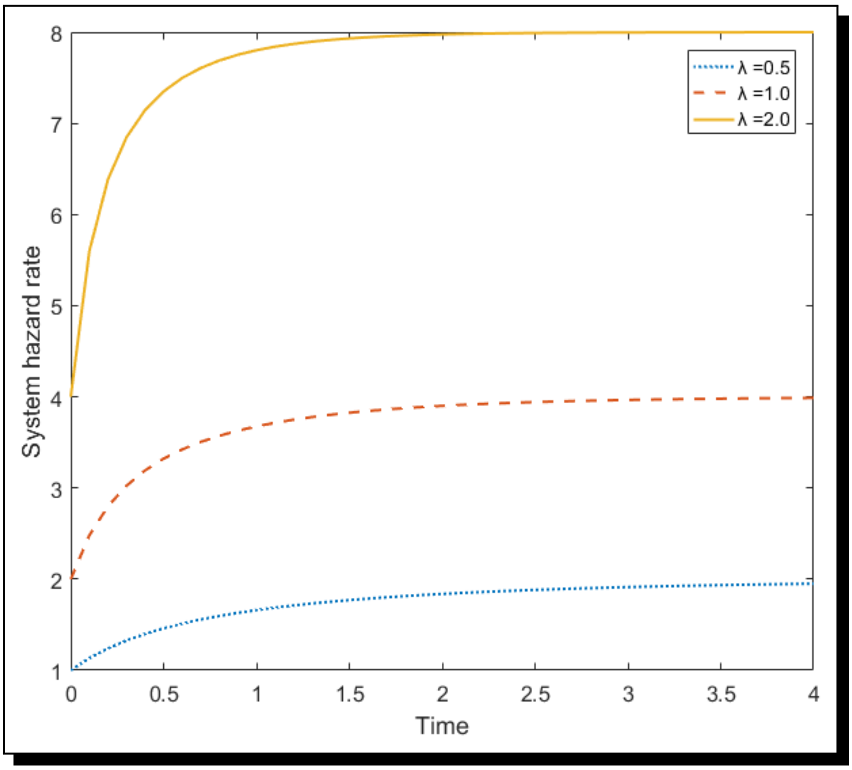

Figure 2. System hazard rate vs. Time

In Table 2, we employed the proposed approach to calculate the survival signature and system reliability function for different $LC/k/n:G$ systems, all featuring independent exponentially distributed components. Zhang [36] assessed the reliability of several $LC/k/n:G$ systems in his research, and our current findings are in line with his outcomes. Table 3 extends our analysis to a few more systems, using the Pareto distribution for calculations with a reliability function $R(t) = (b/t)^a$, where 'b' is lower bound of data and 'a' is shape parameter.

Table 2. Reliability of $LC/k/n:G$ systems with exponentially distributed components

System	Survival signature $(\Phi(0), \Phi(1), \dots, \Phi(n))$	System reliability
$LC/3/4:G$	$\left(0,0,0,\frac{1}{2},1\right)$	$(e^{-4\lambda t})(2e^{\lambda t}-1)$
$LC/3/5:G$	$\left(0,0,0,\frac{3}{10},\frac{4}{5},1\right)$	$(e^{-4\lambda t})(3e^{\lambda t}-2)$
$LC/4/7:G$	$\left(0,0,0,0,\frac{4}{35},\frac{3}{7},\frac{6}{7},1\right)$	$(e^{-5\lambda t})(4e^{\lambda t}-3)$
$LC/5/10:G$	$\left(0,0,0,0,0,\frac{1}{42},\frac{5}{42},\frac{1}{3},\frac{2}{3},1,1\right)$	$(e^{-6\lambda t})(6e^{\lambda t}-5)$
$LC/7/12:G$	$\left(0,0,0,0,0,0,0,\frac{1}{132},\frac{5}{99},\frac{2}{11},\frac{5}{11},\frac{5}{6},1\right)$	$(e^{-8\lambda t})(6e^{\lambda t}-5)$
$LC/9/14:G$	$\left(0,0,0,0,0,0,0,0,0,\frac{3}{1001},\frac{25}{1001},\frac{10}{91},\frac{30}{91},\frac{5}{7},1\right)$	$(e^{-10\lambda t})(6e^{\lambda t}-5)$

Table 3. Reliability of $LC/k/n:G$ systems with Pareto distributed components

System	Survival signature $(\Phi(0),\Phi(1),\dots,\Phi(n))$	System reliability
$LC/2/3:G$	$\left(0,0,\frac{2}{3},1\right)$	$-\left(\frac{b}{t}\right)^{2a}\left(\left(\frac{b}{t}\right)^{a}-2\right)$
$LC/4/6:G$	$\left(0,0,0,\frac{1}{5},\frac{2}{3},1\right)$	$-\left(\frac{b}{t}\right)^{4a}\left(2\left(\frac{b}{t}\right)^{a}-3\right)$
$LC/5/9:G$	$\left(0,0,0,0,0,\frac{5}{126},\frac{4}{21},\frac{1}{2},\frac{8}{9},1\right)$	$-\left(\frac{b}{t}\right)^{5a}\left(4\left(\frac{b}{t}\right)^{a}-5\right)$
$LC/6/11:G$	$\left(0,0,0,0,0,0,\frac{1}{77},\frac{5}{66},\frac{8}{33},\frac{6}{11},\frac{10}{11},1\right)$	$-\left(\frac{b}{t}\right)^{6a}\left(5\left(\frac{b}{t}\right)^{a}-6\right)$
$LC/8/13:G$	$\left(0,0,0,0,0,0,0,0,\frac{2}{429},\frac{5}{143},\frac{20}{143},\frac{5}{13},\frac{10}{13},1\right)$	$-\left(\frac{b}{t}\right)^{8a}\left(5\left(\frac{b}{t}\right)^{a}-6\right)$
$LC/11/14:G$	$\left(0,0,0,0,0,0,0,0,0,0,0,\frac{1}{91},\frac{9}{91},\frac{3}{7},1\right)$	$-\left(\frac{b}{t}\right)^{11a}\left(3\left(\frac{b}{t}\right)^{a}-4\right)$

5. Conclusion

Survival signature has many advantages over structure function in the system reliability analysis owing to its easy applicability to large complex systems with heterogeneous components. The survival signature methodology has practical applications in evaluating system reliability and hazard rate function. In the present study, firstly we have established the expression concerning the survival signature for the $LC/k/n:G$ system. The closed formula for $LC/k/n:G$ system reliability and hazard rate has also been worked out using survival signature. The obtained representations are non-recursive and can be conveniently adopted for the higher-order $LC/k/n:G$ systems. To demonstrate the results, we examined a numerical example pertaining to the quality control system. Further, we have evaluated the survival signature and reliability function across various $LC/k/n:G$ systems by assuming exponential and Pareto distributions. The findings are very much relevant for industry people as the studied $LC/k/n:G$ system has enormous applications in modern critical systems. The present study also indicates that the survival signature is an influential tool and can be used easily for the performance analysis of more coherent systems in the future.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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