Communications in Mathematics and Applications
Vol. 15, No. 1, pp. 191–202, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i1.2430

http://www.rgnpublications.com

Research Article

A Novel Approximation on the Solution of Systems of Ordinary Differential Equations

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Received: September 26, 2023 **Accepted:** January 1, 2024

Abstract. In this paper, the initial-value problem for the system of first-order differential equations is considered. To solve this problem, we construct a fitted difference scheme using the finite difference method, which is based on integral identities for the quadrature formula with integral term remainder terms. Next, we prove first-order convergence for the method in the discrete maximum norm. Although this scheme has the same rate of convergence, it has more efficiency and accuracy compared to the classical Euler scheme. Two test problems are solved by using the proposed method and the classical Euler method, which confirm the theoretical findings. The numerical results obtained from here show that the proposed method is reliable, efficient, and accurate.

Keywords. System of differential equation, Finite difference method, Convergence

Mathematics Subject Classification (2020). 34A30, 65L05, 65L12, 65L20, 65L70

1. Introduction

Systems of Ordinary Differential Equations (SODEs) appear in widely in many scientific process ranging from physics and engineering to medicine and biology. SODEs play a significant role in modeling of applied sciences. Traditionally, these models occur in the mathematical formulations of the various problems such as deep neural network, electric circuits, feedback control systems, iodate-arsenous acid reactions, cell growth, etc. (Batiha [5], Goldberg and Schwartz [13], Goodwine, [14], Logemann and Ryan [20], Perko [21], and Sideris [25]).

Many approaches about SODEs were proposed which are analytical or numerically solutions. Researchers studied and improved some methods for getting exact solution except for classic methods such as direct integration method (Kydyraliev and Urdaletova [18]). In addition, Adrianova [2], Grigorian [15], Ross [22], and Seiler and Seiß [24] discussed existence, uniqueness and stability analysis for SODEs.

Although there are many analytical solution techniques, finding exact solution can not be always possible for SODEs. Therefore, in recent years numerically methods catch on with mathematicians and numerous numerically approaches were suggested (Lambert [19]). For example, parallel direct method (Amodio and Trigiante [4]), differential transformation method (Abdel-Halim Hassan [1]), variational iteration methods (Biazar and Ghazvini [6], and Darvishi *et al*. [9]), Adomian decomposition method (Biazar *et al*. [7], and Kaya [17]), Laplace transform-Adomian decomposition method (Dogan [10]), vectorized algorithm method (Dufera [11]), the Monte Carlo method (Ermakov and Smilovitskiy [12]), Sawi transformation (Higazy and Aggarwal [16]), psuedo-spectral method (Saravi *et al*. [23]), etc.

Motivated by the above works, we consider the following system of first order linear initial value problem:

$$Lu_1 \equiv u_1'(t) + a_1(t)u_1(t) + b_1(t)u_2(t) = f_1(t), \quad t \in I, \tag{1.1}$$

$$Lu_2 \equiv u_2'(t) + a_2(t)u_2(t) + b_2(t)u_1(t) = f_2(t), \quad t \in I, \tag{1.2}$$

$$u_1(0) = \gamma_1,\ u_2(0) = \gamma_2, \tag{1.3}$$

where $I = (0,T]$ and $a_k(t) \ge \alpha_k > 0$, $b_k(t)$, $f_k(t)$ are assumed to be continuous in $\bar{I} = [0,T]$ and such that (1.1)-(1.3) has a unique solution $u_k(t) \in C^1(\bar{I})$ satisfying the given initial conditions and γ_k are real constants ($k = 1,2$) (more details see, Ross [22]).

The present paper aim is to present an efficient numerical approach for solving (1.1)-(1.3). The method derives from a finite difference scheme given on a uniform mesh that is constituted by appropriate quadrature formulas with the weight, whose remainder terms are in integral form. This ends up with in a local truncation error including only first order derivatives of the exact solution, which is the most important advantage of the method compared to the classical methods such as Euler, and hence allows of analysis of the convergence.

The schedule of the present paper is as follows: In Section 2, some properties for the exact solution of (1.1)-(1.3) and its derivative are given. The difference scheme is constructed on uniform mesh in Section 3. In Section 4, error estimates and stability analysis are proved and convergence is addressed. In Section 5, we formulate the difference scheme by an algorithm and present two examples for illustrate the theoretical results provided. In the last section, summary of the main conclusions are given.

2. The Continuous Problem

Here we give a few properties of the solutions of (1.1)-(1.3), which are needed in the next sections for the analysis of the appropriate numerical solutions. Furthermore, for any continuous function $g(t)$, we use $\|g\|_\infty = \max_{0 \le t \le T} |g(t)|$ and C denotes a generic positive constant throughout the paper.

Lemma 2.1. *Let a_k, b_k, $f_k \in C(\bar{I})$ and*

$$\alpha_1^{-1}\alpha_2^{-1}\|b_1\|_\infty\|b_2\|_\infty < 1.$$

Then for the solution $u_k(t)$ of (1.1)-(1.3) *the following estimates hold:*

$$\|u_k\| \le C, \tag{2.1}$$

$$\|u_k'\| \le C, \tag{2.2}$$

for $k = 1,2$.

Proof. We can rewrite eqn. (1.1):

$$u_1'(t) + a_1(t)u_1(t) = f_1(t) - b_1(t)u_2(t),$$

$$u_1(t) = \gamma_1 \exp\left(-\int_0^t a_1(\tau)d\tau\right) + \int_0^t (f_1(\tau) - b_1(\tau)u_2(\tau))\exp\left(-\int_\tau^t a_1(\eta)d\eta\right)d\tau,$$

$$\begin{aligned}|u_1(t)| &\le |\gamma_1|\exp\left(-\int_0^t \alpha_1 d\tau\right) + \int_0^t (|f_1(\tau)| + |b_1(\tau)||u_2(\tau)|)\exp\left(-\int_\tau^t \alpha_1 d\eta\right)d\tau \\ &\le |\gamma_1|\exp(-\alpha_1 t) + \alpha_1^{-1}(\|f_1\|_\infty + \|b_1\|_\infty\|u_2\|_\infty)(1-\exp(-\alpha_1 t)) \\ &\le |\gamma_1| + \alpha_1^{-1}(\|f_1\|_\infty + \|b_1\|_\infty\|u_2\|_\infty).\end{aligned} \tag{2.3}$$

Also, the similar relation can be written for (1.2):

$$\|u_2\| \le |\gamma_2| + \alpha_2^{-1}(\|f_2\|_\infty + \|b_2\|_\infty\|u_1\|_\infty). \tag{2.4}$$

Considering (2.3) and (2.4) inequalities together, we obtain

$$\|u_1\| \le |\gamma_1| + \alpha_1^{-1}\{\|f_1\|_\infty + (|\gamma_2| + \alpha_2^{-1}\|f_2\|_\infty)\|b_1\|_\infty\}[1-(\alpha_1\alpha_2)^{-1}\|b_1\|_\infty\|b_2\|_\infty]^{-1}.$$

Taking account with this last inequality in (2.4), we have

$$\|u_2\| \le |\gamma_2| + \alpha_2^{-1}\{\|f_2\|_\infty + (|\gamma_1| + \alpha_1^{-1}\|f_1\|_\infty)\|b_2\|_\infty\}[1-(\alpha_1\alpha_2)^{-1}\|b_1\|_\infty\|b_2\|_\infty]^{-1},$$

which lead to (2.1) for $k = 1,2$. Hereby, from (1.1) we can write

$$\begin{aligned}|u_1'(t)| &\le |f_1(t)| + |a_1(t)||u_1(t)| + |b_1(t)||u_2(t)| \\ &\le \|f_1\|_\infty + \|a_1\|_\infty\|u_1\|_\infty + \|b_1\|_\infty\|u_2\|_\infty,\end{aligned}$$

and from (2.1) we arrive at (2.2) for $k = 1$. From (1.2), evaluation for $k = 2$ can be found similarly. Thus the proof is complete. □

3. The Difference Scheme and Mesh

3.1 The Mesh

Let ω_N be a uniform mesh on $\bar{I}$:

$$\omega_N = \{t_i = ih,\ 1 \le i \le N,\ h = T/N\}$$

and

$$\bar{\omega}_N = \omega_N \cup \{t_0 = 0\}.$$

For any mesh function $g(t)$, we use $g_i = g(t_i)$ and moreover $y_i^{(k)}$ to denote approximations of $u_k(t)$ at t_i (for $k = 1,2$) and

$$g_{\bar{t},i} = (g_i - g_{i-1})/h, \quad \|g\|_{\infty,\omega} = \|g\|_{\infty,\omega_N} := \max_{1\le i\le N}|g_i|.$$

3.2 The Difference Scheme

We multiply (1.1) and (1.2) system equation by $\Phi_{ki}(t)$ (for $k=1,2$) basis functions, respectively, and later integrate over (t_{i-1},t_i) to obtain approximation for the problem. Then we get following identity:

$$h^{-1}\int_{t_{i-1}}^{t_i} Lu_k(t)\Phi_{ki}(t)dt = h^{-1}\int_{t_{i-1}}^{t_i} f_k(t)\Phi_{ki}(t)dt, \quad 1\le i\le N. \tag{3.1}$$

The basis functions

$$\Phi_{ki}(t) = e^{-\int_t^{t_i} a_k(t)d\tau}, \quad t_{i-1}<t<t_i,$$

which are solutions of the following problems:

$$\begin{cases} -\frac{d}{dt}\Phi_{ki}(t) + a_k(t)\Phi_{ki}(t) = 0, & t_{i-1}<t<t_i, \\ \Phi_{ki}(t_i) = 1. \end{cases}$$

The relation (3.1) is rewritten as for $t\in(t_{i-1},t_i)$ and $(k=1,2)$,

$$\begin{aligned} &h^{-1}\int_{t_{i-1}}^{t_i} u_k'(t)\Phi_{ki}(t)dt + h^{-1}\int_{t_{i-1}}^{t_i} a_k(t)u_k(t)\Phi_{ki}(t)dt + h^{-1}\int_{t_{i-1}}^{t_i} b_k(t)u_{3-k}(t)\Phi_{ki}(t)dt \\ &\quad = h^{-1}\int_{t_{i-1}}^{t_i} f_k(t)\Phi_{ki}(t)dt. \end{aligned} \tag{3.2}$$

Using the formulas (2.1) and (2.2) from [3] on interval (t_{i-1},t_i) taking into account (3.2), we have following precise relation:

$$\ell u_{ki} \equiv A_{ki}u_{k\bar{t},i} + B_{ki}u_{(3-k)\bar{t},i} + C_{ki}u_{ki} + D_{ki}u_{(3-k)ki} = F_{ki} - R_{ki}, \quad 1\le i\le N,$$

with

$$A_{ki} = h^{-1}\int_{t_{i-1}}^{t_i} [1+(t-t_i)a_k(t)]\Phi_{ki}(t)dt,$$

$$B_{ki} = h^{-1}\int_{t_{i-1}}^{t_i} (t-t_i)b_k(t)\Phi_{ki}(t)dt,$$

$$C_{ki} = h^{-1}\int_{t_{i-1}}^{t_i} a_k(t)\Phi_{ki}(t)dt,$$

$$D_{ki} = h^{-1}\int_{t_{i-1}}^{t_i} b_k(t)\Phi_{ki}(t)dt,$$

$$F_{ki} = h^{-1}\int_{t_{i-1}}^{t_i} f_k(t)\Phi_{ki}(t)dt,$$

and remainder terms

$$R_{ki} = h^{-1}\int_{t_{i-1}}^{t_i} dt b_k(t)\Phi_{ki}(t)\int_{t_{i-1}}^{t_i} u_{3-k}'(\xi)K_0(t,\xi)d\xi, \tag{3.3}$$

$$K_0(t,\xi) = T_0(t-\xi) - h^{-1}(t-t_{i-1}), \;\; T_0(\lambda)=1, \;\; \lambda\ge 0; \;\; T_0(\lambda)=0, \;\; \lambda<0.$$

Eventually, we propose the following difference scheme for approximation for (1.1)-(1.3) where y_{ki} the solution of (1.1)-(1.3) system at mesh point t_i:

$$\ell y_{ki} \equiv A_{ki}y_{k\bar{t},i} + B_{ki}y_{(3-k)\bar{t},i} + C_{ki}y_{ki} + D_{ki}y_{(3-k)i} = F_{ki}, \quad 1\le i\le N, \tag{3.4}$$

$$\ell y_{k0} \equiv \gamma_k, \tag{3.5}$$

for $k=1,2$.

In addition, we suggest another difference scheme, which can be easily obtained using the implicit Euler method, known as the classical method, as an alternative to the approximate solution of system (1.1)-(1.3):

$$\ell y_{ki} \equiv y_{k\bar{t},i} + a_{ki} y_{ki} + b_{ki} y_{(3-k)i} = f_{ki}, \quad 1 \le i \le N, \tag{3.6}$$

$$\ell y_0^{(k)} \equiv \gamma_k. \tag{3.7}$$

4. Convergence Analysis

We define $z_{ki} = y_{ki} - u_{ki}$, $1 \le i \le N$ error functions in order to investigate the convergence of presented method whose the solution of the following discrete problem

$$\ell z_{ki} \equiv R_{ki}, \quad 1 \le i \le N, \tag{4.1}$$

$$z_{k0} = 0, \tag{4.2}$$

for $k = 1,2$.

Lemma 4.1. *If $a_k, b_k, f_k \in C(I)$, then for the truncation errors we get*

$$\|R_k\|_{\infty,\omega_N} \le CN^{-1},$$

for $k = 1,2$.

Proof. From (3.3), we can write

$$|R_{ki}| \le h^{-1} \int_{t_{i-1}}^{t_i} dt |b_k(t)||\Phi_{ki}(t)| \int_{t_{i-1}}^{t_i} |u'_{3-k}(\xi)||K_0(t,\xi)| d\xi$$

$$\le Ch^{-1} \int_{t_{i-1}}^{t_i} dt |\Phi_{ki}(t)| \int_{t_{i-1}}^{t_i} |u'_{3-k}(\xi)| d\xi,$$

and by virtue of Lemma 2.1 and $0 < \Phi_{ki}(t) \le C$, $k = 1,2$, we easily obtain

$$|R_{ki}| \le Ch.$$

□

Lemma 4.2. *Let z_{ki} be the solution* (4.1)-(4.2) *holds true and*

$$\alpha_1^{-1}\alpha_2^{-1}(\|B_1\|_{\infty,\omega_N} + \|D_1\|_{\infty,\omega_N})(\|B_2\|_{\infty,\omega_N} + \|D_2\|_{\infty,\omega_N}) < 1.$$

Then

$$\|z_k\|_{\infty,\omega_N} \le C\{\|R_1\|_{\infty,\omega_N} + \|R_2\|_{\infty,\omega_N}\},$$

for $k = 1,2$.

Proof. We can rewrite eqn. (4.1) as follows

$$\ell z_{ki} \equiv A_{ki} z_{k\bar{t},i} + B_{ki} z_{(3-k)\bar{t},i} + C_{ki} z_{ki} + D_{ki} z_{(3-k)i} = R_{ki}, \quad 1 \le i \le N.$$

For $k = 1$:

$$A_{1i} z_{1\bar{t},i} + B_{1i} z_{2\bar{t},i} + C_{1i} z_{1i} + D_{1i} z_{2i} = R_{1i},$$

$$A_{1i} z_{1\bar{t},i} + C_{1i} z_{1i} = Q_i \tag{4.3}$$

with

$$Q_i = R_{1i} - B_{1i} z_{2\bar{t},i} - D_{1i} z_{2i}.$$

Applying maximum principle [8], from (4.3), we get

$$\begin{aligned} |z_{1i}| &\le \alpha_1^{-1} \|Q\|_{\infty,\omega_N} \\ &\le \alpha_1^{-1} \|R_1\|_{\infty,\omega_N} + \alpha_1^{-1}(\|B_1\|_{\infty,\omega_N} + \|D_1\|_{\infty,\omega_N}) \|z_2\|_{\infty,\omega_N}. \end{aligned} \tag{4.4}$$

We can write similar relation for z_{2i}:

$$|z_{2i}| \le \alpha_2^{-1} \|R_2\|_{\infty,\omega_N} + \alpha_2^{-1}(\|B_2\|_{\infty,\omega_N} + \|D_2\|_{\infty,\omega_N}) \|z_1\|_{\infty,\omega_N}. \tag{4.5}$$

Substituting (4.5) inequality in (4.4):

$$\begin{aligned} \|z_1\|_{\infty,\omega_N} \le \alpha_1^{-1} \|R_1\|_{\infty,\omega_N} &+ \alpha_1^{-1}(\|B_1\|_{\infty,\omega_N} + \|D_1\|_{\infty,\omega_N}) \\ &\times [\alpha_2^{-1} \|R_2\|_{\infty,\omega_N} + \alpha_2^{-1}(\|B_2\|_{\infty,\omega_N} + \|D_2\|_{\infty,\omega_N}) \|z_1\|_{\infty,\omega_N}]. \end{aligned}$$

Hereby, we can obtain the following estimate for $z_i^{(1)}$:

$$\|z_1\|_{\infty,\omega_N} \le C\{\|R_1\|_{\infty,\omega_N} + \|R_2\|_{\infty,\omega_N}\}.$$

In a similar way, the estimation for z_{2i} is easily achieved. □

Now we give the main convergence result.

Theorem 4.1. *Let u_k be the solution of* (1.1)-(1.3) *and y_k the solution* (3.4)-(3.5). *Then*

$$\|y_k - u_k\|_{\infty,\bar{\omega}_N} \le C N^{-1}.$$

Proof. This follows immediately by combining the previous lemmas. □

5. Algorithm and Numerical Results

In this section, we present numerical results obtained by applying the novel numerical scheme (3.4)-(3.5) to two particular problems. Also, we present numerical results obtained by using implicit Euler method in (3.6)-(3.7).

We rewrite difference scheme (3.4)-(3.5):

$$y_{ki} = \frac{hF_{ki}}{A_{ki} + hC_{ki}} - \frac{B_{ki} + hD_{ki}}{A_{ki} + hC_{ki}} y_{(3-k)i} + \frac{A_{ki}}{A_{ki} + hC_{ki}} y_{k(i-1)} + \frac{B_{ki}}{A_{ki} + hC_{ki}} y_{(3-k)(i-1)},$$

and for $k = 1$, we obtain

$$y_{1i} = \frac{hF_{1i}}{A_{1i} + hC_{1i}} - \frac{B_{1i} + hD_{1i}}{A_{1i} + hC_{1i}} y_{2i} + \frac{A_{1i}}{A_{1i} + hC_{1i}} y_{1(i-1)} + \frac{B_{1i}}{A_{1i} + hC_{1i}} y_{2(i-1)}, \tag{5.1}$$

and for $k = 2$, we obtain

$$y_{2i} = \frac{hF_{2i}}{A_{2i} + hC_{2i}} - \frac{B_{2i} + hD_{2i}}{A_{2i} + hC_{2i}} y_{1i} + \frac{A_{2i}}{A_{2i} + hC_{2i}} y_{2(i-1)} + \frac{B_{2i}}{A_{2i} + hC_{2i}} y_{1(i-1)}. \tag{5.2}$$

If we use the substitution method (or Cramer's rule) to solve (5.1)-(5.2), we have

$$y_{ki} = \frac{G_{ki} + M_{ki} y_{k(i-1)} + N_{ki} y_{(3-k)(i-1)}}{P_{ki}},$$

where

$$G_{ki} = (A_{(3-k)i} + hC_{(3-k)i}) hF_{ki} - (B_{ki} + hD_{ki}) hF_{(3-k)i},$$

$$M_{ki} = A_{ki}(A_{(3-k)i} + hC_{(3-k)i}) - B_{(3-k)i}(B_{ki} + hD_{ki}),$$
$$N_{ki} = hB_{ki}C_{(3-k)i} - hD_{ki}A_{(3-k)i},$$
$$P_{ki} = (A_{ki} + hC_{ki})(A_{(3-k)i} + hC_{(3-k)i}) - (B_{ki} + hD_{ki})(B_{(3-k)i} + hD_{(3-k)i}),$$

for $k = 1,2$.

Example 5.1. We consider the test problem:

$$u_1'(t) + 2u_1(t) - u_2(t) = e^{-4t}, \quad t \in (0,1],$$
$$u_2'(t) + 4u_2(t) - 3u_1(t) = 6e^t, \quad t \in (0,1],$$
$$u_1(0) = 0, \quad u_2(0) = 1,$$

whose exact solution is given by

$$u_1(t) = \frac{1}{4}(2e^t - e^{-t} - e^{-5t}),$$
$$u_2(t) = \frac{1}{4}(3e^{-5t} - e^{-t} + 6e^t - 4e^{-4t}).$$

The computational results for the test problem are presented in Table 1. We define the exact errors $E_i^{(k;N)}$ and the computed maximum pointwise errors $E^{(k;N)}$ for any N as follows:

$$E_i^{(k;N)} = |y_{ki} - u_{ki}|, \quad 0 \le i \le N; \quad E^{(k;N)} = \max_{0\le i\le N} E_i^{(k;N)}, \quad k = 1,2.$$

Example 5.2. We consider the another test problem:

$$u_1'(t) + 2u_1(t) + (1+t)u_2(t) = e^t, \quad t \in (0,1],$$
$$u_2'(t) + 4u_2(t) + e^{-t}u_1(t) = e^{-2t}, \quad t \in (0,1],$$
$$u_1(0) = 0, \quad u_2(0) = 1,$$

whose exact solution is unknown.

Therefore, in order to calculate the maximum pointwise error, we use the double mesh principle. Define the double mesh differences to be:

$$e_i^{(k;N)} = |y_{ki}^N - y_{k(2i)}^{2N}|, \quad 0 \le i \le N, \; k = 1,2,$$

where y_{ki}^N and $y_{k(2i)}^{2N}$, respectively, denote the numerical solutions obtained using N and $2N$. Thus, the maximum pointwise-errors are taken as

$$e^{(k;N)} = \max_{0\le i\le N} e_i^{(k;N)}, \quad k = 1,2.$$

The computational results for this test problem are presented in Table 2.

To validate the applicability of the method, two test problems are considered for numerical experimentation for different values of the mesh points. The numerical results are listed in terms of the approximate errors (see Tables 1-2). From the results in these tables, we observe that the maximum pointwise errors ($E^{(k;N)}$ and $e^{(k;N)}$) decreases monotonically and the increases. Further, the convergence of the method is shown by the log-log plot (see Figures 1-4). From Figures 1-4, we notice that the maximum pointwise errors are bounded by $O(N^{-1})$, which is proved in Theorem 4.1.

Table 1. Comparison of $E^{(k;N)}$ of both methods for Example 5.1

N	$E^{(1;N)}$(PM)	$E^{(1;N)}$(EM)	$E^{(2;N)}$(PM)	$E^{(2;N)}$(EM)
32	8.959E-5	8.511E-3	1.070E-4	1.027E-2
64	2.240E-5	4.362E-3	2.674E-5	5.107E-3
128	5.601E-6	2.207E-3	6.689E-6	2.546E-3
256	1.400E-6	1.110E-3	1.672E-6	1.271E-3
512	3.500E-7	5.566E-4	4.180E-7	6.350E-4
1024	8.751E-8	2.787E-4	1.045E-7	3.174E-4

Table 2. Comparison of $e^{(k;N)}$ of both methods for Example 5.2

N	$e^{(1;N)}$(PM)	$e^{(1;N)}$(EM)	$e^{(2;N)}$(PM)	$e^{(2;N)}$(EM)
32	8.949E-5	1.791E-3	2.856E-6	6.690E-3
64	2.237E-5	9.368E-4	7.157E-7	3.464E-3
128	5.592E-6	4.794E-4	1.788E-7	1.763E-3
256	1.398E-6	2.426E-4	4.472E-8	8.895E-4
512	3.495E-7	1.220E-4	1.118E-8	4.468E-4
1024	8.738E-8	6.119E-5	2.795E-9	2.239E-4

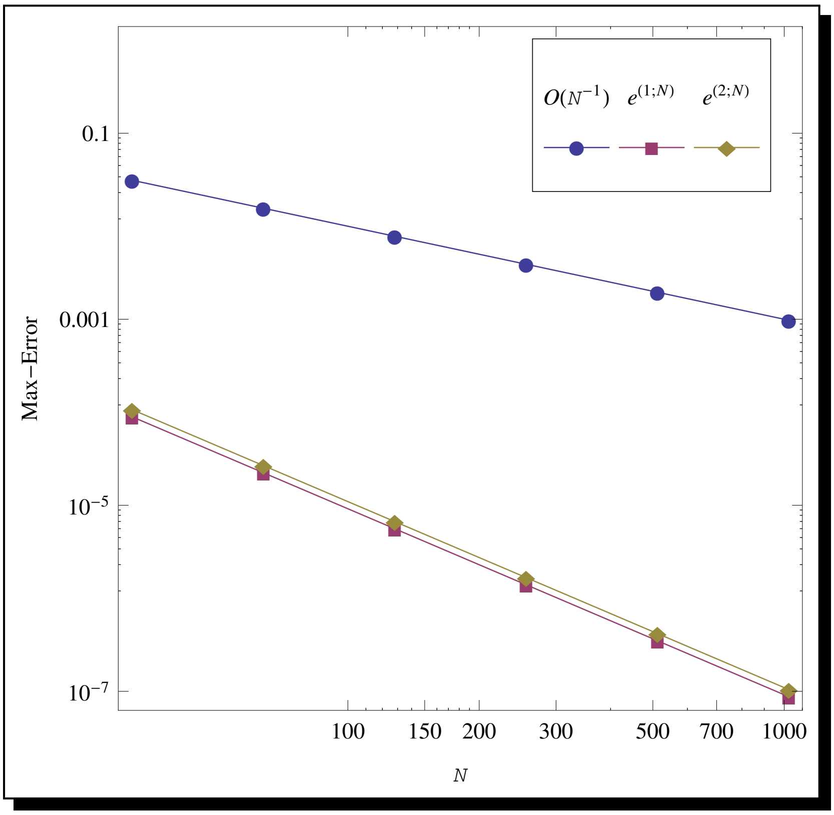

Figure 1. Maximum point-wise errors of log-log plot for Example 5.1 (PM)

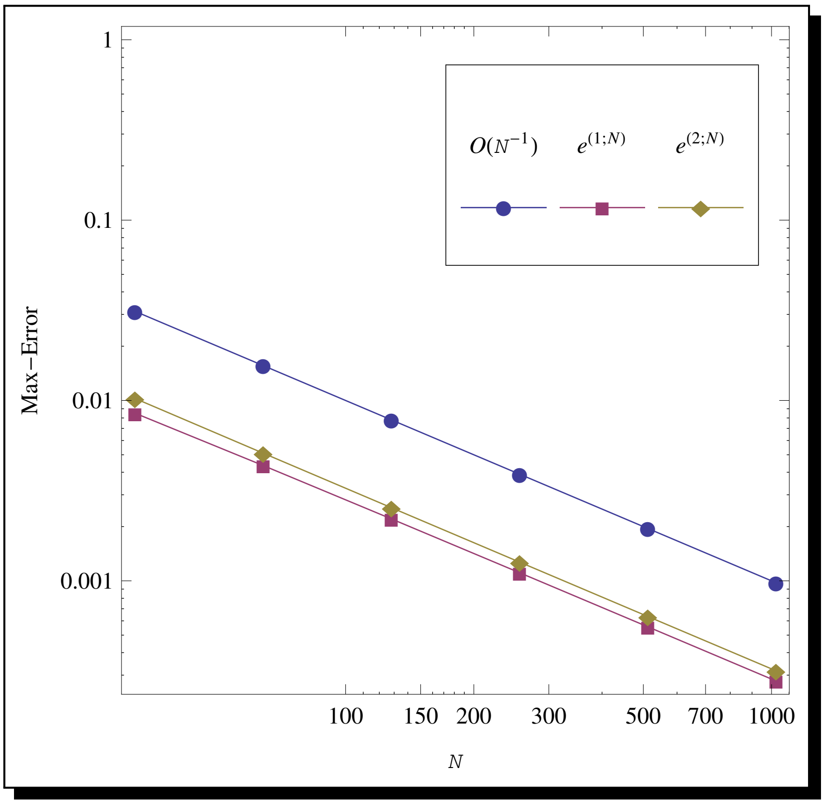

Figure 2. Maximum point-wise errors of log-log plot for Example 5.1 (EM)

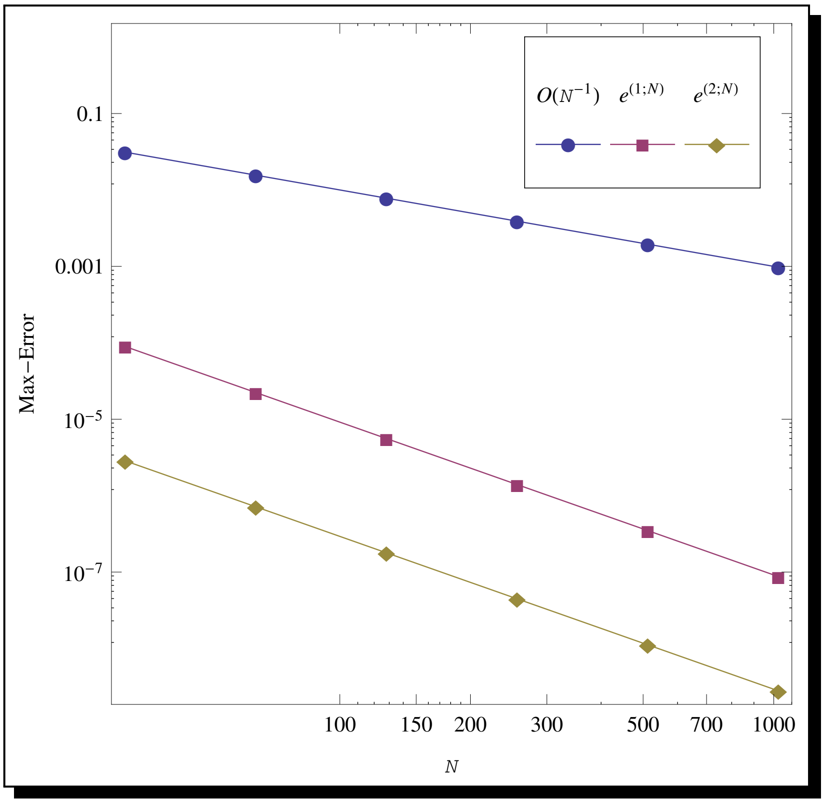

Figure 3. Maximum point-wise errors of log-log plot for Example 5.2 (PM)

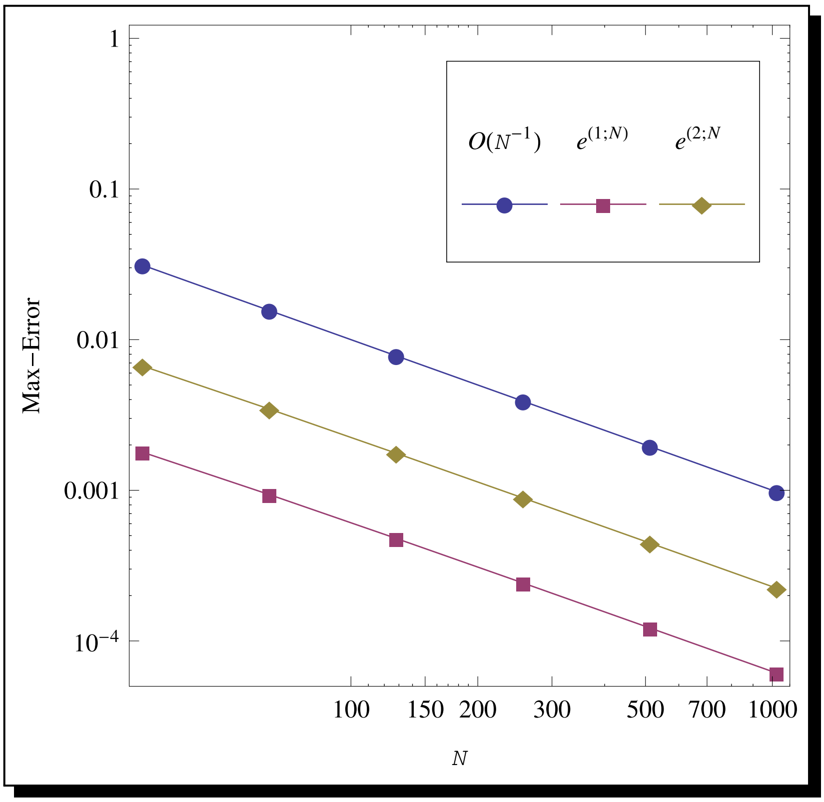

Figure 4. Maximum point-wise errors of log-log plot for Example 5.2 (EM)

6. Conclusion

In this paper, using the finite difference method, we have proposed a novel and efficient scheme for solving the initial value problem of the inhomogeneous linear coupled system of ordinary differential equations. We have proved that this scheme convergence first-order in the discrete maximum norm. Two test problems solved using the present method and the classical method (Euler) are discussed. The comparison of the maximum error values obtained from both methods is presented in Tables 1-2 and demonstrated in Figures 1-4. Considering these tables and figures it is observed that the presented method is more effective than the classical method, although they have the same convergence order ($O(N^{-1})$). Theoretical results represents an undergoing studies within a further research such as the systems of delay differential equations and the singularly perturbed systems of differential equations.

Acknowledgement

We thank the editor(s) and the referee(s) for their valuable comments.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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