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Research Article

Rough Ideal Statistical Convergence via Generalized Difference Operators in Intuitionistic Fuzzy Normed Spaces

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Abstract. This study focuses on investigating the concept of rough ideal statistical convergence for generalized difference sequences in intuitionistic fuzzy normed spaces. We have studied the algebraic and topological properties of rough ideal statistical limit points for generalized difference sequence. Apart from this, we also investigated rough ideal statistical cluster points, the relation between rough I-statistical limit points and rough I-statistical cluster points for generalized difference sequence in intuitionistic fuzzy normed spaces.

Keywords. Ideal statistical convergence, Rough ideal statistical convergence, Intuitionistic fuzzy normed space, Difference Sequence

Mathematics Subject Classification (2020). 0A05, 40A35, 46S50, 54E70

1. Introduction

Statistical convergence has applications in various areas of analysis and number theory, and it has been studied extensively to understand the convergence behavior of different types of sequences and series. It allows mathematicians to explore convergence in more general and flexible settings, which can lead to new insights and results in various branches of mathematics. The first edition of Zygmund's monograph [36], which was published in Warsaw in 1935, introduced the concept of statistical convergence. Initially, Steinhaus [34] and Fast [15] developed the idea of statistical convergence, and then later reintroduced by Schuenberg [32]. It is based on the concept of natural density. The Natural density of the set K_p of natural numbers, denoted by $\delta(K_p)$, is defined as $\delta(K_p)=\lim_{p\to\infty}\frac{|K_p|}{n}$ such that $|K_p|$ signifies the number of elements in $K_p=\{p\in N: p\leq n\}$. A sequence $y=\{y_p\}$ in R is said to be statistical convergent to $y_0\in R$ if for every $\epsilon>0$, the natural density of the set $\{p\in N:\|y_p-y_0\|\geq\epsilon\}$ is zero.

Initially, Kizmaz [19] proposed the concept of difference sequence spaces as $Z(\Delta)=\{y=(y_p):(\Delta y_p)\in Z\}$ for $Z=l_\infty,c,c_0$ i.e. spaces of all bounded sequences, convergent sequences and null sequences respectively, where $\Delta y=(\Delta y_p)=(y_p-y_{p+1})$. In particular, $l(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are also Banach spaces, relative to a norm induced by $\|y_p\|_\Delta=|y_1|+\sup_k|\Delta y_p|$ and the generalized difference sequence spaces was defined as (see, Et and Çolak [14]): $Z(\Delta^m y_p)=\{y=(y_p):(\Delta^m y_p)\in Z\}$ for $Z=l_\infty,c,c_0$, where $\Delta^m y=(\Delta^m y_p)=(\Delta^{m-1}y_p-\Delta^{m-1}y_{p+1})$ so that $\Delta^m y_p=\sum_{r=0}^{m}(-1)^r\binom{m}{r}y_{k+r}$.

The proposal of rough convergence was first suggested by Phu [28] for sequences on finite-dimensional normed linear spaces in 2001. After that, numerous authors were inspired to work on various sequence-spaces, including those for double sequences (Malik and Maity [24, 25]), triple sequences [11], lacunary sequences (Kişi and Dündar [18]), ideals (Malik and Maity [25], Pal *et al.* [26]) etc. Despite this, it has been established in a variety of spaces, see Antal *et al.* [2], Arslan and Dündar [3], Banerjee and Paul [9], Debnath and Subramanian [11] etc. It was later expanded to infinite-dimensional normed linear spaces (Phu [29]). In 2008, Aytar [5] also worked on same and introduced new generalized convergence named rough statistical convergence.

In 2000, Kostryko *et al.* [21] proposed the idea of ideal convergence (I-convergence) by generalizing the statistical convergence with the helps of ideals, for more details, we refer Balcerzak *et al.* [6], Banerjee and Banerjee [7, 8], Kostruko *et al.* [20], Lahiri and Das [23]. In 2011, with the help of ideals a new generalization named rough ideal statistical convergent in normed spaces was defined by Das *et al.* [10]. They studied its fundamental properties. Demir and Gümüş [12] have examined the idea of rough convergence in difference sequences. On the other hand, Demir and Gümüş [13] defined the rough statistical convergence of the (Δy_p) sequences and examined some topological and algebraic properties for the set of rough statistical limit points. Finally, Karabacak and Or [16] introduced the concept of rough convergence and rough statistical convergence for generalized difference sequences in normed linear spaces.

Zadeh [35] proposed the idea of fuzzy sets as an extension of classical sets to study the vague qualitative or quantitative data. Fuzzy set theory is an efficient technique for describing uncertainty and ambiguity. As a generalisation of fuzzy sets that can deal with both the degree of non-membership and the degree of membership of the components for the given set, Atanassov [4] introduced intuitionistic fuzzy sets in 1986. Park [27] was the first to develop intuitionistic fuzzy metric spaces and the concept of Cauchy sequences in the same spaces using continuous t-norm and continuous t-conorms. Saadati and Park [30] expanded this idea to intuitionistic fuzzy topological spaces. In 2008, Karakus *et al.* [17] defined statistical convergence in intuitionistic fuzzy normed spaces (*IFNS*) and established some topological as well as algebraic results in the same space. Recently, Antal *et al.* [1] introduced rough statistical convergence in intuitionistic fuzzy normed spaces and examined some fundamental results.

In this article, we have proposed the concept of rough statistical convergence and rough ideal statistical convergence for generalized difference sequences in the setup of intuitionistic fuzzy normed spaces. We studied the algebraic and topological properties of rough ideal statistical limit points for generalized difference sequences. Also, investigated rough ideal statistical cluster points, and the relation between rough I-statistical limit points and rough I-cluster points for generalized difference sequence in intuitionistic fuzzy normed spaces.

2. Preliminaries

In this section, we recall the basic definitions of *IFNS*, Statistical convergence and rough statistical convergence in *IFNS*. We should use the following definition of *IFNS* in this paper given by Lael and Nourouzi [22] as follows:

Definition 2.1 ([22]). Let Y be a vector space and ψ,η be two fuzzy sets on $Y \times R$, then the triplet (Y,ψ,η) is called an *Intuitionistic fuzzy normed space* (*IFNS*) if for each $y,z \in Y$ and $p,q \in R$, the following conditions are satisfied:

(i) $\psi(y,q) = 0$ and $\eta(y,q) = 1$ for $q \notin R^+$,

(ii) $\psi(y,q) = 1$ and $\eta(y,q) = 0$ for $q \in R^+$ iff $y = 0$,

(iii) $\psi(\alpha y;q) = \psi\left(y;\frac{q}{|\alpha|}\right)$ and $\eta(\alpha y;q) = \psi\left(y;\frac{q}{|\alpha|}\right)$, $\alpha \neq 0$ is a real number,

(iv) $\min\{\psi(y,p),\psi(z,q)\} \le \psi(y+z;p+q)$ and $\max\{\eta(y,p),\eta(z,q)\} \ge \eta(y+z;p+q)$,

(v) $\lim_{q\to\infty}\psi(y,q) = 1$, $\lim_{q\to 0}\psi(y,q) = 0$, $\lim_{q\to\infty}\eta(y,q) = 0$, $\lim_{q\to 0}\eta(y,q) = 1$.

Throughout the article, we will denote intuitionistic fuzzy normed spaces as (Y,ψ,η) with (ψ,η) intuitionistic fuzzy norms.

Antal *et al*. [1] defined statistical convergence and rough statistical convergence in *IFNS* (Y,ψ,η) as follow:

Definition 2.2 ([1]). Let (Y,ψ,η) be an *intuitionistic fuzzy normed space*. A sequence $y = \{y_p\}$ in Y is said to be *statistical convergent* to $\xi \in Y$ with respect to norm (ψ,η) if for every $\epsilon > 0$ and $\lambda \in (0,1)$

$$\lim_{n\to\infty}\frac{1}{n}|\{p \le n : \psi(y_p-\xi;\epsilon) \le 1-\lambda \text{ or } \eta(y_p-\xi;\epsilon) \ge \lambda\}| = 0$$

or

$$\delta(\{p \le n : \psi(y_p-\xi;\epsilon) \le 1-\lambda \text{ or } \eta(y_p-\xi;\epsilon) \ge \lambda\}) = 0.$$

It is denoted by $y_p \xrightarrow{st_{(\psi,\eta)}} \xi$ or $st_{(\psi,\eta)}\text{-}\lim_{p\to\infty} y_p = \xi$.

Definition 2.3 ([1]). Let (Y,ψ,η) be an *intuitionistic fuzzy normed space*. A sequence $y = \{y_p\}$ in Y is said to be *rough statistical convergent* to $\xi \in Y$ with respect to norm (ψ,η) for some non-negative number r if for every $\epsilon > 0$ and $\lambda \in (0,1)$

$$\lim_{n\to\infty}\frac{1}{n}|\{p \le n : \psi(y_p-\xi;r+\epsilon) \le 1-\lambda \text{ or } \eta(y_p-\xi;r+\epsilon) \ge \lambda\}| = 0$$

or

$$\delta(\{p \le n : \psi(y_p - \xi; r+\epsilon) \le 1-\lambda \text{ or } \eta(y_p - \xi; r+\epsilon) \ge \lambda\}) = 0.$$

It is denoted by $y_p \xrightarrow{r\text{-}st_{(\psi,\eta)}} \xi$ or $r\text{-}st_{(\psi,\eta)}\text{-}\lim_{p\to\infty} y_p = \xi$.

Let $st_{(\psi,\eta)}\text{-}LIM^r_{y_p}$ denotes the the set of all rough statistical limit points of the sequence $y = \{y_p\}$.

Next, we have mentioned *ideal statistical convergent* in intuitionistic fuzzy normed spaces.

Definition 2.4 ([31])**.** Let (Y,ψ,η) be an *intuitionistic fuzzy normed space*. A sequence $y = \{y_m\}$ in Y is said to be *ideal statistical convergent* to $\rho \in Y$ with respect to norm (ψ,η) if for every $\epsilon > 0$ and $\lambda \in (0,1)$:

$$\left\{n : \frac{1}{n}|\{m \le n : \psi(y_m - \rho; \epsilon) \le 1-\lambda \text{ or } \eta(y_m - \rho; \epsilon) \ge \lambda\}| > \delta\right\} \in I.$$

It is denoted by $y_m \xrightarrow{I\text{-}st_{(\psi,\eta)}} \rho$.

Next the statistical convergence of generalized difference sequences in intuitionistic fuzzy normed spaces is as given below.

Definition 2.5 ([33])**.** Let (Y,ψ,η) be an *intuitionistic fuzzy normed space*, $\Delta^m y = (\Delta^m y_p)$ where $m \in N$, be a generalized difference sequence in Y is said to be Δ^m*-statistical convergent* to $\xi \in Y$ with respect to norm (ψ,η) if for every $\epsilon > 0$ and $\lambda \in (0,1)$:

$$\lim_{n\to\infty} \frac{1}{n}|\{p \le n : \psi(\Delta^m y_p - \xi; \epsilon) \le 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; \epsilon) \ge \lambda\}| = 0$$

or

$$\delta(\{p \le n : \psi(\Delta^m y_p - \xi; \epsilon) \le 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; \epsilon) \ge \lambda\}) = 0.$$

It is denoted by $\Delta^m y_p \xrightarrow{st_{(\psi,\eta)}} \xi$ or $st_{(\psi,\eta)}\text{-}\lim_{p\to\infty} \Delta^m y_p = \xi$.

3. Main Results

In this section, we introduce the notions of *Rough Statistical Convergence* and *Rough Ideal Statistical Convergence* for generalized difference sequences in *IFNS* and then examined some results for r-I-statistical limit points for a generalized difference sequence. Throughout the article, I is an admissible ideal and $(\Delta^m y_p) = \Delta^{m-1} y_p - \Delta^{m-1} y_{p+1}$, where $m \in N$, be the generalized difference sequence.

Definition 3.1. Let (Y,ψ,η) be an *intuitionistic fuzzy normed space*, $\Delta^m y = (\Delta^m y_p)$ where $m \in N$, be a generalized difference sequence in Y is said to be *rough* Δ^m*-statistical convergent* to $\xi \in Y$ with respect to norm (ψ,η) for some non-negative number r if for every $\epsilon > 0$ and $\lambda \in (0,1)$:

$$\lim_{n\to\infty} \frac{1}{n}|\{p \le n : \psi(\Delta^m y_p - \xi; r+\epsilon) \le 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; r+\epsilon) \ge \lambda\}| = 0$$

or

$$\delta(\{p \le n : \psi(\Delta^m y_p - \xi; r+\epsilon) \le 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; r+\epsilon) \ge \lambda\}) = 0.$$

It is denoted by $\Delta^m y_p \xrightarrow{r\text{-}st_{(\psi,\eta)}} \xi$ or $r\text{-}st_{(\psi,\eta)}\text{-}\lim_{p\to\infty} \Delta^m y_p = \xi$.
Let $st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$ denotes the the set of all rough statistical limit points of the generalized difference sequence $(\Delta^m y_p)$.

Remark 3.1. For $r = 0$, the notion rough Δ^m-statistical convergence is equivalent to the Δ^m-statistical convergence for the generalized sequence $(\Delta^m y_p)$ in an *IFNS*.

The $r\text{-}st_{(\psi,\eta)}$-limit of a generalized difference sequence may not be unique in *IFNS*. So, consider the set of rough statistical limit points of $(\Delta^m y_p)$ as $st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} = \left[\xi : \Delta^m y_p \xrightarrow{r\text{-}st_{(\psi,\eta)}} \xi\right]$. Note that the sequence $(\Delta^m y_p)$ is $r_{(\psi,\eta)}$-convergent if $LIM^{r(\psi,\eta)}_{\Delta^m y_p} \ne \phi$, where

$$LIM^{r(\psi,\eta)}_{\Delta^m y_p} = \left[\xi^* \in Y : \Delta^m y_p \xrightarrow{r_{(\psi,\eta)}} \xi^*\right].$$

For unbounded sequences, $LIM^{r(\psi,\eta)}_{\Delta^m y_p} = \phi$, but $st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} \ne \phi$, that is the sequence might be rough statistical convergent. Following example shows that a generalized sequence can be rough statistical convergent but not rough convergent.

Example 3.1 ([30]). Let $(Y, \|\cdot\|)$ be any real normed space. For every $q > 0$ and for all $\Delta^m y = (\Delta^m y_p) \in Y$, define $\psi(\Delta^m y_p, q) = \frac{q}{q+\|\Delta^m y_p\|}$, $\eta(\Delta^m y_p, q) = \frac{\Delta^m y_p}{q+\|\Delta^m y_p\|}$. Then (Y, ψ, η) is an *IFNS*. Consider a sequence $(\Delta^m y_p)_{m\in N}$ such that

$$\Delta^m y_p = \begin{cases} (-1)^p, & p \ne n^2, \\ p, & p = n^2. \end{cases}$$

Then $\Delta^m y_p = (-1, 2, 3, 1, 5, 6, 7, 8, -1, \dots)$ and clearly for every $\epsilon > 0$ and $\lambda \in (0,1)$,

$$\lim_{n\to\infty} \frac{1}{n} |\{p \le n : \psi(\Delta^m y_p - \xi; r+\epsilon) \le 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; r+\epsilon) \ge \lambda\}| = 0.$$

Also,

$$st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} = \begin{cases} \phi, & r < 1, \\ [1-r, r-1], & \text{otherwise.} \end{cases}$$

In addition, $LIM^{r(\psi,\eta)}_{\Delta^m y_p} = \phi$, for all $r \ge 0$, since $(\Delta^m y_p)$ is unbounded sequence.

Definition 3.2. Let (Y, ψ, η) be an *intuitionistic fuzzy normed space*, $\Delta^m y = (\Delta^m y_p)$, be a generalized difference sequence in Y is said to be *I-Δ^m-statistical convergent* to $\xi \in Y$ with respect to norm (ψ, η) if for every $\epsilon > 0$ and $\lambda \in (0,1)$:

$$\left\{n \in N : \frac{1}{n} |\{p \le n : \psi(\Delta^m y_p - \xi; \epsilon) \le 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; \epsilon) \ge \lambda\}| \ge \delta\right\} \in I.$$

It is denoted by $\Delta^m y_p \xrightarrow{I\text{-}st_{(\psi,\eta)}} \xi$. Let $I\text{-}st_{(\psi,\eta)}\text{-}LIM_{\Delta^m y_p}$ denotes the set of all ideal statistical limit points of the generalized difference sequence $(\Delta^m y_p)$.

Next, we direct our attention towards the examination of rough ideal statistical convergence for generalized difference sequences in intuitionistic fuzzy normed spaces.

Definition 3.3. Let (Y,ψ,η) be an *intuitionistic fuzzy normed space*, $\Delta^m y = (\Delta^m y_p)$, be a generalized difference sequence in Y is said to be *rough I-Δ^m-statistical convergent* to $\xi \in Y$ with respect to norm (ψ,η) for some non-negative number r if for every $\epsilon > 0$ and $\lambda \in (0,1)$

$$\left\{n \in N : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \xi; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda\}| \geq \delta\right\} \in I.$$

It is denoted by $\Delta^m y_p \xrightarrow{r\text{-}I\text{-}st_{(\psi,\eta)}} \xi$. Let $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$ denotes the the set of all rough ideal statistical limit points of the generalized difference sequence $(\Delta^m y_p)$.

Remark 3.2. For $r = 0$, the notion *rough I-Δ^m-statistical convergence* agrees with the I-Δ^m-*statistical convergence* in an *IFNS*.

Example 3.2. Let $(Y, \|\cdot\|)$ be any real normed space. For every $q > 0$ and for all $\Delta^m y = (\Delta^m y_p) \in Y$, define $\psi(\Delta^m y_p, q) = \frac{q}{q+\|\Delta^m y_p\|}$, $\eta(\Delta^m y_p, q) = \frac{\Delta^m y_p}{q+\|\Delta^m y_p\|}$. Then (Y,ψ,η) is an *IFNS*. If we take $I_d = \{A \in N \text{ with } \delta(A) = 0\}$ where $\delta(A)$ denotes the natural density of set A, then for this admissible ideal, define $(\Delta^m y_p)_{m\in N}$ such that

$$\Delta^m y_p = \begin{cases} p, & p \in A, \\ (-1)^p, & p \notin A. \end{cases}$$

Then

$$I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} = \begin{cases} \phi, & r < 1, \\ [1-r, r-1], & \text{otherwise.} \end{cases}$$

Hence $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} \neq \phi$.

Definition 3.4. A sequence $\Delta^m y = (\Delta^m y_p)$ in *intuitionistic fuzzy normed space* (Y,ψ,η) is I-Δ^m-st bounded if for $\epsilon > 0$, $\lambda \in (0,1)$, there exists $H > 0$ such that

$$\left\{n \in N : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p; H) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p; H) \geq \lambda\}| \geq \delta\right\} \in I.$$

In view of above definitions, we have obtained the following results for generalized difference sequence in *IFNS*.

Definition 3.5. Let (Y,ψ,η) be an *IFNS* with intuitionistic fuzzy norms (ψ,η). A sequence $(\Delta^m y_p)$ in Y is I-Δ^m-st-bounded iff $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} \neq \phi$ for some $r > 0$.

Proof. (*Necessary Part*) Consider the sequence $(\Delta^m y_p)$ which is I-Δ^m-st-bounded in an *intuitionistic fuzzy normed space* (Y,ψ,η). Then, for every $\epsilon > 0$, $\lambda \in (0,1)$ and some $r > 0$, $\exists\, H > 0$ such that

$$\left\{n \in N : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p; H) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p; H) \geq \lambda\}| \geq \delta\right\} \in I.$$

Since I is admissible, therefore $M = N \setminus G$ is a non-empty set, where

$$G = \left\{ n \in N : \frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p; H) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p; H) \geq \lambda\}| \geq \delta \right\}.$$

Choose $p \in M$, then

$$\frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p; H) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p; H) \geq \lambda\}| < \delta$$

$$\implies \frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p; H) > 1 - \lambda \text{ and } \eta(\Delta^m y_p; H) < \lambda\}| \geq 1 - \delta. \tag{3.1}$$

Let $K = \{p \leq n : \psi(\Delta^m y_p; H) > 1 - \lambda \text{ and } \eta(\Delta^m y_p; H) < \lambda\}$.

Also,

$$\begin{aligned} \psi(\Delta^m y_p; r + H) &\geq \min\{\psi(0, r), \psi(\Delta^m y_p, H)\} \\ &= \min\{1, \psi(\Delta^m y_p; H)\} \\ &> 1 - \lambda, \\ \eta(\Delta^m y_p; r + H) &\leq \max\{\psi(0, r), \eta(\Delta^m y_p, H)\} \\ &= \max\{0, \eta(\Delta^m y_p; H)\} \\ &< \lambda. \end{aligned}$$

Thus, $K \subset \{p \leq n : \psi(\Delta^m y_p; r + H) > 1 - \lambda \text{ and } \eta(\Delta^m y_p; r + H) < \lambda\}$.

Using (3.1), we get

$$1 - \delta \leq \frac{|K|}{n} \leq \frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p; r + H) > 1 - \lambda \text{ and } \eta(\Delta^m y_p; r + H) < \lambda\}|.$$

Therefore,

$$\frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p; r + H) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p; r + H) \geq \lambda\}| < 1 - (1 - \delta) < \delta.$$

Then, for $n \in N$,

$$\frac{1}{n} |\{p \leq n : \psi(y_p; r + H) \leq 1 - \lambda \text{ or } \eta(y_p; r + H) \geq \lambda\}| \geq \delta \subset G \in I.$$

Hence $0 \in I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y}$. Therefore, $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y} \neq \phi$.

(*Sufficient Part*) Let $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} \neq \phi$, for some $r > 0$. Then, there exists some $\xi \in Y$ such that $\xi \in I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$.

For every $\epsilon > 0$ and $\lambda \in (0,1)$, we have

$$\left\{ n \in N : \frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p - \xi; r + \epsilon) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p - \xi; r + \epsilon) \geq \lambda\}| \geq \delta \right\} \in I.$$

Therefore, almost all $\Delta^m y_p$'s are enclosed in some ball with centre ξ in *IFNS*, which imply that $(\Delta^m y_p)$ is *I-Δ^m-statistically bounded* in an *IFNS*. Next, we will show that the algebraic characterization also hold for rough ideal statistical convergent sequences for generalized difference sequences in *IFNS*. □

Theorem 3.1. *Let $(\Delta^m y_p)$ and $(\Delta^m y_q)$ be two generalized difference sequences in an IFNS (Y, ψ, η) with I as admissible ideal and r is a non-negative number, then the following results holds:*

(i) *if $\Delta^m y_p \xrightarrow{r\text{-}I\text{-}st_{(\psi,\eta)}} L$ and $\beta \in R$, then $\beta \Delta^m y_p \xrightarrow{r\text{-}I\text{-}st_{(\psi,\eta)}} \beta L$,*

(ii) *if* $\Delta^m y_p \xrightarrow{r\text{-}I\text{-}st_{(\psi,\eta)}} L_1$ *and* $\Delta^m y_q \xrightarrow{r\text{-}I\text{-}st_{(\psi,\eta)}} L_2$*, then* $(\Delta^m y_p + \Delta^m y_q) \xrightarrow{r\text{-}I\text{-}st_{(\psi,\eta)}} (L_1 + L_2)$.

Proof. (i) If $\beta = 0$, then, we have nothing to prove. So, assume $\beta \neq 0$. As $\Delta^m y_p \xrightarrow{r\text{-}I\text{-}st_{(\psi,\eta)}} L$, then for given $\lambda > 0$ and $r \geq 0$,

$$G = \left\{ n \in N : \frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p - L; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - L; r+\epsilon) \geq \lambda\}| \geq \delta \right\} \in I.$$

Since I is admissible, therefore $M = N \setminus G$ is a non-empty set. Choose $m \in M$, then

$$\frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p - L; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - L; r+\epsilon) \geq \lambda\}| < \delta$$

$$\Longrightarrow \quad \frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p - L; r+\epsilon) > 1-\lambda \text{ or } \eta(\Delta^m y_p - L; r+\epsilon) < \lambda\}| \geq 1-\delta$$

$$\Longrightarrow \quad \frac{1}{n} |K| \geq 1-\delta \tag{3.2}$$

where

$$K = \{p \in N : \psi(\Delta^m y_p - L; r+\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - L; r+\epsilon) < \lambda\}.$$

It is sufficient to prove that for each $\lambda > 0$ and $r \geq 0$:

$$K \subset \{m \in N : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) > 1-\lambda \text{ and } \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda\}.$$

Let $k \in K$, then $\psi(\Delta^m y_k - L; r+\epsilon) > 1-\lambda$ and $\eta(\Delta^m y_k - L; r+\epsilon) < \lambda$.
So,

$$\begin{aligned}
\psi(\beta y_p - \beta L; r+\epsilon) &= \psi\left(\Delta^m y_p - L, \frac{r+\epsilon}{|\beta|}\right) \\
&\geq \min\left\{\psi(\Delta^m y_p - L, r+\epsilon), \psi\left(0, \frac{r+\epsilon}{|\beta|} - (r+\epsilon)\right)\right\} \\
&\geq \min\{\psi(\Delta^m y_p - L, r+\epsilon), 1\} \\
&= \psi(\Delta^m y_p - L, r+\epsilon) > 1-\lambda
\end{aligned}$$

and

$$\begin{aligned}
\eta(\beta\Delta^m y_p - \beta L; r+\epsilon) &= \eta\left(\Delta^m y_p - L, \frac{r+\epsilon}{|\beta|}\right) \\
&\leq \max\left\{\eta(\Delta^m y_p - L, r+\epsilon), \eta\left(0, \frac{r+\epsilon}{|\beta|} - (r+\epsilon)\right)\right\} \\
&\leq \max\{\eta(\Delta^m y_p - L, r+\epsilon), 0\} \\
&= \eta(\Delta^m y_p - L, r+\epsilon) < \lambda.
\end{aligned}$$

Hence

$$K \subset \{m \in N : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) > 1-\lambda \text{ and } \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda\}.$$

Using (3.2), we have

$$1-\delta \leq \frac{|K|}{n} \leq \frac{1}{n} |\{p \leq n : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) > 1-\lambda \text{ and } \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda\}|.$$

Therefore,

$$\frac{1}{n} |\{p \leq n : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) \leq 1-\lambda \text{ and } \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) \geq \lambda\}| < 1-(1-\delta) < \delta.$$

Then

$$\left\{n \in N : \frac{1}{n}|\{p \leq n : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) \geq \lambda\}| \geq \delta\right\} \subset G \in I,$$

which shows that $\beta\Delta^m y_p \xrightarrow{r\text{-}I\text{-}st_{(\psi,\eta)}} \beta L$.

(ii) In the similar manner, we can prove (ii) part. So, we are omitting its proof. □

In next result, we will show the set $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$ is closed.

Theorem 3.2. *The set $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$ of a generalized difference sequence $(\Delta^m y_p)$ is a closed set in an IFNS (Y,ψ,η).*

Proof. If $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} = \phi$ then the result is obvious as $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y}$ is either empty set or singleton set.

Let $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} \neq \phi$, for some $r > 0$.

Let $\Delta^m x = (\Delta^m x_p)$ be a convergent sequence in (Y,ψ,η) with respect to (ψ,η), which converges to $x_0 \in Y$. For $\epsilon > 0$ and $\lambda \in (0,1)$ then there exists $m_0 \in N$ such that

$$\psi\left(\Delta^m x_p - x_0; \frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \eta\left(\Delta^m x_p - x_0; \frac{\epsilon}{2}\right) < \lambda, \quad \text{for all } p \geq m_0.$$

Let us take $\Delta^m x_{m_1} \in I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$ with $m_1 > m_0$ such that

$$A = \left\{p \in N : \frac{1}{n}\left|\left\{p \leq n : \psi\left(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right) \leq 1-\lambda \text{ or } \eta\left(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right) \geq \lambda\right\}\right| \geq \delta\right\} \in I.$$

Since I is admissible so $G = N \setminus A$ is a non-empty set. Choose $n \in G$, then

$$\frac{1}{n}\left|\left\{p \leq n : \psi\left(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right) \leq 1-\lambda \text{ or } \eta\left(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right) \geq \lambda\right\}\right| < \delta$$

$$\Rightarrow \quad \frac{1}{n}\left|\left\{p \leq n : \psi\left(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \eta\left(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right) < \lambda\right\}\right| \geq 1-\delta.$$

Put $B_n = \{p \leq n : \psi\left(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \eta\left(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right) < \lambda\}$.

Then, for $j \in B_n$, $j \geq m_0$, we have

$$\psi(\Delta^m y_j - x_0; r+\epsilon) \geq \min\left\{\psi\left(\Delta^m y_j - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right), \psi\left(\Delta^m x_{m_1} - x_0; \frac{\epsilon}{2}\right)\right\} > 1-\lambda$$

and

$$\eta(\Delta^m y_j - x_0; r+\epsilon) \leq \max\left\{\eta\left(\Delta^m y_j - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right), \eta\left(\Delta^m x_{m_1} - x_0; \frac{\epsilon}{2}\right)\right\} < \lambda.$$

Therefore,

$$j \in \{p \in N : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}.$$

Hence

$$B_n \subset \{p \in N : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda\},$$

which implies

$$1-\delta \leq \frac{|B_n|}{n} \leq \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}|.$$

Therefore,

$$\frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - x_0; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - x_0; r+\epsilon) \geq \lambda\}| < 1-(1-\delta) = \delta.$$

Then

$$\left\{n \in N : \frac{1}{n}|\{p \le n : \psi(\Delta^m y_p - x_0; r+\epsilon) \le 1-\lambda \text{ or } \eta(\Delta^m y_p - x_0; r+\epsilon) \ge \lambda\}| \ge \delta\right\} \subset A \in I$$

which shows that $x_0 \in I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$ in (Y,ψ,η). □

The convexity of the set $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y}$ is demonstrated in the following result.

Theorem 3.3. *The set $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$ of a generalized difference sequence in IFNS (Y,ψ,η) is a convex set for some non-negative number r.*

Proof. Let $\varphi_1, \varphi_2 \in I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$. For convexity, we have to show that $(1-\omega)\varphi_1 + \omega\varphi_2 \in I\text{-}st\text{-}LIM^r_{\Delta^m y}$ for any real number $\omega \in (0,1)$.

Since $\varphi_1, \varphi_2 \in I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$, then there exists $p \in N$ for every $\epsilon > 0$ and $\lambda \in (0,1)$ such that

$$A_0 = \left\{p \in N : \psi\left(\Delta^m y_p - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right) \le 1-\lambda \text{ or } \eta\left(\Delta^m y_p - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right) \ge \lambda\right\}$$

and

$$A_1 = \left\{p \in N : \psi\left(\Delta^m y_p - \varphi_2; \frac{r+\epsilon}{2\omega}\right) \le 1-\lambda \text{ or } \eta\left(\Delta^m y_p - \varphi_2; \frac{r+\epsilon}{2\omega}\right) \ge \lambda\right\}.$$

For $\delta > 0$, we have

$$\left\{n \in N : \frac{1}{n}|\{p \le n : p \in A_0 \cup A_1|\} \ge \delta\right\} \in I.$$

Now choose $0 < \delta_1 < 1$ such that $0 < 1-\delta_1 < \delta$.

Let

$$A = \left\{n \in N : \frac{1}{n}|\{p \le n : p \in A_0 \cup A_1|\} \ge \delta_1\right\} \in I.$$

Now for $n \notin A$,

$$\frac{1}{n}|\{p \le n : p \in A_0 \cup A_1|\} < 1-\delta_1,$$

$$\frac{1}{n}|\{p \le n : p \notin A_0 \cup A_1|\} \ge 1-(1-\delta_1) = \delta_1.$$

This implies $\{p \le n : m \notin A_0 \cup A_1\} \ne \phi$.

Let $m_0 \in (A_0 \cup A_1)^c = A_0^c \cap A_1^c$.

Then

$$\begin{aligned}
&\psi(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) \\
&\quad = \psi[(1-\omega)(\Delta^m y_{m_0} - \varphi_1) + \omega(\Delta^m y_{m_0} - \varphi_2); r+\epsilon] \\
&\quad \ge \min\left\{\psi\left((1-\omega)(\Delta^m y_{m_0} - \varphi_1); \frac{r+\epsilon}{2}\right), \psi\left(\omega(\Delta^m y_{m_0} - \varphi_2); \frac{r+\epsilon}{2}\right)\right\} \\
&\quad = \min\left\{\psi\left(\Delta^m y_{m_0} - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right), \psi\left(\Delta^m y_{m_0} - \varphi_2; \frac{r+\epsilon}{2\omega}\right)\right\} \\
&\quad > 1-\lambda
\end{aligned}$$

and

$$\eta(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) = \eta[(1-\omega)(\Delta^m y_{m_0} - \varphi_1) + \omega(\Delta^m y_{m_0} - \varphi_2); r+\epsilon]$$

$$\leq \max\left\{\eta\left((1-\omega)(\Delta^m y_{m_0}-\varphi_1);\frac{r+\epsilon}{2}\right),\eta\left(\omega(\Delta^m y_{m_0}-\varphi_2);\frac{r+\epsilon}{2}\right)\right\}$$
$$=\max\left\{\eta\left(\Delta^m y_{m_0}-\varphi_1;\frac{r+\epsilon}{2(1-\omega)}\right),\eta\left(\Delta^m y_{m_0}-\varphi_2;\frac{r+\epsilon}{2\omega}\right)\right\}$$
$$<\lambda.$$

This implies

$$A_0^c\cap A_1^c\subset B^c,$$

where

$$B=\{p\in N:\psi(\Delta^m y_{m_0}-[(1-\omega)\varphi_1+\omega\varphi_2];r+\epsilon)\leq 1-\lambda \text{ or } \eta(\Delta^m y_{m_0}-[(1-\omega)\varphi_1+\omega\varphi_2];r+\epsilon)\geq\lambda\}.$$

So for $n\notin A$,

$$\delta_1\leq\frac{1}{n}|\{p\leq n:p\notin A_0\cup A_1\}|\leq\frac{1}{n}|\{p\leq n:p\notin B\}|$$

or

$$\frac{1}{n}|\{p\leq n:p\in B\}|<1-\delta_1<\delta.$$

Thus $A^c\subset\left\{n\in N:\frac{1}{n}|p\leq n:p\in B|<\delta\right\}$. Since $A^c\in F(I)$, then $\left\{n\in N:\frac{1}{n}|p\leq n:p\in B|<\delta\right\}\in F(I)$, which implies $\left\{n:\frac{1}{n}|p\leq n:p\in B|\geq\delta\right\}\in I$. Therefore, $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$ is a convex set. □

Theorem 3.4. *A generalized difference sequence $\Delta^m y=(\Delta^m y_p)$ in IFNS (Y,ψ,η) is rough-I-Δ^m-statistically convergent to $\rho\in Y$ with respect to the norm (ψ,η) for some $r>0$ if there exists a sequence $\Delta^m z=(\Delta^m z_p)$ in Y such that $I\text{-}st_{(\psi,\eta)}\text{-}LIM_{\Delta^m z_p}=\rho$ in Y and for every $\lambda\in(0,1)$ have $\psi(\Delta^m y_p-\Delta^m z_p;r+\epsilon)>1-\lambda$ and $\eta(\Delta^m y_p-\Delta^m z_p;r+\epsilon)<\lambda$, for all $p\in N$.*

Proof. Since $\Delta^m z=(\Delta^m z_p)$ be a generalized difference sequence in Y, which is I-Δ^m-*statistically convergent* to $\rho\in Y$ and $\psi(\Delta^m y_p-\Delta^m z_p;r+\epsilon)>1-\lambda$ and $\eta(\Delta^m y_p-\Delta^m z_p;r+\epsilon)<\lambda$, for all $p\in N$ and $\lambda\in(0,1)$.

Then by definition, for any $\epsilon,\delta>0$ and $\lambda\in(0,1)$ the set

$$M=\left\{n\in N:\frac{1}{n}|\{p\leq n:\psi(\Delta^m z_p-\rho;\epsilon)\leq 1-\lambda \text{ or } \eta(\Delta^m z_p-\rho;\epsilon)\geq\lambda\}|\geq\delta\right\}\in I.$$

Define

$$A_1=\{p\in N:\psi(\Delta^m z_p-\rho;\epsilon)\leq 1-\lambda \text{ or } \eta(\Delta^m z_p-\rho;\epsilon)\geq\lambda\},$$
$$A_2=\{p\in N:\psi(\Delta^m y_p-\Delta^m z_p;r)\leq 1-\lambda \text{ or } \eta(\Delta^m y_p-\Delta^m z_p;r)\geq\lambda\}.$$

For $\delta>0$, we have

$$\left\{n\in N:\frac{1}{n}|\{p\leq n:p\in A_1\cup A_2|\}\geq\delta\right\}\in I.$$

Now choose $0<\delta_1<1$ such that $0<1-\delta_1<\delta$ and let

$$A=\left\{n:\frac{1}{n}|\{p\leq n:p\in A_1\cup A_2|\}\geq\delta_1\right\}\in I.$$

Now for $n\notin A$

$$\frac{1}{n}|\{p\leq n:p\in A_1\cup A_2|\}<1-\delta_1,$$

$$\frac{1}{n}|\{p \le n : p \notin A_1 \cup A_2|\} \ge 1-(1-\delta_1) = \delta_1.$$

This implies $\{p \le n : p \notin A_1 \cup A_2\} \neq \phi$.
Let $p \in (A_1 \cup A_2)^c = A_1^c \cap A_2^c$.
Then

$$\psi(\Delta^m y_p - \rho; r+\epsilon) \ge \min\{\psi(\Delta^m y_p - \Delta^m z_p; r), \psi(\Delta^m z_p - \rho; \epsilon)\} > 1-\lambda$$

and

$$\eta(\Delta^m y_p - \rho; r+\epsilon) \le \max\{\eta(\Delta^m y_p - \Delta^m z_p; r), \eta(\Delta^m z_p - \rho; \epsilon)\} < \lambda$$

which gives

$$A_1^c \cap A_2^c \subset B^c,$$

where

$$B = \left\{p \in N : \psi(\Delta^m y_p - \rho; r+\epsilon) \le 1-\lambda \text{ or } \eta(\Delta^m y_p - \rho; r+\epsilon) \ge \lambda\right\}.$$

Thus, for $n \notin A$,

$$\delta_1 \le \frac{1}{n}|\{p \le n : p \notin A_1 \cup A_2\}| \le \frac{1}{n}|\{p \le n : p \notin B\}|$$

or

$$\frac{1}{n}|\{p \le n : p \in B\}| < 1-\delta_1 < \delta.$$

Thus $A^c \subset \left\{n : \frac{1}{n}|p \le n : p \in B| < \delta\right\}$. Since $A^c \in F(I)$, So, $\left\{n : \frac{1}{n}|p \le n : p \in B| < \delta\right\} \in F(I)$, which implies $\left\{n : \frac{1}{n}|p \le n : p \in B| \ge \delta\right\} \in I$. Hence, $\Delta^m y_p \xrightarrow{r\text{-}I\text{-}st_{(\psi,\eta)}} \rho$ in *IFNS* (Y, ψ, η). □

Theorem 3.5. *Let $\Delta^m y = (\Delta^m y_p)$ be a generalized difference sequence in an IFNS (Y, ψ, η) then there does not exist two elements $\alpha_1, \alpha_2 \in I\text{-}st\text{-}LIM^r_{\Delta^m y}$ for $r > 0$ and $\lambda \in (0,1)$ such that $\psi(\alpha_1 - \alpha_2; cr) \le 1-\lambda$ and $\eta(\alpha_1 - \alpha_2; cr) \ge \lambda$ for $c > 2$.*

Proof. If possible, suppose there exists two elements $\alpha_1, \alpha_2 \in I\text{-}st\text{-}LIM^r_{\Delta^m y}$ such that

$$\psi(\alpha_1 - \alpha_2; cr) \le 1-\lambda \text{ and } \eta(\alpha_1 - \alpha_2; cr) \ge \lambda, \quad \text{for } c > 2. \tag{3.3}$$

As $\alpha_1, \alpha_2 \in I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y}$ then for every $\epsilon > 0$ and $\lambda \in (0,1)$. Define,

$$A_1 = \left\{p \in N : \psi\left(\Delta^m y_p - \alpha_1; r+\frac{\epsilon}{2}\right) \le 1-\lambda \text{ or } \eta\left(\Delta^m y_p - \alpha_1; r+\frac{\epsilon}{2}\right) \ge \lambda\right\},$$

$$A_2 = \left\{p \in N : \psi\left(\Delta^m y_p - \alpha_2; r+\frac{\epsilon}{2}\right) \le 1-\lambda \text{ or } \eta\left(\Delta^m y_p - \alpha_2; r+\frac{\epsilon}{2}\right) \ge \lambda\right\}.$$

Then

$$\frac{1}{n}|\{p \le n : p \in A_1 \cup A_2\}| \le \frac{1}{n}|\{p \le n : p \in A_1\}| + \frac{1}{n}|\{p \le n : p \in A_2\}|.$$

So, by the property of *I-convergence*, we have

$$I\text{-}\lim_{n\to\infty}\frac{1}{n}|\{p \le n : p \in A_1 \cup A_2\}| \le I\text{-}\lim_{n\to\infty}\frac{1}{n}|\{p \le n : p \in A_1\}| + I\text{-}\lim_{n\to\infty}\frac{1}{n}|\{p \le n : p \in A_2\}| = 0.$$

Thus

$$\left\{n : \frac{1}{n}|\{p \le n : p \in A_1 \cup A_2|\} \ge \delta\right\} \in I, \quad \text{for all } \delta > 0.$$

Now choose $0 < \delta_1 = 1/2 < 1$ such that $0 < 1-\delta_1 < \delta$.

Let

$$K = \left\{n : \frac{1}{n}|\{p \le n : p \in A_1 \cup A_2|\} \ge \delta_1\right\} \in I.$$

Now for $n \notin K$,

$$\frac{1}{n}|\{p \le n : p \in A_1 \cup A_2|\} < 1-\delta_1 = 1/2,$$

$$\frac{1}{n}|\{p \le n : p \notin A_1 \cup A_2|\} \ge 1-(1-\delta_1) = 1/2.$$

This implies $\{p \le n : p \notin A_1 \cup A_2\} \ne \phi$. Then, for $p \in A_1^c \cap A_2^c$, we have

$$\psi(\alpha_1-\alpha_2;2r+\epsilon) \ge \min\left\{\psi\left(\Delta^m y_p - \alpha_2 : r+\frac{\epsilon}{2}\right), \psi\left(\Delta^m y_p - \alpha_1; r+\frac{\epsilon}{2}\right)\right\} > 1-\lambda$$

and

$$\eta(\alpha_1-\alpha_2;2r+\epsilon) \le \max\left\{\eta\left(\Delta^m y_p - \alpha_2 : r+\frac{\epsilon}{2}\right), \eta\left(\Delta^m y_p - \alpha_1; r+\frac{\epsilon}{2}\right)\right\} < \lambda.$$

Hence,

$$\psi(\alpha_1-\alpha_2;2r+\epsilon) > 1-\lambda \text{ and } \eta(\alpha_1-\alpha_2;2r+\epsilon) < \lambda. \tag{3.4}$$

Then from (3.4), we have $\psi(\alpha_1-\alpha_2;cr) > 1-\lambda$ and $\eta(\alpha_1-\alpha_2;cr) < \lambda$ for $c > 2$, which is contradiction to (3.3). Therefore, there does not exist two elements such that $\psi(\alpha_1-\alpha_2;cr) \le 1-\lambda$ and $\eta(\alpha_1-\alpha_2;cr) \ge \lambda$, for $c > 2$. □

Definition 3.6. Let (Y,ψ,η) be an *intuitionistic fuzzy normed space*. Then $\gamma \in Y$ is called *rough I-Δ^m-statistical cluster point* of the sequence $\Delta^m y = (\Delta^m y_p)$ in Y with respect to norm (ψ,η) for some $r > 0$ if for every $\epsilon > 0$ and $\lambda \in (0,1)$,

$$\delta_I(\{p \in N : \psi(\Delta^m y_p - \gamma; r+\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - \gamma; r+\epsilon) < \lambda\}) \ne 0,$$

where $\delta_I(A) = I\text{-}\lim_{n\to\infty} \frac{1}{n}|\{p \le n : p \in A\}|$ if exists. In this case, γ is known as *r-I-Δ^m-statistical cluster point* of a sequence $(\Delta^m y_p)$.

Let $\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)$ indicates the set of all *r-I-Δ^m-statistical cluster points* with respect to the norm (ψ,η) of a sequence $(\Delta^m y_p)$ in an *IFNS* (Y,ψ,η). If $r = 0$ then the notion stands for only *I-Δ^m-statistical cluster point* with respect to the norm (ψ,η) in an *IFNS* (Y,ψ,η), symbolically; $\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p) = \Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)$.

In the next result, we have derived the closedness of the set $\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)$ of generalized difference sequence $(\Delta^m y_p)$ in Y.

Theorem 3.6. *The set $\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)$ of generalized difference sequence $\Delta^m y = (\Delta^m y_p)$ in an IFNS (Y,ψ,η) is closed for some $r > 0$.*

Proof. If $\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p) = \phi$, then the result is obvious. So nothing to prove.

Let us suppose $\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p) \ne \phi$. Consider $\Delta^m x = (\Delta^m x_p)$ be a generalized difference sequence such that

$$(\Delta^m x) \subseteq \Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p) \text{ and } \Delta^m x_p \xrightarrow{(\psi,\eta)} x_0.$$

To prove closedness, it is sufficient to show that $x_0 \in \Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)$.

As $\Delta^m x_p \xrightarrow{(\psi,\eta)} x_0$, then for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists $p_\epsilon \in N$ such that

$$\psi\left(\Delta^m x_p - x_0; \frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \eta\left(\Delta^m x_p - x_0; \frac{\epsilon}{2}\right) < \lambda, \quad \text{for } p \geq p_\epsilon.$$

Choose some $p_0 \in N$ such that $p_0 \geq p_\epsilon$. Then, we have $\psi\left(\Delta^m x_{p_0} - x_0; \frac{\epsilon}{2}\right) > 1-\lambda$ and $\eta\left(\Delta^m x_{p_0} - x_0; \frac{\epsilon}{2}\right) < \lambda$.

Again as $\Delta^m x = (\Delta^m x_p) \subseteq \Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)$, we have $\Delta^m x_{p_0} \in \Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)$,

$$\implies \quad \delta_I\left(\left\{p \in N : \psi\left(\Delta^m y_p - c; r + \frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \eta\left(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right) < \lambda\right\}\right) \neq 0. \tag{3.5}$$

Consider $G = \left\{p \in N : \psi(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) > 1-\lambda \text{ and } \eta(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) < \lambda\right\}$.

Choose $j \in G$, then we have

$$\psi\left(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \eta\left(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right) < \lambda.$$

Now,

$$\psi(\Delta^m y_j - x_0; r+\epsilon) \geq \min\left\{\psi\left(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right), \psi\left(\Delta^m x_{p_0} - x_0; r + \frac{\epsilon}{2}\right)\right\} > 1-\lambda$$

and

$$\eta(\Delta^m y_j - x_0; r+\epsilon) \leq \max\left\{\eta\left(\Delta^m y_j - x_{p_0}; r + \frac{\epsilon}{2}\right), \eta\left(x_{p_0} - y_0; r + \frac{\epsilon}{2}\right)\right\} < \lambda.$$

Thus

$$j \in \{p \in N : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}.$$

Hence

$$\left\{p \in N : \psi\left(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \eta\left(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right) < \lambda\right\}$$
$$\subseteq \{p \in N : \psi(\Delta^m y_p - y_0; r+\epsilon) > 1-\lambda \text{ and } \eta(y_m - y_0; r+\epsilon) < \lambda\},$$
$$\delta_I\left(\left\{p \in N : \psi\left(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \eta\left(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right) < \lambda\right\}\right)$$
$$\leq \delta_I(\{p \in N : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}). \tag{3.6}$$

Using (3.5), we conclude that

$$\delta_I(\{m \in N : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}) \neq 0,$$

as the set on left side hand of (3.6) possesses natural density more than zero. Therefore, $x_0 \in \Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)$. Hence the result. □

Theorem 3.7. *Let $\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)$ be the set of all I-Δ^m-statistical cluster points of the sequence $\Delta^m y = (\Delta^m y_p)$ in an IFNS (Y,ψ,η). Then for any arbitrary $v \in \Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)$, $r \geq 0$ and $\lambda \in (0,1)$, we have $\psi(\zeta - v; r) > 1-\lambda$ and $\eta(\zeta - v; r) < \lambda$, for all $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)$.*

Proof. Since $v \in \Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)$ then for $\epsilon > 0$ and $\lambda \in (0,1)$, we have

$$\delta_I(\{p \in N : \psi(\Delta^m y_p - v; \epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - v; \epsilon) < \lambda\}) \neq 0. \tag{3.7}$$

Now it is sufficient to show that if any $\zeta \in Y$ satisfying $\psi(\zeta - v; \epsilon) > 1-\lambda$ and $\eta(\zeta - v; \epsilon) < \lambda$, then $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)$.

Suppose $j \in \{p \in N : \psi(\Delta^m y_p - \nu;\epsilon) > 1-\lambda$ and $\eta(\Delta^m y_p - \nu;\epsilon) < \lambda\}$ then $\psi(\Delta^m y_j - \nu;\epsilon) > 1-\lambda$ and $\eta(\Delta^m y_j - \nu;\epsilon) < \lambda$.
On the other side,

$$\psi(\Delta^m y_j - \zeta; r+\epsilon) \geq \min\{\psi(\Delta^m y_j - \nu;\epsilon), \psi(\zeta - \nu; r)\} > 1-\lambda$$

and

$$\eta(\Delta^m y_j - \zeta; r+\epsilon) \leq \max\{\eta(\Delta^m y_j - \nu;\epsilon), \eta(\zeta - \nu; r)\} < \lambda.$$

So, we have $\psi(\Delta^m y_j - \zeta; r+\epsilon) > 1-\lambda$ and $\eta(\Delta^m y_j - \zeta; r+\epsilon) < \lambda$.
Thus $j \in \{p \in N : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda$ and $\eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}$ which gives the inclusion

$$\{p \in N : \psi(\Delta^m y_p - \nu;\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - \nu;\epsilon) < \lambda\}$$
$$\subseteq \{p \in N : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}.$$

Then

$$\delta_I(\{p \in N : \psi(\Delta^m y_p - \nu;\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - \nu;\epsilon) < \lambda\})$$
$$\leq \delta_I(\{p \in N : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}).$$

Therefore, from (3.7),

$$\delta_I(\{p \in N : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}) \neq 0.$$

Hence $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)$. □

Theorem 3.8. *Let $\Delta^m y = (\Delta^m y_p)$ be a generalized difference sequence in an IFNS (Y,ψ,η) and $\overline{B(\rho,\lambda,r)} = \{\Delta^m y \in Y : \psi(\Delta^m y - \rho; r) \geq 1-\lambda, \eta(\Delta^m y - \rho; r) \leq \lambda\}$, represents the closure of the open ball $B(\rho,\lambda,r) = \{\Delta^m y \in Y : \psi(\Delta^m y - \rho; r) > 1-\lambda, \eta(\Delta^m y - \rho; r) < \lambda\}$ for some $r > 0$ and $\lambda \in (0,1)$ and fixed $\rho \in Y$ then $\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p) = \bigcup_{\rho \in \Gamma^I_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$.*

Proof. Let $\zeta \in \bigcup_{\rho \in \Gamma^I_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$ then there exists some $\rho \in \Gamma^I_{st(\psi,\eta)}(\Delta^m y_p)$ for $r > 0$, and $\lambda \in (0,1)$ such that $\psi(\rho - \zeta; r) > 1-\lambda$ and $\eta(\rho - \zeta; r) < \lambda$.
As $\rho \in \Gamma^I_{st(\psi,\eta)}(\Delta^m y_p)$ then there exists a set

$$M = \{p \in N : \psi(\Delta^m y_p - \rho;\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - \rho;\epsilon) < \lambda\}$$

with $\delta_I(M) \neq 0$. For $p \in M$,

$$\psi(\Delta^m y_p - \zeta; r+\epsilon) \geq \min\{\psi(\Delta^m y_p - \rho;\epsilon), \psi(\rho - \zeta; r)\} > 1-\lambda$$

and

$$\eta(\Delta^m y_p - \zeta; r+\epsilon) \leq \max\{\eta(\Delta^m y_p - \rho;\epsilon), \eta(\rho - \zeta; r)\} < \lambda.$$

This implies that

$$\delta_I(\{p \in N : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda \text{ and } \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}) \neq 0.$$

Hence

$$\zeta \in \Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p).$$

So,

$$\bigcup_{\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} \subseteq \Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p).$$

Conversely, take $\zeta\in\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)$ if possible let $\zeta\notin \bigcup_{\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$, i.e., $\zeta\notin\overline{B(\rho,\lambda,r)}$ for all $\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)$.

Then for all $\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)$, we have $\psi(\zeta-\rho;r)\le 1-\lambda$ or $\eta(\zeta-\rho;r)\ge\lambda$ for every $\rho\in\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)$. According to Theorem 3.7 for any arbitrary $\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)$, we have $\psi(\zeta-\rho;r)>1-\lambda$ and $\eta(\zeta-\rho;r)<\lambda$ which is contradiction to our supposition.

Hence $\zeta\in\bigcup_{\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$. Hence $\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p)\subseteq\bigcup_{\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$.

This completes the proof. □

Theorem 3.9. *Let $\Delta^m y=(\Delta^m y_p)$ be a generalized difference sequence in IFNS (Y,ψ,η). Then for $\lambda\in(0,1)$ and $r>0$,*

(i) *If $\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)$ then I-$st_{(\psi,\eta)}$-$LIM^r_{\Delta^m y_p}\subseteq\overline{B(\rho,\lambda,r)}$.*

(ii) I-$st_{(\psi,\eta)}$-$LIM^r_{\Delta^m y_p}=\bigcap_{\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)}\overline{B(\rho,\lambda,r)}=\{\xi\in Y:\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)\subseteq\overline{B(\xi,\lambda,r)}\}$.

Proof. Let $\xi\in I$-$st_{(\psi,\eta)}$-$LIM^r_{\Delta^m y_p}$ and $\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)$.

For $\epsilon>0$ and $\lambda\in(0,1)$.

Consider

$$G=\{p\in N:\psi(\Delta^m y_p-\xi;r+\epsilon)>1-\lambda \text{ and } \eta(\Delta^m y_p-\xi;r+\epsilon)<\lambda\}$$

and

$$H=\{p\in N:\psi(\Delta^m y_p-\rho;\epsilon)>1-\lambda \text{ and } \eta(\Delta^m y_p-\rho;\epsilon)<\lambda\}$$

with $\delta_I(G^c)=0$ and $\delta_I(H)\neq0$, respectively. Now for $p\in G\cap H$,

$$\psi(\xi-\rho;r)\ge\min\{\psi(\Delta^m y_p-\rho;\epsilon),\psi(\Delta^m y_p-\xi;r+\epsilon)\}>1-\lambda$$

and

$$\eta(\xi-\rho;r)\le\max\{\eta(y_m-\rho;\epsilon),\eta(y_m-\xi;r+\epsilon)\}<\lambda.$$

Thus $\xi\in\overline{B(\rho,\lambda,r)}$. Hence I-$st_{(\psi,\eta)}$-$LIM^r_{\Delta^m y_p}\subseteq\overline{B(\rho,\lambda,r)}$.

(ii) It follows from (i) part that I-$st_{(\psi,\eta)}$-$LIM^r_{\Delta^m y_p}\subseteq\bigcap_{\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)}\overline{B(\rho,\lambda,r)}$.

Take $y\in\bigcap_{\rho\in\Gamma^{\alpha}_{st(\psi,\eta)}(I_d)}\overline{B\left(\rho,\lambda,r\right)}$ then

$$\psi(y-\rho;r)\ge1-\lambda \text{ and } \eta(y-\rho;r)\le\lambda,\quad \text{for all } \rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p).$$

This implies that

$$\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)\subseteq\overline{B(y,\lambda,r)}$$

i.e.

$$\bigcap_{\rho\in\Gamma^{\alpha}_{st(\psi,\eta)}(I_d)} \overline{B(\rho,\lambda,r)} \subseteq \{\xi\in Y : \Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)\subseteq \overline{B(\xi,\lambda,r)}\}.$$

Now assume $y\notin I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$, then for $\lambda\in(0,1)$ and $\epsilon>0$, we have

$$\delta_I(\{p\in N : \psi(\Delta^m y_p - y; r+\epsilon)\le 1-\lambda \text{ or } \eta(\Delta^m y_p - y; r+\epsilon)\ge\lambda\})\ne 0.$$

It means there exists some I-statistical cluster point ρ for the sequence $\Delta^m y=(\Delta^m y_p)$ with $\psi(y-\rho;r+\epsilon)\le 1-\lambda$ or $\eta(y-\rho;r+\epsilon)\ge\lambda$.

Thus $\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)$ does not contained in $\overline{B(y,\lambda,r)}$ and $y\notin\{\xi\in Y : \Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)\subseteq\overline{B(\xi,\lambda,r)}\}$.

Hence

$$\{\xi\in Y : \Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)\subseteq\overline{B(\xi,\lambda,r)}\}\subseteq I_d - st\text{-}LIM^r_{\Delta^m y_p}$$

and

$$\bigcap_{\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} \subseteq I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}.$$

So,

$$I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} = \bigcap_{\rho\in\Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} = \{\xi\in Y : \Gamma^{I}_{st(\psi,\eta)}(\Delta^m y_p)\subseteq\overline{B(\xi,\lambda,r)}\}. \qquad \square$$

Theorem 3.10. *$\Delta^m y=(\Delta^m y_p)$ be a generalized difference sequence in IFNS (Y,ψ,η), which is ideal statistically convergent to ρ then $\overline{B(\rho,\lambda,r)} = I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$.*

Proof. Since $(\Delta^m y_p)$ is ideal statistical convergent to ρ with respect to the norms (ψ,η) i.e. $(\Delta^m y_p)\xrightarrow{I\text{-}st(\psi,\eta)}\rho$, then there is a set A such that

$$A=\left\{n : \frac{1}{n}|\{p\le n : \psi(\Delta^m y_p-\rho;\epsilon)\le 1-\lambda \text{ or } \eta(\Delta^m y_p-\rho;\epsilon)\ge\lambda\}|>\delta\right\}\in I.$$

Since I is admissible so $G=N\setminus A$ is a non-empty set, then for $p\in G^c$,

$$\frac{1}{n}|\{p\le n : \psi(\Delta^m y_p-\rho;\epsilon)\le 1-\lambda \text{ or } \eta(\Delta^m y_p-\rho;\epsilon)\ge\lambda\}|<\delta$$

$$\Rightarrow\quad \frac{1}{n}|\{p\le n : \psi(\Delta^m y_p-\rho;\epsilon)> 1-\lambda \text{ and } \eta(\Delta^m y_p-\rho;\epsilon)<\lambda\}|\ge 1-\delta.$$

Put $B_n=\{p\le n : \psi(\Delta^m y_p-\rho;\epsilon)>1-\lambda \text{ and } \eta(\Delta^m y_p-\rho;\epsilon)<\lambda\}$, for $j\ge m$.

Now for $j\in B_n$, we have $\psi(\Delta^m y_j-\rho;\epsilon)>1-\lambda$ and $\eta(\Delta^m y_j-\rho;\epsilon)<\lambda$.

Let $y\in\overline{B(\rho,\lambda,r)}$. We will prove $y\in I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$,

$$\psi(\Delta^m y_j-y;r+\epsilon)\ge\min\{\psi(\Delta^m y_j-\rho,\epsilon),\psi(y-\rho,r)\}>1-\lambda,$$

$$\eta(\Delta^m y_j-y;r+\epsilon)\le\max\{\psi(\Delta^m y_j-\rho,\epsilon),\eta(y-\rho,r)\}<\lambda.$$

Hence

$$B_n\subset\{p\in N : \psi(\Delta^m y_p-y;r+\epsilon)>1-\lambda \text{ and } \eta(\Delta^m y_p-y;r+\epsilon)<\lambda\},$$

which implies

$$1-\delta\le\frac{|B_n|}{n}\le\frac{1}{n}|\{p\le n : \psi(\Delta^m y_p-y;r+\epsilon)>1-\lambda \text{ and } \eta(\Delta^m y_p-y;r+\epsilon)<\lambda\}|.$$

Therefore,

$$\frac{1}{n}|\{p \le n : \psi(\Delta^m y_p - y; r+\epsilon) \le 1-\lambda \text{ or } \eta(\Delta^m y_p - y; r+\epsilon) \ge \lambda\}| < 1-(1-\delta) = \delta.$$

Then

$$\left\{n \in N : \frac{1}{n}|\{p \le n : \psi(\Delta^m y_p - y; r+\epsilon) \le 1-\lambda \text{ or } \eta(\Delta^m y_p - y; r+\epsilon) \ge \lambda\}| \ge \delta\right\} \subset A \in I$$

which shows that $y \in I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$ in (Y,ψ,η).

Hence $\overline{B(\rho,\lambda,r)} \subseteq I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$. Also $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} \subseteq \overline{B(\rho,\lambda,r)}$.

Hence,

$$I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} = \overline{B(\rho,\lambda,r)}.$$ □

In the next result, we examined the relation between rough I-statistical limit points and rough I-statistical cluster points of a generalized difference sequence in Intuitionistic fuzzy norm space.

Theorem 3.11. *Let $\Delta^m y = (\Delta^m y_p)$ be a generalized difference sequence in IFNS (Y,ψ,η), which is ideal statistically convergent to ξ then $\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p) = I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$.*

Proof. Firstly, assume $y_p \xrightarrow{I\text{-}st_{(\psi,\eta)}} \xi$, which gives $\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p) = \{\xi\}$. Then for $r > 0$ and $\lambda \in (0,1)$ by Theorem 3.8, we have $\Gamma^{r(I)}_{st_{(\psi,\eta)}}(\Delta^m y_p) = \overline{B(\xi,\lambda,r)}$. Also, from Theorem 3.10, $\overline{B(\xi,\lambda,r)} = I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$.

Hence $\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p) = I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$.

Conversely, assume $\Gamma^{r(I)}_{st(\psi,\eta)}(\Delta^m y_p) = I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p}$, then by Theorems 3.8 and 3.9(ii),

$$\bigcap_{\xi \in \Gamma^I_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} = \bigcup_{\xi \in \Gamma^I_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}.$$

This is possible only if either $\Gamma^I_{st(\psi,\eta)}(\Delta^m y_p) = \phi$ or $\Gamma^I_{st(\psi,\eta)}(\Delta^m y_p)$ is a singlton set. Then

$$\begin{aligned} I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} &= \bigcap_{\rho \in \Gamma^I_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} \\ &= \overline{B(\xi,\lambda,r)}, \quad \text{for some } \xi \in \Gamma^I_{st(\psi,\eta)}(\Delta^m y_p). \end{aligned}$$

Also, by Theorem 3.10, $I\text{-}st_{(\psi,\eta)}\text{-}LIM^r_{\Delta^m y_p} = \{\xi\}$. □

4. Conclusions

The present article is devoted to study the concept of rough ideal statistical convergence for the generalized difference sequences on the intuitionistic fuzzy normed spaces. The various topological and algebraic properties of the set of rough ideal statistical limit points as well as rough ideal statistical cluster points has been discussed for the generalized difference sequences.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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