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Research Article

# Semi Analytic-Numerical Solution of Imbibition Phenomenon in Homogeneous Porous Medium Using Hybrid Differential Transform Finite Difference Method

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**Abstract.** Analysis of the counter current imbibition phenomenon in a two-phase flow in a homogeneous porous media under specific conditions is the primary goal of the current work. Imbibition is said to occur when a wetting fluid in a porous medium displaces a non-wetting fluid. The phenomena of imbibition are significant in natural and man-made systems. When oil and water form the two immiscible liquid phases, it is assumed that water is the wetting phase. The partial differential equation that governs this imbibition phenomenon is highly non-linear It is solved using the *Hybrid Differential Transform Finite Difference Method* (HDTFDM) which gives the solution in the form of an infinite series emphasizing the semi analytic nature of this method. The solution to this equation enables the measurement of the saturation of the injected water in a double phase flow at different distances and time. HDTFDM is a combination of the *Differential Transform Method* (DTM) and *Finite Difference Method* (FDM). The flexibility of the DTM is integrated with the efficiency of the FDM which speeds up computation compared to the conventional DTM. This approach has been discovered to be reliable and effective. Further, to overcome the shortcomings of this method for large values of time, the *Multistep Differential Transform Method* (MDTM) and *Finite Difference Method* (FDM) have been used to achieve the solution for large values of time. Using MATLAB, the numerical solution and graphical representation were obtained. The results obtained were compared with the existing results and found to be in close agreement.

**Keywords.** Imbibition phenomenon, Hybrid Differential Transform Finite Difference Method (HDTFDM), Multistep differential transform method, Counter-current

**Mathematics Subject Classification (2020).** 35A25, 35A22,35Q35

## 1. Introduction

The primary objective is to investigate the counter current imbibition phenomenon in a homogeneous porous medium. Imbibition which is counter current in nature is said to occur when capillary pressure causes a wetting fluid (water) to spontaneously displace a non-wetting fluid (oil), because of which in the opposite direction a counterflow of oil occurs.

This capillary pressure is caused due to the discontinuity between the pressures of both the wetting and non-wetting fluid, when two immiscible fluids flow in a two-phase flow. This imbibition that occurs because of capillary pressure is called "spontaneous" or "natural" imbibition. The process of oil recovery, groundwater hydrology, geophysics, and petroleum technology all depend on this imbibition phenomenon. Primary oil recovery process uses only natural pressure to recover oil with no external force. Secondary oil recovery is the process of recovering the remnant oil from reservoir, by displacing oil toward the production well wherein water or gas is injected. It is essential to create a mathematical model of this phenomenon because it is challenging to analyse it in the real world.

This phenomenon for homogenous porous media was studied by Alazmi and Vafai [1], Scheidegger and Johnson [15], and Scheidegger [14]. Generalised separable solution was used by Parikh *et al*. [9] to solve this phenomenon in horizontal direction and it was solved by Pathak and Singh [13] in inclined homogenous porous medium. A Homotopy Series Solution in a inclined homogenous Porous Medium was studied by Patel and Desai [10]. *Hybrid Differential Transform Finite Difference Method* (HDTFDM) is used to solve this problem (Süngü and Demir [16]).

## 2. Mathematical Formulation

A finite cylindrical piece of length $L$ of homogenous porous media is considered for study. This cylindrical piece is fully saturated with oil ($o$) which is termed as native fluid. It is surrounded entirely by an impervious surface, with the exception of the end, which is called as the cylinder's imbibition surface ($x = 0$), where wetting fluid, water ($w$), will enter the cylinder. The injected liquid is assumed to be water, whereas the native liquid is oil. Water preferentially wets the medium, resulting in counter-current imbibition, which causes the wetting fluid (water) to flow spontaneously into the porous medium, resulting in a flow of the oil (native liquid) in the opposite direction (Figure 1) .

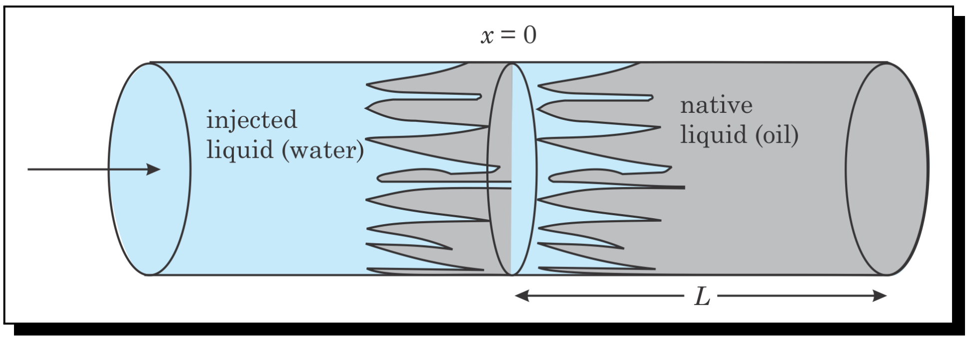


**Figure 1.** Illustration of linear counter-current imbibition phenomenon

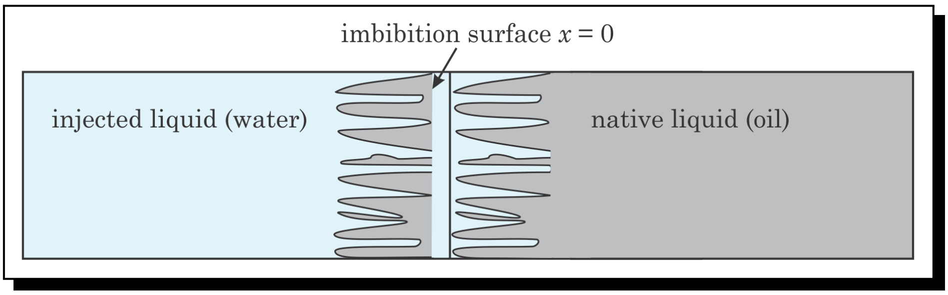


**Figure 2.** Schematic representation of imbibition phenomenon (*Source:* Parikh *et al.* [9])

The seepage velocity of oil and water can be written as Scheidegger [14]

$$V_w = -\frac{k_w}{\mu_w} k \frac{\partial P_w}{\partial x}, \tag{1}$$

$$V_o = -\frac{k_o}{\mu_o} k \frac{\partial P_o}{\partial x}. \tag{2}$$

The equations of continuity are,

$$\emptyset \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0, \quad \emptyset \frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0. \tag{3}$$

From equation (1),

$$\emptyset \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{k_w}{\mu_w} k \frac{\partial P_w}{\partial x}\right) = 0, \tag{4}$$

$$\emptyset \frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0, \tag{5}$$

where

| | |
|---|---|
| $k$ | permeability in m$^2$ |
| $k_w$ | relative permeability of water |
| $k_o$ | relative permeability of oil |
| $L$ | length (m) |
| $P_w$ | pressure of water (injected fluid) |
| $P_o$ | pressure of oil (native fluid) |
| $V_w$ | seepage velocity of water (m/s) |
| $V_o$ | seepage velocity of oil (m/s) |
| $\mu_w$ | constant kinematic viscosity of water |
| $\mu_o$ | constant kinematic viscosity of oil |
| $\emptyset$ | porosity of a medium |

$x$ is measured along the horizontal direction from the imbibition surface which acts as origin. The capillary pressure $P_c$ (Graham and Richardson [4]) is given by

$$P_c = P_o - P_w. \tag{6}$$

Because of the spontaneous flow which occurs in the counter current direction due to the contact of two phases, the sum of the velocities water and oil of is zero (Scheidegger and Johnson [15]). Therefore,

$$V_w + V_o = 0, \tag{7}$$

$$V_w = -V_o. \tag{8}$$

From (1) and (2), we get

$$\frac{k_w}{\mu_w}k\frac{\partial P_w}{\partial x}+\frac{k_o}{\mu_o}k\frac{\partial P_o}{\partial x}=0. \tag{9}$$

Substituting for $P_o$ from equation (6),

$$\frac{k_w}{\mu_w}k\frac{\partial P_w}{\partial x}+\frac{k_o}{\mu_o}k\frac{\partial}{\partial x}(P_c+P_w)=0, \tag{10}$$

$$\frac{k_w}{\mu_w}k\frac{\partial P_w}{\partial x}+\frac{k_o}{\mu_o}k\frac{\partial P_c}{\partial x}+\frac{k_o}{\mu_o}k\frac{\partial P_w}{\partial x}=0. \tag{11}$$

Simplifying,

$$k\frac{\partial P_w}{\partial x}\left(\frac{k_w}{\mu_w}+\frac{k_o}{\mu_o}\right)+\frac{k_o}{\mu_o}k\frac{\partial P_c}{\partial x}=0, \tag{12}$$

$$k\frac{\partial P_w}{\partial x}=-\frac{\frac{k_o}{\mu_o}k\frac{\partial P_c}{\partial x}}{\left(\frac{k_w}{\mu_w}+\frac{k_o}{\mu_o}\right)}. \tag{13}$$

Substituting in equation (4)

$$\emptyset\frac{\partial S_w}{\partial t}+\frac{\partial}{\partial x}\left(\frac{k_w}{\mu_w}k\frac{\frac{k_o}{\mu_o}k\frac{\partial P_c}{\partial x}}{\frac{k_w}{\mu_w}+\frac{k_o}{\mu_o}}\right)=0. \tag{14}$$

Using,

$$\frac{\partial P_c}{\partial x}=\frac{\partial P_c}{\partial s_w}\frac{\partial s_w}{\partial x},$$

we get

$$\emptyset\frac{\partial S_w}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\frac{k_w}{\mu_w}k\frac{k_o}{\mu_o}\frac{\partial P_c}{\partial s_w}\frac{\partial s_w}{\partial x}}{\frac{k_w}{\mu_w}+\frac{k_o}{\mu_o}}\right)=0. \tag{15}$$

According to Schreidegger [14],

$$\frac{\frac{k_w}{\mu_w}\frac{k_o}{\mu_o}}{\left(\frac{k_w}{\mu_w}+\frac{k_o}{\mu_o}\right)}\approx\frac{k_o}{\mu_o}. \tag{16}$$

Therefore, equation (15) reduces to,

$$\emptyset\frac{\partial S_w}{\partial t}+k\frac{\partial}{\partial x}\left(\frac{k_o}{\mu_o}\frac{\partial P_c}{\partial s_w}\frac{\partial s_w}{\partial x}\right)=0. \tag{17}$$

As per Scheidegger and Johnson [15], where relative permeability is assumed to be linear for water and oil,

$$k_w=S_w,$$

$$k_o=1-\alpha S_w,\ \ \text{where } \alpha \text{ is a constant} \tag{18}$$

The capillary pressure is expressed in terms of saturation as,

$$P_c=-\beta S_w, \tag{19}$$

where $\beta$ is a constant. From equations (16), (17) and (18), we get

$$\emptyset\frac{\partial S_w}{\partial t}=\frac{k\beta}{\mu_o}\frac{\partial}{\partial x}\left((1-\alpha S_w)\frac{\partial S_w}{\partial x}\right). \tag{20}$$

The imbibition phenomenon is given by the non-linear partial differential equation given in (20). The initial condition is,

$$S_w(x,0) = f(x), \quad 0 \le x \le l.$$

The boundary conditions are,

$$S_w(0,t) = f(t), \quad 0 < t \le 1,$$
$$S_w(l,t) = g(t), \quad 0 < t \le 1.$$

The saturation of water is dependent on time $t > 0$ at $x = 0$ and $x = l$, where $x = l$ is very near to common interface ($x = 0$) and $l \ll L$. Converting equation (20) into dimension less form, using dimensionless parameters

$$X = \frac{x}{l}, \quad T = \frac{k\beta}{\emptyset \mu_o L^2} t.$$

Equation (20) is transformed to,

$$\frac{\partial S_w}{\partial T} = (1 - \alpha S_w)\frac{\partial^2 S_w}{\partial X^2} - \alpha \left(\frac{\partial S_w}{\partial X}\right)^2. \tag{21}$$

Transforming the initial and boundary conditions, we get

$$S_w(X,0) = f(X), \quad 0 \le X \le 1,$$
$$S_w(0,T) = f(T), \quad 0 < T \le 1,$$
$$S_w(l,T) = g(T). \quad 0 < T \le 1.$$

## 3. Problem Solution

Consider imbibition phenomenon equation

$$\frac{\partial S_w}{\partial T} = (1 - \alpha S_w)\frac{\partial^2 S_w}{\partial X^2} - \alpha \left(\frac{\partial S_w}{\partial X}\right)^2.$$

The initial condition [9] is,

$$S_w(X,0) = \frac{(X + X^2)}{6}, \quad 0 \le X \le 1, \tag{22}$$

where the saturation of injected liquid is chosen to be a quadratic function of '$X$', which is an increasing function of '$X$' for $X > 0$, in agreement with the physical phenomenon where saturation of injected water increases with distance and time.

Taking appropriate boundary conditions [9],

$$\left.\begin{aligned} S_w(0,T) &= 0, & 0 < T \le 1, \\ S_w(1,T) &= \tfrac{(1+T)}{3}, & 0 < T \le 1. \end{aligned}\right\} \tag{23}$$

Equation (21) is solved by the *Hybrid Differential Transform and Finite Difference Method* (HDTFDM) for small values of $T$ and Multistep Hybrid Differential Transform Method for large values of $T$.

### 3.1 Methodology

The non-linear partial differential equation (21) is solved using HDTFDM. This method is a combine of the *Differential Transform Method* (DTM) and the *Finite Difference Method* (FDM).

The spatial variables are approximated using the FDM, while the time variable is approximated by the DTM.

Zhou [20] proposed the DTM to solve differential equations both linear and non linear in electrical circuit analysis Yu and Chen [19] used the hybrid method in mechanical engineering problems.

For the $T$ variable, the differential transform is used, and on the $X$ variable, the finite difference method is used. This combined or hybrid method is efficient in solving linear and nonlinear partial differential equations as it is found to converge rapidly with a few iterations.

## 3.2 Preliminaries

The differential transform of the $k$th derivative of $u(x,t)$ applied to the '$t$' variable is given as

$$U(i,k) = \frac{1}{k!}\left[\frac{d^k u(x,t)}{dt^k}\right]_{t=0}, \tag{24}$$

$$k = 0,1,2,\dots \quad \text{and} \quad i = 0,1,2.$$

The inverse transform of $U(i,k)$ is given as,

$$u(x,t) = \sum_{k=0}^{\infty} U(i,k)t^k, \tag{25}$$

where $u(x,t)$ in lower case letters represents the original function and $U(i,k)$ in upper case letters represents transformed function

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k u(x,t)}{dt^k}\right]_{t=0} t^k, \tag{26}$$

where $U(i,k) = U(x_i,k)$, $x_i = ih$, $i = 0,1,2,3,\dots$; $h$ denotes the step size in spatial direction.

The theorems stated below follow from [2] and [3]:

**Theorem 3.1.** *If $f(x,t) = \frac{\partial m}{\partial t}$, then $F(i,k) = (k+1)M(i,k+1)$.*

**Theorem 3.2.** *If $f(x,t) = \frac{\partial^2 m}{\partial t^2}$, then $F(i,k) = (k+1)(k+2)M(i,k+2)$.*

**Theorem 3.3.** *If $f(x,t) = xe^{-t}$, then $F(i,k) = x\frac{(-1)^k}{k!}$.*

**Theorem 3.4.** *If $f(x,t) = \sin x$, then $F(i,k) = \sin x$.*

**Theorem 3.5.** *If $f(x,t) = \sin t$, then $F(i,k) = \sin\left(\frac{\pi k}{2}\right)\frac{1^k}{k!}$.*

**Theorem 3.6.** *If $f(x,t) = \frac{\partial m}{\partial x}(x,t)$, then $F(i,k) = \frac{M(i+1,m)-M(i-1,m)}{2h}$.*

**Theorem 3.7.** *If $f(x,t) = m(x,t)\frac{\partial m}{\partial x}(x,t)$, then $F(i,k) = \sum_{m=0}^{k} M(i,k-m)\frac{M(i+1,m)-M(i-1,m)}{2h}$.*

## Convergence Criteria for HDTFDM

From eq. (25) we get the series solution for the nonlinear PDE as

$$u(x,t) = \sum_{k=0}^{\infty} U(i,k)t^k,$$

then the convergence of the power series in '$t$' can be found as per the following theorem [7]:

**Theorem 3.8.** *If $\varphi_k(x,t) = U(i,k)(t-t_0)^k$, then the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$, stated in equation above, $\forall\ k \in N \cup \{0\}$ follows the following criteria:*

(i) *Series is convergent if $\exists\ 0 < \lambda < 1$, such that $\|\varphi_{k+1}\| \le \lambda \|\varphi_k\|$,*

(ii) *Series divergent if $\exists\ \lambda > 1$, such that $\|\varphi_{k+1}\| \ge \lambda \|\varphi_k\|$.*

**Theorem 3.9.** *If the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$, where $\varphi_k(x,t) = U(i,k)(t-t_0)^k$ converges to the solution $u(x,t)$ and the truncated series $\sum_{k=0}^{m} \varphi_k(x,t)$ is an approximation to the solution $u(x,t)$, then the maximum absolute truncated error estimated is given as*

$$\left[\!\!\left[ u(x,t) - \sum_{k=0}^{m} \varphi_k(x,t) \right]\!\!\right] \le \frac{1}{1-\lambda} \lambda^{m+1} [\![\varphi_0]\!].$$

### 3.3 Numerical Solution

The *Hybrid Differential Transform and Finite Difference Method* (HDTFDM) and theorems given above are applied to eq. (21)

$$\frac{\partial S_w}{\partial T} = (1-\alpha S_w)\frac{\partial^2 S_w}{\partial X^2} - \alpha \left(\frac{\partial S_w}{\partial X}\right)^2 \tag{27}$$

$$\begin{aligned} \text{subject to } S_w(X,0) &= \frac{(X+X^2)}{6}, \quad && 0 \le X \le 1; \\ S_w(0,T) &= 0, && 0 < T \le 1; \\ S_w(1,T) &= \frac{(1+T)}{3}, && 0 < T \le 1. \end{aligned} \tag{28}$$

Taking $\alpha = 1.11$.

Applying, differential transformation to the $T$ variable and finite difference to the $X$ variable and using theorems given above, we have

$$\begin{aligned} \frac{\partial S_w}{\partial T} &= (k+1)S(i,k+1), \\ S_w \frac{\partial^2 S_w}{\partial X^2} &= \sum_{r=0}^{k} S(i,k-r)\frac{S(i+1,r) - 2S(i,r) + S(i-1,r)}{h^2}, \\ \frac{\partial^2 S_w}{\partial X^2} &= \frac{S(i+1,k) - 2S(i,k) + S(i-1,k)}{h^2}, \\ \left(\frac{\partial S_w}{\partial X}\right)^2 &= \sum_{r=0}^{k} \frac{S(i+1,r) - S(i-1,r)}{2h} \frac{S(i+1,k-r) - S(i-1,k-r)}{2h}, \end{aligned} \tag{29}$$

where $S_w(X,T)$ is the original function and, $S(i,k) = S(X_i,k)$, $X_i = ih$, $i = 0,1,2,3,\dots$ is transformed function.

Transforming the initial and boundary conditions,

$$S(i,0) = S(X_i,0) = f(X_i), \quad X_i = ih,\ i = 0,1,2,3,\dots$$

$$S(0,k) = 0, \quad \text{for } k = 1,2,3,\dots,$$

$$S(N,k) = \frac{(\delta(k) + \delta(k-1))}{3} = \begin{cases} \frac{1}{3}, & \text{for } k = 0, \\ \frac{1}{3}, & \text{for } k = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $N$ is the number of spatial segments. Substituting in eq. (21) we get, according to the hybrid method, the following recurrence relation,

$$\begin{aligned}(k+1)S(i,k+1) &= \frac{S(i+1,k)-2S(i,k)+S(i-1,k)}{h^2}\\ &\quad -\alpha\sum_{r=0}^{k} S(i,k-r))\frac{S(i+1,r)-2S(i,r)+S(i-1,r)}{h^2}\\ &\quad -\alpha\sum_{r=0}^{k}\frac{S(i+1,r)-S(i-1,r)}{2h}\frac{S(i+1,k-r)-S(i-1,k-r)}{2h}.\end{aligned} \tag{30}$$

For $k=0,1,2,3,\dots$, $S(i,0),S(i,1),S(i,2),\dots$ are obtained. The approximate solutions for various values $X$ and $T$ are found using the inverse transformation,

$$S_w(x,t)=\sum_{k=0}^{\infty} S(i,k)T^k$$

when $X_i=0$,

$$\begin{aligned}S_w(0,T) &= \sum_{k=0}^{\infty} S(0,k)T^k = S(0,0)+S(0,1)T+S(0,2)T^2+S(0,3)T^3+\dots\\ &= 0, \quad \text{where } X_i = ih \text{ for } h=0.1,\ i=0,1,2,\dots.\end{aligned} \tag{31}$$

## 4. Results and Discussion

The numerical values of saturation obtained from equation (30) for various distances $X$ and fixed time $T = .001, .002, .003, .004, .005, .006, .007, .008, .009$ and $.1$ are obtained by using MATLAB and presented in Table 1.

**Table 1.** Saturation $S_w(X,T)$ for $X$ and $T$ (HDTFDM)

| HDTFDM | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $X$ \ $T$ | 0.001 | 0.002 | 0.003 | 0.004 | 0.005 | 0.006 | 0.007 | 0.008 | 0.009 | 0.01 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.0186 | 0.0188 | 0.0191 | 0.0193 | 0.0195 | 0.0196 | 0.0198 | 0.0199 | 0.0201 | 0.0202 |
| 0.2 | 0.0403 | 0.0405 | 0.0408 | 0.041 | 0.0412 | 0.0415 | 0.0417 | 0.0419 | 0.0421 | 0.0423 |
| 0.3 | 0.0652 | 0.0655 | 0.0657 | 0.0659 | 0.0661 | 0.0664 | 0.0666 | 0.0668 | 0.067 | 0.0672 |
| 0.4 | 0.0935 | 0.0937 | 0.0939 | 0.0941 | 0.0943 | 0.0945 | 0.0947 | 0.0949 | 0.0951 | 0.0953 |
| 0.5 | 0.1252 | 0.1253 | 0.1255 | 0.1257 | 0.1258 | 0.126 | 0.1261 | 0.1263 | 0.1265 | 0.1266 |
| 0.6 | 0.1601 | 0.1602 | 0.1604 | 0.1605 | 0.1606 | 0.1607 | 0.1609 | 0.161 | 0.1611 | 0.1613 |
| 0.7 | 0.1984 | 0.1985 | 0.1986 | 0.1987 | 0.1987 | 0.1988 | 0.1989 | 0.199 | 0.1991 | 0.1992 |
| 0.8 | 0.24 | 0.2401 | 0.2401 | 0.2402 | 0.2402 | 0.2403 | 0.2403 | 0.2404 | 0.2405 | 0.2405 |
| 0.9 | 0.285 | 0.285 | 0.2851 | 0.2851 | 0.2852 | 0.2853 | 0.2854 | 0.2855 | 0.2856 | 0.2857 |
| 1 | 0.3337 | 0.334 | 0.3343 | 0.3347 | 0.335 | 0.3353 | 0.3357 | 0.336 | 0.3363 | 0.3367 |

From Table 1 we observe that, for fixed time $T$, the saturation $S_w(X,T)$ of injected water increases with $X$ and saturation is also increasing when $T$ is increasing for fixed distance $X$.

## Graphical Representation

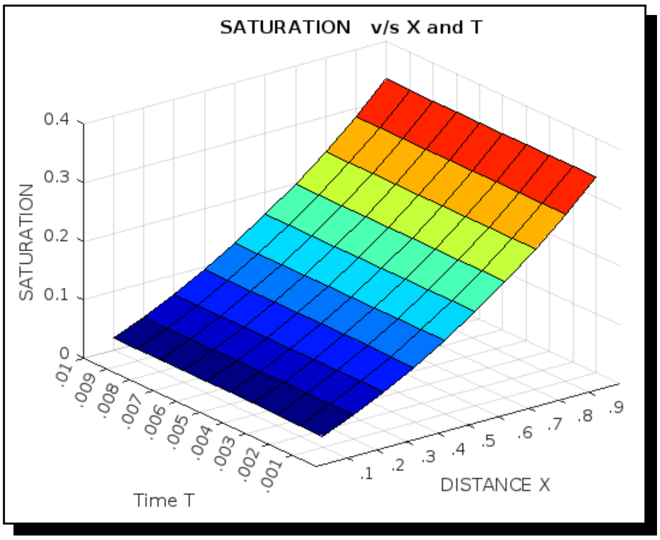


**Figure 3.** 3D plot of saturation $S_w(X,T)$ for $X$ and $T$ by HDTFDM

Figure 3 illustrates that the saturation $S_w(X,T)$ of injected water is increasing when distance $X$ is increasing for fixed time $T$ and saturation is also increasing when time $T$ is increasing for fixed distance $X$.

### Limitation of Hybrid Differential Transform Method

Though the HDFDTM, offers approximate solutions to a vast number of nonlinear problems, it has its drawbacks. In a small region, the series solution always converges quickly, while in a larger region it converges slowly. The *Multi-step DTM* (MDTM) combined with *Finite Difference Method* (FDM) is an improved method which helps to accelerate the series solution convergence over a large region while also improving DTM accuracy.

## 5. Multistep Differential Transform Method

The computation interval $[0,T]$ is not always small, hence the domain $T$ is partitioned into $N$ subdomains to speed convergence and increase calculation accuracy.

The key advantage of partitioning the domain is that the solution can be obtained in a small interval of time using a few Taylor series terms. It is important to remember that, if necessary, the time interval might be chosen to be arbitrarily small. The differential equations in each subdomain can be thus be solved.

### 5.1 Methodology

Multi-step DTM solutions have a large interval of convergence as compared to DTM solutions whose interval of convergence is small. This demonstrates how the MDTM helps in increasing the interval of convergence. Moreover, we can improve the accuracy by decreasing $h$ the interval of differencing and increasing the number of terms in the series in each subinterval.

**Definition 5.1.** Differential Transform of $x(t)$ given as

$$X(k)=\frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}, \tag{32}$$

where $x(t)$ is called the original function and $X(k)$ is called the transformed function.

The inverse transform is given as,

$$x(t)=\sum_{k=0}^{k=\infty} X(k)(t-t_0)^k \tag{33}$$

$$=\sum_{k=0}^{k=\infty}\frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}(t-t_0)^k \tag{34}$$

and the considering a finite number of terms $x(t)$ is considered as below

$$x(t)=\sum_{k=0}^{k=m}\frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}(t-t_0)^k, \tag{35}$$

where $m$ represents number of Taylor series components. Increasing the number of terms can improve the accuracy of the solution.

### 5.2 Solution of Partial Differential Equation in $u(x,t)$ in Domain $[0,T]$

We divide the domain $[0,T]$ into $n$ sections where $H=T/N$ is the length of each subdomain. Thus, a separate function is considered for each sub domain.

$$u(x,t)=\begin{cases} u_1(j,t), & t\in[t_1,t_2],\\ u_2(j,t), & t\in[t_2,t_3], \qquad \text{here } t_i=(i-1)H,\\ u_N(j,t), & t\in[t_N,t_{N+1}],\end{cases} \tag{36}$$

$$u_i(j,t)=\sum_{k=0}^{m} U_i(j,k)\left(\frac{t-t_i}{H}\right)^k, \tag{37}$$

where

$$U_i(j,k)=\frac{H^k}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=t_0}, \quad k=0,1,2,\dots \text{ and } i=0,1,2. \tag{38}$$

The solution of equation (21) is of the form

$$S_{wi}(j,t)=\sum_{k=0}^{m} S_i(j,k)\left(\frac{t-t_i}{H}\right)^k, \quad \text{for } t\in[t_i,t_{i+1}],$$

$$S_i(j,k)=\frac{H^k}{k!}\left[\frac{\partial^k S_w(x,t)}{\partial t^k}\right]_{t=t_0}.$$

Applying central finite difference to the spatial variable $X$ and MDTM to the variable $T$, to eq. (21), the following recurrence relation can be obtained

$$\begin{aligned}\frac{(k+1)S_i(j,k+1)}{H}=&\frac{S_i(j+1,k)-2S_i(j,k)+S_i(j-1,k)}{h^2}\\ &-\alpha\sum_{r=0}^{k} S_i(j,k-r)\frac{S_i(j+1,r)-2S_i(j,r)+S_i(j-1,r)}{h^2}\\ &-\alpha\sum_{r=0}^{k}\frac{S_i(j+1,r)-S_i(j-1,r)}{2h}\,\frac{S_i(j+1,k-r)-S_i(j-1,k-r)}{2h},\end{aligned} \tag{39}$$

where $S_i(j,k)$ is the differential transform of $S_{wi}(j,t)$.

Here, separate functions are considered in each sub domain, i.e.,

$$S_w(x,t) = \begin{cases} S_{w_1}(x,t), & t \in [t_1,t_2], \\ S_{w_2}(x,t), & t \in [t_2,t_3], \\ S_{w_N}(x,t), & t \in [t_N,t_{N+1}], \end{cases} \tag{40}$$

where $t_i = (i-1)H$.

*Applying Multistep Differential Transform Method* (MDTM) to the $T$ variable and *Finite Difference Method* (FDM) on the initial conditions, we have

$$S_i(X_j,0) = \frac{X_j + X_j^2}{6}.$$

Boundary conditions are:

$$S_i(0,k) = 0 \quad \text{for } k = 1,2,3,\dots,$$

$$S_i(n,k) = \frac{(\delta(k)+\delta(k-1))}{3} = \begin{cases} \frac{1}{3}, & \text{for } k = 0, \\ \frac{1}{3}, & \text{for } k = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $n$ is the number of spatial segments. Here $n = 10$, $T = 1$, $H = .01$.

Firstly, the MDTM is applied to the given PDE over the interval $[0,.01]$. For the next time step, the value at $t = .01$ is used as an initial condition in the interval $[.01,.02]$, i.e., in general, the continuity condition should be used in each time subdomain.

MDTM is implemented by dividing the solution interval $[0,1]$ into 100 subintervals of equal step size given by $H = 0.01$.

## 6. Results and Discussion

The numerical values of the saturation $S_{w1}(x,t)$ are obtained from equation (39) for various distances $X$ at fixed time $T = 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09$ and $.1$ by using MATLAB and presented in Table 2.

**Table 2.** Saturation $S_{w1}(X,T)$ for distance $X$ and time $T$ by *Hybrid Multistep Differential Transform and Finite Difference Method* for $T \in [0.01,.1]$

| $X$ \ $T$ | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 | 0.1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.0202 | 0.0213 | 0.0221 | 0.0227 | 0.0232 | 0.0237 | 0.0241 | 0.0244 | 0.0247 | 0.025 |
| 0.2 | 0.0423 | 0.0441 | 0.0455 | 0.0466 | 0.0476 | 0.0484 | 0.0492 | 0.0498 | 0.0504 | 0.051 |
| 0.3 | 0.0672 | 0.0692 | 0.0709 | 0.0724 | 0.0736 | 0.0748 | 0.0758 | 0.0767 | 0.0776 | 0.0784 |
| 0.4 | 0.0953 | 0.0972 | 0.0989 | 0.1004 | 0.1019 | 0.1032 | 0.1044 | 0.1056 | 0.1067 | 0.1078 |
| 0.5 | 0.1266 | 0.1282 | 0.1298 | 0.1313 | 0.1327 | 0.1341 | 0.1355 | 0.1369 | 0.1382 | 0.1395 |
| 0.6 | 0.1612 | 0.1625 | 0.1638 | 0.1652 | 0.1666 | 0.1681 | 0.1696 | 0.1712 | 0.1727 | 0.1743 |
| 0.7 | 0.1992 | 0.2001 | 0.2012 | 0.2025 | 0.204 | 0.2057 | 0.2074 | 0.2092 | 0.211 | 0.2128 |
| 0.8 | 0.2404 | 0.2412 | 0.2424 | 0.244 | 0.2458 | 0.2477 | 0.2497 | 0.2518 | 0.254 | 0.2562 |
| 0.9 | 0.285 | 0.2864 | 0.2883 | 0.2905 | 0.2929 | 0.2954 | 0.298 | 0.3006 | 0.3032 | 0.3059 |
| 1 | 0.3367 | 0.34 | 0.3433 | 0.3467 | 0.35 | 0.3533 | 0.3567 | 0.36 | 0.3633 | 0.3667 |

From Table 2, we observe that the values of saturation converge in this interval by using MDTM. We continue finding $S_{w2}(x,t), S_{w3}(x,t), \dots$ for the remaining intervals [.1,.2], [.2,.3] and so on and saturation at $T = .1, .2, .3 \dots$ is shown in Table 3.

**Table 3.** Saturation $S_w(X,T)$ for distance $X$ and time $T$ by *Hybrid Multistep Differential Transform and Finite Difference Method* for $T \in [0.1, 1]$

| $X$ \ $T$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.025 | 0.0274 | 0.0294 | 0.0314 | 0.0332 | 0.0349 | 0.0364 | 0.0379 | 0.0393 | 0.0405 |
| 0.2 | 0.051 | 0.0558 | 0.0601 | 0.0641 | 0.0679 | 0.0714 | 0.0746 | 0.0777 | 0.0805 | 0.0831 |
| 0.3 | 0.0784 | 0.0858 | 0.0924 | 0.0986 | 0.1045 | 0.11 | 0.1151 | 0.1199 | 0.1243 | 0.1284 |
| 0.4 | 0.1078 | 0.1177 | 0.1268 | 0.1354 | 0.1436 | 0.1512 | 0.1583 | 0.165 | 0.1712 | 0.1769 |
| 0.5 | 0.1395 | 0.1521 | 0.1639 | 0.1751 | 0.1857 | 0.1958 | 0.2051 | 0.2139 | 0.2221 | 0.2296 |
| 0.6 | 0.1743 | 0.1897 | 0.2044 | 0.2185 | 0.2319 | 0.2447 | 0.2566 | 0.2678 | 0.2783 | 0.2879 |
| 0.7 | 0.2128 | 0.2315 | 0.2495 | 0.2668 | 0.2835 | 0.2993 | 0.3143 | 0.3284 | 0.3416 | 0.3539 |
| 0.8 | 0.2562 | 0.2786 | 0.3005 | 0.3218 | 0.3423 | 0.362 | 0.3808 | 0.3987 | 0.4155 | 0.4311 |
| 0.9 | 0.3059 | 0.333 | 0.3598 | 0.386 | 0.4117 | 0.4366 | 0.4608 | 0.484 | 0.5062 | 0.5272 |
| 1 | 0.3667 | 0.4 | 0.4333 | 0.4667 | 0.5 | 0.5333 | 0.5667 | 0.6 | 0.6333 | 0.6667 |

It is observed that, MDTM gives a series solution which converges for wide time region, which is not possible by the traditional DTM.

From Table 3, we observe that the saturation $S_w(X,T)$ of injected water is increasing when distance $X$ is increasing for fixed time $T$ and saturation is also increasing when time $T$ is increasing for fixed distance $X$.

## Graphical Representation

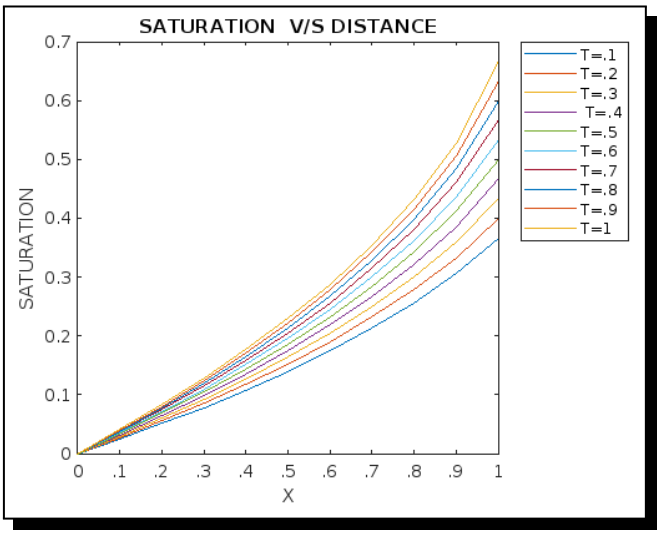


**Figure 4.** Saturation $S_w(X,T)$ for distance $X$ and time $T$ by *Hybrid Multistep Differential Transform Finite Difference Method*

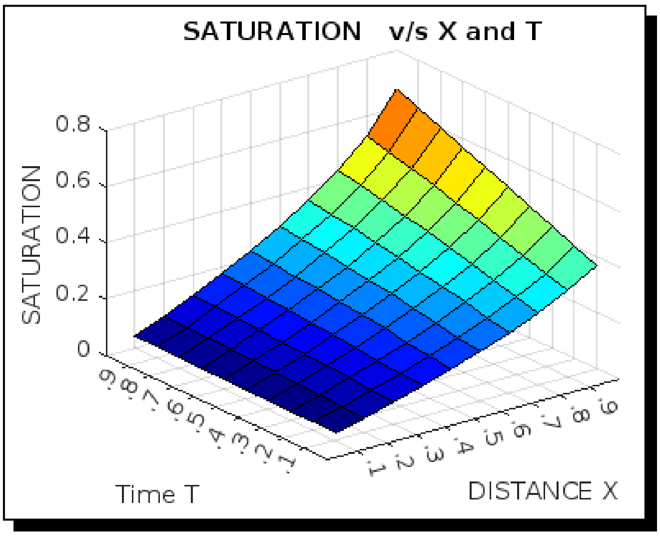


**Figure 5.** 3D plot of saturation $S_w(X,T)$ for different values of distance $X$ and time $T$ by *Hybrid Multistep Differential Transform Finite Difference Method*

Figure 5 is a 3D plot of Saturation versus Distance and Time. Here, we can clearly observe that the saturation $S_w(X,T)$ increases with increasing distance $X$ for fixed time $T$ and also increasing with increasing time $T$ for fixed distance $X$.

The above solution was compared with *Homotopy Analysis Method* [10] and the results were found to be in close agreement.

**Table 4.** Comparison of *Hybrid Multistep Differential Transform Method* (HDTFDM) and *Homotopy Analysis Method* (HAM) [10]

| $X$ \ $T$ | 0.1 | | 0.3 | | 0.5 | | 0.7 | | 0.9 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | HDTFDM | HAM | HDTFDM | HAM | HDTFDM | HAM | HDTFDM | HAM | HDTFDM | HAM |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.025 | 0.0254 | 0.0294 | 0.0296 | 0.0332 | 0.0333 | 0.0364 | 0.0365 | 0.0393 | 0.0393 |
| 0.2 | 0.051 | 0.0518 | 0.0601 | 0.0604 | 0.0679 | 0.068 | 0.0746 | 0.0748 | 0.0805 | 0.0805 |
| 0.3 | 0.0784 | 0.0796 | 0.0924 | 0.0928 | 0.1045 | 0.1047 | 0.1151 | 0.1152 | 0.1243 | 0.1243 |
| 0.4 | 0.1078 | 0.1091 | 0.1268 | 0.1274 | 0.1436 | 0.1439 | 0.1583 | 0.1585 | 0.1712 | 0.1712 |
| 0.5 | 0.1395 | 0.1409 | 0.1639 | 0.1646 | 0.1857 | 0.1861 | 0.2051 | 0.2054 | 0.2221 | 0.2221 |
| 0.6 | 0.1743 | 0.1756 | 0.2044 | 0.2052 | 0.2319 | 0.2325 | 0.2566 | 0.2569 | 0.2783 | 0.2783 |
| 0.7 | 0.2128 | 0.2141 | 0.2495 | 0.2504 | 0.2835 | 0.2841 | 0.3143 | 0.3147 | 0.3416 | 0.3416 |
| 0.8 | 0.2562 | 0.2575 | 0.3005 | 0.3016 | 0.3423 | 0.3431 | 0.3808 | 0.3813 | 0.4155 | 0.4155 |
| 0.9 | 0.3059 | 0.3075 | 0.3598 | 0.3612 | 0.4117 | 0.4128 | 0.4608 | 0.4616 | 0.5062 | 0.5065 |
| 1 | 0.3667 | 0.3667 | 0.4333 | 0.4333 | 0.5 | 0.5 | 0.5667 | 0.5667 | 0.6333 | 0.6333 |

Table 4 shows the comparison of Hybrid Multistep Differential Transform Finite Difference Method with Homotopy Analysis Method and the results are found to be in close agreement.

## 7. Conclusions

Numerical solution and graphs are obtained using MATLAB. The graph given by Figure 2 shows that saturation of water increases with $X$ for given time $T$. The saturation of water increases with distance as well as with time, which is consistent with physical phenomena.

Multi-step DTM, was used in this work for the variable $T$, and it was found to improve the solution for large values of $T$. This method can be used to solve linear and nonlinear differential equations to obtain approximate numerical solutions. Table 1 shows that using DTM we get solutions which converge over a small interval of convergence, whereas Multi-step DTM solutions have a wide interval of convergence, as shown in Table 3. This demonstrates that the MDTM increases the interval of convergence for the series solution.

Comparative study of obtaining the result by the two methods, namely, HDTFDM and *Homotopy Analysis Method* (HAM) shows that results closely agree with each other.

Complex symbolic computation is not necessary because the HDTFDM calculates numerical solutions through an iterative procedure. It has been shown that the suggested approach can produce very precise numerical approximations and that Multistep DTM improves the obtained solution. Most importantly there is no need of using linearization.

As a result, this method can help to solve a wide range of difficult partial differential equations, linear and also nonlinear, without any need for linearization or perturbation.

### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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