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Research Article

Meshless Radial Basis Function Pseudo-Spectral Method for Solving Non-linear KdV Equation

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Abstract. The Korteweg-De Vries (KdV) problem is solved in this study using a meshless strategy based on the radial basis function. The nonlinear KdV equation is solved using the radial basis function in conjunction with the pseudo-spectral method. With the aid of a radial basis function, the method transforms the problem into a system of ODEs, which are subsequently solved by an ODE solver. The usefulness and efficiency of the strategy are assessed using two numerical examples. The numerical results are well-aligned with the exact solutions found in the literature.

Keywords. Collocation method, KdV equation, Radial basis function, Shape parameter

Mathematics Subject Classification (2020). 35A25, 35C08, 35Q53

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1. Introduction

The well-known KdV equation is

$$\frac{\partial u(x,y)}{\partial t}+\alpha u(x,t)+\beta\frac{\partial^2 u(x,t)}{\partial x^2}=0, \tag{1.1}$$

where α,β are non-zero real constants. It was introduced by Joseph Boussinesq (cf. Guan and Kuksin [9]) as a model for shallow water wave propagation. It gained notoriety after being utilized by Diederik Korteweg and Gustav De Vries (cf. Guan and Kuksin [9]) to explain the occurrence of a soliton water wave. The KdV equation is vital for understanding how low amplitude water waves propagate. The solutions to the KdV PDE are soliton or solitary waves (Kolebaje and Oyewande [14]). A key component of a model for waves over shallow water is the Korteweg-de Vries equation, which provides an example of the propagation of weakly dispersive and weakly nonlinear waves[1]. The KdV equation can be used to explain a wide range of important scientific phenomena. This research focuses on two wave types: shallow-water waves and ion-acoustic plasma waves. The numerical solution of the KdV equation is crucial since it is used in the investigation of non-linear dispersive waves (Zabusky [18]). Numerous numerical techniques were employed to tackle the KdV problem, with the methodology falling into the following groups: spectral methods (Helal [11]), finite difference methods (Qu and Wang [15], and Kolebaje and Oyewande [14]), and finite element methods (Arnold and Winther [3]).

The Kansa approach is one of the meshless methods that Kansa [12, 13] developed in 1990 for estimating the solutions of nonlinear partial differential equations. It has recently grown in acceptance among academics (Arghand and Amirfakhrian [2], Arora and Bhatia [4], Dehghan and Shokri [6], and Safdari-Vaighani *et al*. [16]). The RBFs, especially the multiquadric RBFs, are combined to quantitatively approximate the response (MQ). RBF collocation techniques are meshless and mathematically easier than conventional mesh-based techniques (Fasshauer [7], and Trefethen [17]). The pseudo-spectral approach and the radial basis function are used in this study to numerically solve the third order non-linear KdV problem.

Due to the way this study is structured, the radial basis functions approximation method is first introduced in Section 2. In Section 3, this methodology is then applied to the KdV equation using the R-K method for temporal discretization. Section 4 presents the numerical experiments, and Section 5 provides a succinct summary of the findings.

2. Approximation using Radial Basis Function

Radial Basis Function (RBF) is a mathematical tool used to represent real-valued functions. It is characterized by its dependence on the radial distance between its input and a set of predefined centers or prototypes in a multi-dimensional space. therefore $\phi(x) = \hat{\phi}(\|x\|)$. If c is a fixed point, then $\phi(x) = \hat{\phi}(\|x - c\|)$ (Buhmann [5]).

Alternative metrics are occasionally used, however Euclidean distance is typically how the distance is calculated. They are typically used as a collocation that forms the basis of a function space. Sums calculated using RBF are widely used to approximate functions. The RBFs most frequently utilized are listed in Table 1 (Fornberg and Flyer [8]).

RBFs are so widely used to produce approximation of functions of the kind

$$u(x) = \sum_{i=1}^{n} w_i \phi(\|x - x_i\|). \tag{2.1}$$

The approximation function u can be represented as the sum of n Radial Basis Functions (RBFs), each with a distinct Centre and weighted by an estimated coefficient. The weights of the RBFs can be calculated using a matrix of linear least squares, which can be employed for this purpose. A function can be approximated using this technique by combining various RBFs, each with its own unique weights.

[1]E. M. De Jager, On the origin of the Korteweg–de Vries equation, (2006), pp. 1 – 25, URL: http://arxiv.org/abs/math/0602661.

Table 1. Commonly used RBFs

Multiquadric	$\phi(r) = \sqrt{1+\varepsilon^2 r^2}$
Inverse Quadric	$\phi(r) = 1/(1+\varepsilon^2 r^2)$
Inverse Multiquadric	$\phi(r) = 1/\sqrt{1+\varepsilon^2 r^2}$
Gaussian	$\phi(r) = e^{-\varepsilon^2 r^2}$
Spline (Polyharmonic)	$\phi(r) = r^k$, $k = 1,3,5,\dots$ $\phi(r) = r^k \log(r)$, $k = 2,4,6,\dots$
Spline (Thin plate)	$\phi(r) = r^2 \log(r)$

$r = \|x - x_j\|$ and ε are the shape parameters for sizing the input to the radial kernel

3. RBF-PS Method for the KdV Equation

To solve

$$u_t + \alpha u u_x + \beta u_{xxx} = 0. \tag{3.1}$$

Let the RBF approximation be

$$\tilde{u}(x,t) = \sum_{j=1}^{n} \lambda_i \phi(\|x - x_j\|), \tag{3.2}$$

where $\phi(\|x - x_j\|)$ is a function of known RBF and $\|\cdot\|$ is the Euclidean Norm, λ_i, $i = 1,2,3,\dots$ are the expansion coefficients, which can be obtained at the nodes.

In RBF meshless method approach for each collocation points x_i, $i = 1,2,\dots$. We may represent equation (3.2) in matrix form as

$$u = A\lambda. \tag{3.3}$$

The matrix A entries are $A_{ij} = \phi(\|x - x_j\|)$.
Differentiating equation (3.3) with respect to x,

$$u_x = A_x \lambda, \tag{3.4}$$

where $A_x = \frac{d}{dx}\phi(\|x - x_j\|)_{x=x_i}$.

By solving equations (3.3) and (3.4) for the value of λ, we get the differentiation matrix.

Thus, we have

$$u_x = A_x A^{-1} u \quad \text{or} \quad u_x = D_x u, \tag{3.5}$$

where $D_{xx} = A_{xx} A^{-1}$ is the differentiation matrix.

Similarly, we can write

$$u_{xx} = A_{xx} A^{-1} u = D_{xx}, \tag{3.6}$$

where $D_{xx} = A_{xx} A^{-1}$ and the entries of the matrix A_{xx} are $\frac{d^2}{dx^2}\phi(\|x - x_j\|)_{x-x_i}$.

The same method can be used to compute the higher-order differentiation matrices.

The RBF-PS approach corresponding to equation (3.1) is given by using the above differentiation matrices.

$$\frac{du}{dt} = -\alpha u * D_x u - \beta D_{xxx} u. \tag{3.7}$$

This equation is of the form

$$\frac{du}{dt} = F(u). \tag{3.8}$$

To discretize the equation (3.8) in the time domain, we can use ODE solver for example the fourth-order R-K method.

4. Numerical Examples and Comparisons

This section displays the numerical outcomes of the proposed method for resolving the KdV equation. We show the precision and flexibility of the proposed approach on two different challenges. As indicated by the following, we analyze numerical errors using three different types of norms.

$$L_\infty = \max|u-\tilde{u}|, \tag{4.1}$$

$$L_2 = \sqrt{h\sum(u-\tilde{u})^2}, \tag{4.2}$$

$$RMS = \sqrt{\frac{\sum(u-\tilde{u})^2}{n}}, \tag{4.3}$$

where $\tilde{u}$ is numerical solution of u, and u is the exact solution.

Example 4.1. Using the proposed method, we solve the 3rd order non-linear KdV equation (3.1) with $\alpha = \beta = 1$ and with the initial condition

$$u(x,0) = 3A^2 \operatorname{sech}^2\left\{AL\frac{(x-x_0/L)}{2}\right\},$$

where $A = \frac{1}{\sqrt{6}}$, $x_0 = 0$, $L = 1$ with zero flux boundary conditions.

The exact solution is

$$u(x,t) = 3A^2 \operatorname{sech}^2\left\{AL\frac{(x-x_0/L)}{2} - \frac{A^3 t}{2}\right\} \quad \text{(Haberman [10])}$$

Table 2. L_∞, L_2 and RMS errors of Example 4.1

RBF	L_∞	L_2	RMS	t
GA	7.28E-14	2.55E-13	1.14E-14	1
MQ	0.0019	0.0081	3.62E-04	
IMQ	0.0019	0.0081	3.62E-04	
GA	1.40E-13	4.79E-13	2.14E-14	2
MQ	0.0038	0.0162	7.24E-04	
IMQ	0.0038	0.0162	7.24E-04	
GA	2.09E132	6.77E-13	3.03E-14	3
MQ	0.0057	0.0243	0.0011	
IMQ	0.0057	0.0243	0.0011	
GA	2.54E-13	8.72E-13	3.90E-14	4
MQ	0.0075	0.0323	0.0014	
IMQ	0.0075	0.0323	0.0014	
GA	2.94E-13	1.07E-12	4.78E-14	5
MQ	0.0094	0.0404	0.0018	
IMQ	0.0094	0.0404	0.0018	

(GA: Gaussian, MQ: Multiquadric, IMQ: Inverse Multiquadric, RMS: Root Mean Square)

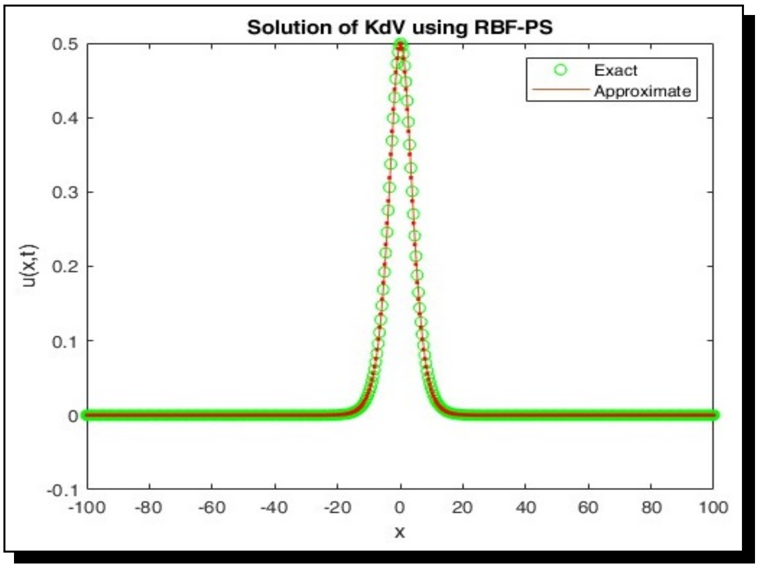

Figure 1. Approximate solution vs exact solution of Example 4.1

Using the suggested approach to numerically solve the equation, the various errors at $t = 1,2,3,4,5$ for different RBFs (Gaussian (GA), Multiquadric (MQ), and Inverse Multiquadric (IMQ)) are shown in Table 2. The graphs of the approximate and exact solutions at $t = 5$, are shown in Figure 1, proving the proposed method's excellent accuracy.

Example 4.2. Now, we solve the KdV equation (3.1) with $\alpha = 1$, $\beta = 4.84 \times 1-^{-4}$ and with initial condition $u(x,0) = 3C\,\text{sech}^2\{Ax + D\}$ where $A = \frac{1}{2}\left(\frac{\alpha C}{\beta}\right)^{\frac{1}{2}}$, $C = 0.3$, $D = -6$ with zero flux boundary conditions. The exact solution is $u(x,t) = 3C\,\text{sech}^2\{Ax - Bt + D\}$ where $B = \alpha CA/2$ (Haberman [10]).

Table 3. L_∞, L_2 and RMS errors of Example 4.2

RBF	L_∞	L_2	RMS	t
GA	7.37E-06	1.83E-05	1.83E-06	1
MQ	7.37E-06	2.63E-05	2.63E-06	
IMQ	7.37E-06	2.39E-05	2.39E-06	
GA	1.47E-05	3.64E-05	3.64E-06	2
MQ	1.47E-05	5.24E-05	5.24E-06	
IMQ	1.47E-05	4.77E-05	4.77E-05	
GA	2.20E-05	5.44E-05	5.44E-06	3
MQ	2.20E-05	7.83E-05	7.83E-06	
IMQ	2.20E-05	7.14E-05	7.14E-06	
GA	2.92E-05	7.22E-05	7.22E-06	4
MQ	2.92E-05	1.04E-04	1.04E-05	
IMQ	2.92E-05	9.48E-05	9.48E-06	
GA	3.64E-05	8.99E-05	8.99E-06	5
MQ	3.64E-05	1.30E-04	1.30E-05	
IMQ	3.64E-05	1.18E-04	1.18E-05	

(GA: Gaussian, MQ: Multiquadric, IMQ: Inverse Multiquadric, RMS: Root Mean Square)

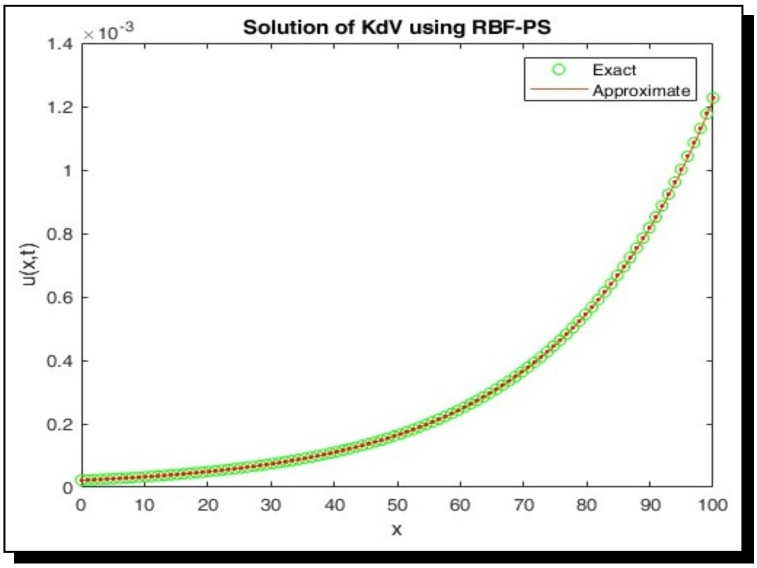

Figure 2. Approximate solution vs exact solution of Example 4.2

Using the suggested approach to numerically solve the equation, the various errors at $t = 1,2,3,4,5$ for different RBFs (Gaussian (GA), Multiquadric (MQ), and Inverse Multiquadric (IMQ)) are shown in Table 3. The graphs of the approximate and exact solutions at $t = 5$, are shown in Figure 2, proving the proposed method's excellent accuracy.

5. Conclusion

The third order non-linear KdV problem was solved using a meshless radial basis function pseudo-spectral approach. This method was found to be highly accurate, efficient, and cost-effective in terms of computing power. The temporal discretization of the problem was done using the R-K technique, and the radial basis functions used to approximate the numerical solution were *Multiquadric* (MQ), *Inverse Multiquadratic* (IMQ), and *Gaussian* (GA). The results of this study demonstrate the effectiveness of the meshless radial basis function pseudo-spectral technique for solving non-linear problems, and provide a potential avenue for further exploration.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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