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Research Article

Some Properties of Kenmotsu Manifolds Admitting a New Type of Semi-Symmetric Non-Metric Connection

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Abstract. In this paper, we study some properties of Kenmotsu manifolds admitting a semi-symmetric non-metric connection. Some curvature's properties of Kenmotsu manifolds that admits a semi-symmetric non-metric connection are obtained. Semi-symmetric, Ricci semi-symmetric and locally ϕ-symmetric conditions for Kenmotsu manifolds with respect to semi-symmetric non-metric connection are also studied. It is proved that the manifold endowed with a semi-symmetric non-metric connection is regular. We obtain some conditions for semi-symmetric and Ricci semi-symmetric Kenmotsu manifolds endowed with semi-symmetric non-metric connection $\tilde{\nabla}$. It is further observed that the Ricci soliton of data (g,ξ,Θ) are expanding and shrinking respectively for semi-symmetric and Ricci semi-symmetric Kenmotsu manifolds admitting a semi-symmetric non-metric connection.

Keywords. Kenmotsu manifold, Semi-symmetric non-metric connection, Semi-symmetric manifold, Ricci semi-symmetric manifold, Locally ϕ-symmetric Kenmotsu manifold, Curvature tensor, Ricci tensor, Einstein manifold

Mathematics Subject Classification (2020). 53C15, 53C25, 53C05, 53D10

1. Introduction

The investigations of a differentiable manifold with contact and almost contact metric structures had been initiated by Boothby and Wang [5]. A class of almost contact metric manifold and named as Kenmotsu manifold has been initiated by Kenmotsu [16]. Further, the characteristics of Kenmotsu manifolds have been investigated by many authors such as Sinha and Srivastava [23], Chaubey and Yildiz [6], Chaubey and Ojha [8], Chaubey and Yadav [9], Özgür and De [19] and many others. The main invariants of an affine connection are its torsion and curvature (Friedmann and Schouten [14]). A torsion tensor of a connection is defined as under

$$\mathcal{T}(\mathcal{K}_1,\mathcal{K}_2) = \nabla_{\mathcal{K}_1}\mathcal{K}_2 - \nabla_{\mathcal{K}_2}\mathcal{K}_1 - [\mathcal{K}_1,\mathcal{K}_2], \quad \text{for all } \mathcal{K}_1,\mathcal{K}_2 \in \mathfrak{X}(\mathfrak{M}), \tag{1.1}$$

where $\mathfrak{X}(\mathfrak{M})$ is a set of all smooth vector fields on $\mathfrak{M}$. A connection ∇ is symmetric and non-symmetric according as $\mathcal{T}(\mathcal{K}_1,\mathcal{K}_2) = 0$, and $\mathcal{T}(\mathcal{K}_1,\mathcal{K}_2) \neq 0$, respectively. The idea of semi-symmetric connection on a differentiable manifold has been proposed by Friedmann and Schouten [14]. A linear connection $\widetilde{\nabla}$ on $\mathfrak{M}^n$ is called semi-symmetric if

$$\widetilde{\mathcal{T}}(\mathcal{K}_1,\mathcal{K}_2) = \eta(\mathcal{K}_2)\mathcal{K}_1 - \eta(\mathcal{K}_1)\mathcal{K}_2, \quad \text{for all } \mathcal{K}_1,\mathcal{K}_2 \in \mathfrak{X}(\mathfrak{M}), \tag{1.2}$$

where η is a 1-form associated with the vector field ξ and satisfies

$$\eta(\mathcal{K}_1) = g(\mathcal{K}_1,\xi), \quad \text{for all } \mathcal{K}_1,\mathcal{K}_2 \in \mathfrak{X}(\mathfrak{M}). \tag{1.3}$$

Further, the idea of metric connection with torsion on a Riemannian manifold has been initiated by Hayden. A connection ∇ is metric and non-metric on $\mathfrak{M}$ according as $\nabla g = 0$ and $\nabla g \neq 0$, respectively, where g is a Reimannian metric in $\mathfrak{M}$. The idea of semi-symmetric non-metric connection has been initiated by Agashe and Chafle [1]. The quarter-symmetric connection in a differentiable manifold with affine connection has been investigated by Golab [15]. Further, characteristics of quarter-symmetric metric connection have been investigated by several geometers like Rastogi [22], Mishra and Pandey [18], Yano and Imai [28], Kumar *et al*. [17], and many others. The semi-symmetric non-metric connection in a Kenmotsu manifold has been investigated by Tripathi and Nakkar [26]. In line with this, Chaubey and Yildiz [6] initiated another semi-symmetric non-metric connection. Later on some other authors, like De and Pathak [10], Pankaj *et al*. [20,21] studied several connections. Tripathi [25] has justified the presence of a new connection and showed that in special cases.

We have decided by above investigations to study characteristics of Kenmotsu manifolds admitting a semi-symmetric non-metric connection. Section 1 is introductory. Section 2 is concerned with some basic results of Kenmotsu manifolds. The necessary results of a semi-symmetric non-metric connection are given in Section 3. Basic characteristics of Riemannian curvature tensor with respect to a semi-symmetric non-metric connection have been investigated in Section 4. Semi-symmetric Kenmotsu manifolds admitting a semi-symmetric non-metric connection have been investigated in Section 5. Ricci semi-symmetric Kenmotsu manifolds admitting a semi-symmetric non-metric connection have been investigated in Section 6. Locally ϕ-symmetric Kenmotsu manifolds admitting a semi-symmetric non-metric connection have been investigated in Section 7. In Section 8, we give an example.

2. Preliminaries

Suppose $\mathfrak{M}$ be an $(2n+1)$-dimensional almost contact metric manifolds with an almost contact metric quartet (ϕ,ξ,η,g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and the Riemannian metric g on $\mathfrak{M}$ satisfying [3,27]:

$$\eta(\xi)=1, \quad \phi(\xi)=0, \quad \eta(\phi(\mathcal{K}_1))=0, \quad g(\mathcal{K}_1,\xi)=\eta(\mathcal{K}_1), \tag{2.1}$$

$$\phi^2(\mathcal{K}_1)=-\mathcal{K}_1+\eta(\mathcal{K}_1)\xi, \quad g(\mathcal{K}_1,\phi\mathcal{K}_2)=-g(\phi\mathcal{K}_1,\mathcal{K}_2), \tag{2.2}$$

$$g(\phi\mathcal{K}_1,\phi\mathcal{K}_2)=g(\mathcal{K}_1,\mathcal{K}_2)-\eta(\mathcal{K}_1)\eta(\mathcal{K}_2). \tag{2.3}$$

An almost contact metric quartet (ϕ,ξ,η,g) is a Kenmotsu manifolds [16] iff

$$(\nabla_{\mathcal{K}_1}\phi)(\mathcal{K}_2)=g(\phi\mathcal{K}_1,\mathcal{K}_2)\xi-\eta(\mathcal{K}_2)\phi\mathcal{K}_1. \tag{2.4}$$

It is also defined by above investigations.

$$\nabla_{\mathcal{K}_1}\xi=\mathcal{K}_1-\eta(\mathcal{K}_1)\xi, \tag{2.5}$$

$$(\nabla_{\mathcal{K}_1}\eta)(\mathcal{K}_2)=g(\mathcal{K}_1,\mathcal{K}_2)-\eta(\mathcal{K}_1)\eta(\mathcal{K}_2)=g(\phi\mathcal{K}_1,\phi\mathcal{K}_1), \tag{2.6}$$

$$\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\xi=\eta(\mathcal{K}_1)\mathcal{K}_2-\eta(\mathcal{K}_2)\mathcal{K}_1, \tag{2.7}$$

$$\mathfrak{R}(\xi,\mathcal{K}_1)\mathcal{K}_2=\eta(\mathcal{K}_2)\mathcal{K}_1-g(\mathcal{K}_1,\mathcal{K}_2)\xi, \tag{2.8}$$

$$\mathfrak{R}(\xi,\mathcal{K}_1)\xi=\mathcal{K}_1-\eta(\mathcal{K}_1)\xi, \tag{2.9}$$

$$\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3)=g(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)-g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1), \tag{2.10}$$

$$\mathcal{S}(\phi\mathcal{K}_1,\phi\mathcal{K}_2)=\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)+2n\eta(\mathcal{K}_1)\eta(\mathcal{K}_2), \tag{2.11}$$

$$\mathcal{S}(\mathcal{K}_1,\xi)=-2n\eta(\mathcal{K}_1), \tag{2.12}$$

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)=g(\mathfrak{Q}\mathcal{K}_1,\mathcal{K}_2), \tag{2.13}$$

$\forall\ \mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3\in\mathfrak{X}(\mathfrak{M})$, where $\mathfrak{X}(\mathfrak{M})$ is a set of all smooth vector fields on $\mathfrak{M}$ and $\mathfrak{R}$, $\mathcal{S}$ and $\mathfrak{Q}$ represent the curvature tensor, Ricci tensor and Ricci operator of the manifold $\mathfrak{M}$, respectively, with respect to the Levi-Civita connection ∇.

Definition 2.1. An almost contact metric manifold $\mathfrak{M}$ is said to be an η-Einstein manifolds if there exists the real valued functions Θ_1, Θ_2 such that

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)=\Theta_1 g(\mathcal{K}_1,\mathcal{K}_2)+\Theta_2\eta(\mathcal{K}_1)\eta(\mathcal{K}_2). \tag{2.14}$$

For $\Theta_2=0$, the manifold $\mathfrak{M}$ is an Einstein manifolds.

Definition 2.2. A Ricci soliton $(g,\mathcal{V},\Theta)$ on a Riemannian manifold is defined by

$$\mathfrak{L}_{\mathcal{V}}g+2\mathcal{S}+2\Theta g=0, \tag{2.15}$$

on $\mathfrak{M}$, where $\mathfrak{L}_{\mathcal{V}}g$ is a Lie-derivative along the vector field $\mathcal{V}$ of metric g and $\Theta\in\mathbb{R}$. The Ricci soliton $(g,\mathcal{V},\Theta)$ is shrinking, steady and expanding whenever, $\Theta<0$, $\Theta=0$ and $\Theta>0$, respectively [2].

Definition 2.3. The Ricci tensor $\mathcal{S}$ of a Kenmotsu manifolds is said to be η-parallel if it satisfies

$$(\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\mathcal{K}_2,\phi\mathcal{K}_3)=0. \tag{2.16}$$

The idea of Ricci η-parallelity for Sasakian manifolds was investigated by Yano and Kon [27]. In [11] the authors proved that a 3-dimensional Kenmotsu manifold has η-parallel Ricci tensor iff it is of constant scalar curvature.

3. A Semi-Symmetric Non-Metric Connection

Let us define, a linear connection $\widetilde{\nabla}$ [4, 13] as

$$\widetilde{\nabla}_{\mathcal{K}_1}\mathcal{K}_2 = \nabla_{\mathcal{K}_1}\mathcal{K}_2 + \frac{1}{2}[\eta(\mathcal{K}_2)\mathcal{K}_1 - \eta(\mathcal{K}_1)\mathcal{K}_2] \tag{3.1}$$

satisfying

$$\widetilde{\mathcal{T}}(\mathcal{K}_1,\mathcal{K}_2) = \eta(\mathcal{K}_2)\mathcal{K}_1 - \eta(\mathcal{K}_1)\mathcal{K}_2, \tag{3.2}$$

and

$$(\widetilde{\nabla}_{\mathcal{K}_1}g)(\mathcal{K}_2,\mathcal{K}_3) = \frac{1}{2}[2\eta(\mathcal{K}_1)g(\mathcal{K}_2,\mathcal{K}_3) - \eta(\mathcal{K}_2)g(\mathcal{K}_1,\mathcal{K}_3) - \eta(\mathcal{K}_3)g(\mathcal{K}_1,\mathcal{K}_2)]. \tag{3.3}$$

for arbitrary vector fields $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ is said to be a semi-symmetric non-metric connection. Also, we have

$$(\widetilde{\nabla}_{\mathcal{K}_1}\phi)(\mathcal{K}_2) = \frac{1}{2}[2(\nabla_{\mathcal{K}_1}\phi)(\mathcal{K}_2) - \eta(\mathcal{K}_2)\phi(\mathcal{K}_1)], \tag{3.4}$$

$$(\widetilde{\nabla}_{\mathcal{K}_1}\eta)(\mathcal{K}_2) = (\nabla_{\mathcal{K}_1}\eta)(\mathcal{K}_2), \tag{3.5}$$

$$(\widetilde{\nabla}_{\mathcal{K}_1}g)(\phi\mathcal{K}_1,\mathcal{K}_3) = \frac{1}{2}[2\eta(\mathcal{K}_1)g(\phi\mathcal{K}_2,\mathcal{K}_3) - \eta(\mathcal{K}_3)g(\mathcal{K}_1,\phi\mathcal{K}_2)]. \tag{3.6}$$

On replacing $\mathcal{K}_2$ by ξ in the equation (3.1), we have

$$\widetilde{\nabla}_{\mathcal{K}_1}\xi = \frac{3}{2}\nabla_{\mathcal{K}_1}\xi. \tag{3.7}$$

On replacing $\mathcal{K}_1$ by ξ in the equation (3.3), we have

$$(\widetilde{\nabla}_{\xi}g)(\mathcal{K}_2,\mathcal{K}_3) = g(\phi\mathcal{K}_2,\phi\mathcal{K}_3) = (\nabla_{\mathcal{K}_2}\eta)(\mathcal{K}_3). \tag{3.8}$$

Hence we have the following propositions:

Proposition 3.1. *The vector field ξ with respect to ∇ and $\widetilde{\nabla}$ is related by equation* (3.7).

Proposition 3.2. *Co-variant differentiation of g with respect to contra-variant vector field ξ is given by the equation* (3.8) *in a contact metric manifold admitting connection $\widetilde{\nabla}$.*

The curvature tensor $\widetilde{\mathfrak{R}}$ of $\widetilde{\nabla}$ defined as follows

$$\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \widetilde{\nabla}_{\mathcal{K}_1}\widetilde{\nabla}_{\mathcal{K}_2}\mathcal{K}_3 - \widetilde{\nabla}_{\mathcal{K}_2}\widetilde{\nabla}_{\mathcal{K}_1}\mathcal{K}_3 - \widetilde{\nabla}_{[\mathcal{K}_1,\mathcal{K}_2]}\mathcal{K}_3, \tag{3.9}$$

where $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3 \in \mathfrak{X}(\mathfrak{M})$.
Using equation (3.1) in (3.9), we have

$$\begin{aligned}\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 &+ \frac{1}{2}[(\nabla_{\mathcal{K}_1}\eta)(\mathcal{K}_3)\mathcal{K}_2 - (\nabla_{\mathcal{K}_1}\eta)(\mathcal{K}_2)\mathcal{K}_3 \\ &- (\nabla_{\mathcal{K}_2}\eta)(\mathcal{K}_3)\mathcal{K}_1 + (\nabla_{\mathcal{K}_2}\eta)(\mathcal{K}_1)\mathcal{K}_3] \\ &+ \frac{1}{4}[\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{K}_1 - \eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{K}_2],\end{aligned} \tag{3.10}$$

where

$$\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \nabla_{\mathcal{K}_1}\nabla_{\mathcal{K}_2}\mathcal{K}_3 - \nabla_{\mathcal{K}_2}\nabla_{\mathcal{K}_1}\mathcal{K}_3 - \nabla_{[\mathcal{K}_1,\mathcal{K}_2]}\mathcal{K}_3 \tag{3.11}$$

is the Riemannian curvature tensor [3] of ∇.

Proposition 3.3. *The relation between Riemannian curvature tensors $\widetilde{\mathfrak{R}}$ and $\mathfrak{R}$ with respect to connections $\widetilde{\nabla}$ and ∇, respectively is given by the equation* (3.10).

4. Some Curvature Tensor of Kenmotsu Manifolds With a Semi-Symmetric Non-Metric Connection

Now using equation (2.6) in equation (3.10), we have

$$\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 + \frac{1}{2}[g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_2 - g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1] + \frac{3}{4}[\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{K}_1 - \eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{K}_2]. \tag{4.1}$$

Contracting of (4.1) with respect to $\mathcal{K}_1$, we have

$$\widetilde{\mathcal{S}}(\mathcal{K}_2,\mathcal{K}_3) = \mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) - ng(\mathcal{K}_2,\mathcal{K}_3) + \frac{3}{2}n\eta(\mathcal{K}_2)\eta(\mathcal{K}_3). \tag{4.2}$$

Using (2.13) in equation (4.2), we have

$$\widetilde{\mathfrak{Q}}\mathcal{K}_2 = \mathfrak{Q}\mathcal{K}_2 - n(\mathcal{K}_1) + \frac{3}{2}n\eta(\mathcal{K}_2)\xi. \tag{4.3}$$

Again contracting equation (4.2), we have

$$\widetilde{\mathfrak{r}} = \mathfrak{r} - \frac{n}{2}(4n-1), \tag{4.4}$$

where $\widetilde{\mathcal{S}}(\mathcal{K}_2,\mathcal{K}_3)$; $\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3)$, $\widetilde{\mathfrak{Q}}$; $\mathfrak{Q}$ and $\widetilde{\mathfrak{r}}$; $\mathfrak{r}$ are the Ricci tensors, Ricci operators and scalar curvatures of $\widetilde{\nabla}$ and ∇.

On replacing $\mathcal{K}_1$ by ξ in (4.1) and using (2.1), (2.2), we have

$$\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\mathcal{K}_3 = \mathfrak{R}(\xi,\mathcal{K}_2)\mathcal{K}_3 - \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_3)\xi + \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\xi - \frac{1}{4}\eta(\mathcal{K}_3)\mathcal{K}_2. \tag{4.5}$$

In view of (2.8) and (4.5), we have

$$\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\mathcal{K}_3 = \frac{3}{4}[-2g(\mathcal{K}_2,\mathcal{K}_3)\xi + \eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\xi + \eta(\mathcal{K}_3)\mathcal{K}_2]. \tag{4.6}$$

Again on replacing $\mathcal{K}_3$ by ξ in (4.1) and using (2.1), (2.7), we have

$$\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\xi = \frac{3}{4}\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\xi = -\frac{3}{4}\widetilde{\mathcal{T}}(\mathcal{K}_1,\mathcal{K}_2) \neq 0. \tag{4.7}$$

Thus, we have the following theorem:

Theorem 4.1. *Every $(2n+1)$-dimensional Kenmotsu manifold admitting connection $\widetilde{\nabla}$ is regular.*

Now operating η on both sides of equation (4.1) and using equation (2.1), we have

$$\eta(\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3) = \frac{1}{2}[2g(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2) - 2g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1) + g(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2) - g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)]. \tag{4.8}$$

On contracting of (4.7) with respect to $\mathcal{K}_1$, we have

$$\tilde{\mathcal{S}}(\mathcal{K}_2,\xi) = -\frac{3}{2}n\eta(\mathcal{K}_2). \tag{4.9}$$

In view of equations (4.2), (4.3) and (4.4), we have the following lemma:

Lemma 4.1. *In a Kenmotsu manifold Ricci tensor, Ricci operator and scalar curvature with respect to connections* $\tilde{\nabla}$ *and* ∇ *are related by the equations* (4.2)*,* (4.3) *and* (4.4)*.*

Proof. On taking $\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = 0$ in the equation (4.1), we have

$$\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = -\frac{1}{2}[g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_2 - g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1] - \frac{3}{4}[\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{K}_1 - \eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{K}_2]. \tag{4.10}$$

Thus

$$\begin{aligned} {}'\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4) = &-\frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_3)g(\mathcal{K}_2,\mathcal{K}_4) + \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_3)g(\mathcal{K}_1,\mathcal{K}_4) \\ &- \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)g(\mathcal{K}_1,\mathcal{K}_4) + \frac{3}{4}\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)g(\mathcal{K}_2,\mathcal{K}_4). \end{aligned} \tag{4.11}$$

Contracting of (4.11) with respect to vector field $\mathcal{K}_1$, we have

$$\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) = ng(\mathcal{K}_2,\mathcal{K}_3) - \frac{3}{2}n\eta(\mathcal{K}_2)\eta(\mathcal{K}_3). \tag{4.12}$$

Using equation (2.13) in equation (4.12), we have

$$\mathfrak{Q}\mathcal{K}_2 = n\mathcal{K}_2 - \frac{3}{2}n\eta(\mathcal{K}_2)\xi. \tag{4.13}$$

Again contracting equation (4.12), we have

$$\mathfrak{r} = \frac{n}{2}(4n-1). \tag{4.14}$$

□

By virtue of Definition 2.1 and equation (4.12), we state the theorem:

Theorem 4.2. *If Riemannian curvature tensor with respect to connection* $\tilde{\nabla}$ *in a Kenmotsu manifold vanishes, then the manifold is an* η*-Einstein manifold.*

5. Semi-Symmetric Kenmotsu Manifolds

A $(2n+1)$-dimensional Kenmotsu manifold $\mathfrak{M}$ with $\tilde{\nabla}$ is said to be semi-symmetric [20] if

$$(\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\tilde{\mathfrak{R}})(\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5 = 0,$$

i.e.

$$\begin{aligned} &\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\tilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5 - \tilde{\mathfrak{R}}(\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5 \\ &- \tilde{\mathfrak{R}}(\mathcal{K}_3,\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4)\mathcal{K}_5 - \tilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\tilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_5 = 0. \end{aligned} \tag{5.1}$$

On replacing $\mathcal{K}_1$ by ξ, we have

$$\begin{aligned} &\tilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\tilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5 - \tilde{\mathfrak{R}}(\tilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5 \\ &- \tilde{\mathfrak{R}}(\mathcal{K}_3,\tilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\mathcal{K}_4)\mathcal{K}_5 - \tilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\tilde{\mathfrak{R}}(\xi,\mathcal{K}_2)\mathcal{K}_5 = 0. \end{aligned} \tag{5.2}$$

In view of equations (2.1), (2.2), (4.6), (4.7) and (4.8), we have

$$\begin{aligned}'\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4,\mathcal{K}_5,\mathcal{K}_2) &= g(\mathcal{K}_2,\mathcal{K}_3)\eta(\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_4)\mathcal{K}_5) - \frac{1}{2}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\eta(\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_4)\mathcal{K}_5) \\ &\quad + \frac{1}{2}\eta(\mathcal{K}_3)\eta(\widetilde{\mathfrak{R}}(\mathcal{K}_2,\mathcal{K}_4)\mathcal{K}_5) - g(\mathcal{K}_2,\mathcal{K}_4)\eta(\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_3)\mathcal{K}_5) \\ &\quad + \frac{1}{2}\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\eta(\widetilde{\mathfrak{R}}(\xi,\mathcal{K}_3)\mathcal{K}_5) + \frac{1}{2}\eta(\mathcal{K}_4)\eta(\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_2)\mathcal{K}_5) \\ &\quad + g(\mathcal{K}_2,\mathcal{K}_5)\eta(\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\xi) - \frac{1}{2}\eta(\mathcal{K}_2)\eta(\mathcal{K}_5)\eta(\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\xi) \\ &\quad + \frac{1}{2}\eta(\mathcal{K}_5)\eta(\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_2). \end{aligned} \tag{5.3}$$

By using equations (2.1), (2.2), (4.6), (4.7) and (4.8), we have

$$\begin{aligned}'\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4,\mathcal{K}_5,\mathcal{K}_2) &= -\frac{3}{2}g(\mathcal{K}_2,\mathcal{K}_3)g(\mathcal{K}_4,\mathcal{K}_5) + \frac{3}{2}g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_4)\eta(\mathcal{K}_5) - \frac{3}{4}\eta(\mathcal{K}_4)\eta(\mathcal{K}_5) \\ &\quad + \frac{3}{2}g(\mathcal{K}_4,\mathcal{K}_5)g(\mathcal{K}_5,\mathcal{K}_3) - \frac{9}{4}g(\mathcal{K}_4,\mathcal{K}_2)\eta(\mathcal{K}_5)\eta(\mathcal{K}_3). \end{aligned} \tag{5.4}$$

Hence, we have

$$\begin{aligned}\widetilde{\mathfrak{R}}(\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5 &= -\frac{3}{2}g(\mathcal{K}_4,B_5)\mathcal{K}_3 + \frac{3}{2}\eta(\mathcal{K}_4)\eta(\mathcal{K}_5)\mathcal{K}_3 - \frac{3}{4}\eta(\mathcal{K}_4)\eta(\mathcal{K}_5) + \frac{3}{2}g(\mathcal{K}_5,\mathcal{K}_3)\mathcal{K}_4 \\ &\quad - \frac{9}{4}\eta(\mathcal{K}_5)\eta(\mathcal{K}_3)\mathcal{K}_4. \end{aligned} \tag{5.5}$$

Contracting equation (5.5) with respect to $\mathcal{K}_3$, we have

$$\widetilde{\mathcal{S}}(\mathcal{K}_4,\mathcal{K}_5) = -3ng(\mathcal{K}_4,\mathcal{K}_5) + \frac{3}{2}(2n-1)\eta(\mathcal{K}_4)\eta(\mathcal{K}_5). \tag{5.6}$$

Using equation (4.2) in above equation, we obtain

$$\mathcal{S}(\mathcal{K}_4,\mathcal{K}_5) = -2ng(\mathcal{K}_4,\mathcal{K}_5) + \frac{3}{2}(n-1)\eta(\mathcal{K}_4)\eta(\mathcal{K}_5). \tag{5.7}$$

Using equation (2.13) in above equation, we have

$$\mathfrak{Q}\mathcal{K}_4 = -2n\mathcal{K}_4 + \frac{3}{2}(n-1)\eta(\mathcal{K}_4)\xi. \tag{5.8}$$

Again contracting equation (5.7), we obtain

$$\mathfrak{r} = -\frac{1}{2}(8n^2 + n + 3). \tag{5.9}$$

By virtue of Definition 2.1 and equation (5.7), we can state

Theorem 5.1. *A semi-symmetric Kenmotsu manifold admitting connection $\widetilde{\nabla}$ is an η-Einstein manifold.*

The Ricci soliton of data $(g,\mathcal{V},\Theta)$ is defined by (2.15), where g, $\mathcal{V}$, Θ are Riemannian metric, a vector field and a real constant. Here two conditions come out with regard to the $\mathcal{V}:\mathcal{V}\in\operatorname{span}\{\xi\}$ and $\mathcal{V}\perp\operatorname{span}\{\xi\}$. Now taking $\mathcal{V}\in\operatorname{span}\{\xi\}$. The Ricci soliton of data (g,ξ,Θ) on a Kenmotsu manifold admitting connection $\widetilde{\nabla}$ defined as under:

$$(\widetilde{\mathfrak{L}}_\xi g)(\mathcal{K}_1,\mathcal{K}_2) + 2\widetilde{\mathcal{S}}(\mathcal{K}_1,\mathcal{K}_2) + 2\Theta g(\mathcal{K}_1,\mathcal{K}_2) = 0. \tag{5.10}$$

$\forall\ \mathcal{K}_1, \mathcal{K}_2 \in \mathfrak{X}(\mathfrak{M})$. Here $\widetilde{\mathfrak{L}}_\xi g$, the Lie-derivative of g with respect to ξ admitting connection $\widetilde{\nabla}$, is defined as under

$$(\widetilde{\mathfrak{L}}_\xi g)(\mathcal{K}_1, \mathcal{K}_2) = g(\widetilde{\nabla}_{\mathcal{K}_1}\xi, \mathcal{K}_2) + g(\mathcal{K}_1, \widetilde{\nabla}_{\mathcal{K}_2}\xi) - 2g(\phi K_1, \phi K_2). \tag{5.11}$$

Now, using equations (2.1), (2.3), (2.5), (3.7) and (5.11), we have

$$(\widetilde{\mathfrak{L}}_\xi g)(\mathcal{K}_1, \mathcal{K}_2) = g(\phi\mathcal{K}_1, \phi\mathcal{K}_2). \tag{5.12}$$

Using equations (5.6) and (5.12) in the equation (5.10), we have

$$g(\phi\mathcal{K}_1, \phi\mathcal{K}_2) - 6ng(\mathcal{K}_1, \mathcal{K}_2) + 3(2n-1)\eta(\mathcal{K}_1)\eta(\mathcal{K}_2) + 2\Theta g(\mathcal{K}_1, \mathcal{K}_2) = 0. \tag{5.13}$$

On taking $\mathcal{K}_1 = \mathcal{K}_2 = \xi$ and using (2.1) in (5.13), we have

$$\Theta = \frac{3}{2} > 0. \tag{5.14}$$

Thus, we state the theorem:

Theorem 5.2. *A semi-symmetric Kenmotsu manifold admitting connection $\widetilde{\nabla}$, the Ricci soliton of data (g, ξ, Θ) is always expanding.*

6. Ricci Semi-Symmetric Kenmotsu Manifolds

A $(2n+1)$-dimensional contact metric manifolds $\mathfrak{M}$ with respect to connection $\widetilde{\nabla}$ is said to be Ricci semi-symmetric [20] if

$$(\widetilde{\mathfrak{R}}(\mathcal{K}_1, \mathcal{K}_2) \cdot \widetilde{\mathcal{S}})(\mathcal{K}_3, \mathcal{K}_4) = 0.$$

i.e.

$$\widetilde{\mathcal{S}}(\widetilde{\mathfrak{R}}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3, \mathcal{K}_4) + \widetilde{\mathcal{S}}(\mathcal{K}_3, \widetilde{\mathfrak{R}}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_4) = 0. \tag{6.1}$$

On replacing $\mathcal{K}_1$ by ξ and using (4.6) in (6.1), we have

$$\widetilde{\mathcal{S}}(\widetilde{\mathfrak{R}}(\xi, \mathcal{K}_2)\mathcal{K}_3, \mathcal{K}_4) + \widetilde{\mathcal{S}}(\mathcal{K}_3, \widetilde{\mathfrak{R}}(\xi, \mathcal{K}_2)\mathcal{K}_4) = 0, \tag{6.2}$$

i.e.

$$\begin{aligned} &-\frac{3}{2}g(\mathcal{K}_2, \mathcal{K}_3)\widetilde{\mathcal{S}}(\xi, \mathcal{K}_4) + \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\widetilde{\mathcal{S}}(\xi, \mathcal{K}_4) - \frac{3}{4}\eta(\mathcal{K}_3)\widetilde{\mathcal{S}}(\mathcal{K}_2, \mathcal{K}_4) - \frac{3}{2}g(\mathcal{K}_2, \mathcal{K}_4)\widetilde{\mathcal{S}}(\mathcal{K}_3, \xi) \\ &\quad + \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\widetilde{\mathcal{S}}(\mathcal{K}_3, \xi) - \frac{3}{4}\eta(\mathcal{K}_4)\widetilde{\mathcal{S}}(\mathcal{K}_3, \mathcal{K}_2) = 0. \end{aligned} \tag{6.3}$$

In view of equation (4.9), the above equation yields

$$\begin{aligned} &\frac{9}{4}ng(\mathcal{K}_2, \mathcal{K}_3)\eta(\mathcal{K}_4) - \frac{9}{8}n\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\eta(\mathcal{K}_4) - \frac{3}{4}\eta(\mathcal{K}_3)\widetilde{\mathcal{S}}(\mathcal{K}_2, \mathcal{K}_4) + \frac{9}{4}ng(\mathcal{K}_2, \mathcal{K}_4)\eta(\mathcal{K}_3) \\ &\quad - \frac{9}{8}n\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\eta(\mathcal{K}_4) - \frac{3}{4}\eta(\mathcal{K}_4)\widetilde{\mathcal{S}}(\mathcal{K}_3, \mathcal{K}_2) = 0. \end{aligned} \tag{6.4}$$

Again replacing $\mathcal{K}_4$ by ξ and using (4.9) in (6.4), we have

$$\widetilde{\mathcal{S}}(\mathcal{K}_2, \mathcal{K}_3) = 3ng(\mathcal{K}_2, \mathcal{K}_3) + \frac{3}{2}n\eta(\mathcal{K}_2)\eta(\mathcal{K}_3). \tag{6.5}$$

Using (4.2) in (6.5), we have

$$\mathcal{S}(\mathcal{K}_2, \mathcal{K}_3) = 4ng(\mathcal{K}_2, \mathcal{K}_3). \tag{6.6}$$

On contracting equation (6.6), we have

$$\mathfrak{r} = 4n(2n+1), \tag{6.7}$$

with the help of equation (6.7), equation (4.4) takes the form

$$\widetilde{\mathfrak{r}} = \frac{3n}{2}(4n+3). \tag{6.8}$$

In view of equation (6.6), we can state following:

Theorem 6.1. *A Ricci semi-symmetric Kenmotsu manifold equipped with connection $\widetilde{\nabla}$ is an Einstein manifold.*

Using equation (4.1) in the given below equation

$$(\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\cdot\widetilde{\mathcal{S}})(\mathcal{K}_3,\mathcal{K}_4) = -\widetilde{\mathcal{S}}(\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4) - \widetilde{\mathcal{S}}(\mathcal{K}_3,\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4), \tag{6.9}$$

we have

$$\begin{aligned}(\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\cdot\widetilde{\mathcal{S}})(\mathcal{K}_3,\mathcal{K}_4) &= (\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\cdot\mathcal{S})(\mathcal{K}_3,\mathcal{K}_4) - \frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_4)\\ &\quad + \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\mathcal{K}_4) - \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\mathcal{K}_4)\\ &\quad + \frac{3}{4}\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_4) - \frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_4)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3)\\ &\quad + \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_4)\mathcal{S}(\mathcal{K}_3,\mathcal{K}_1) - \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\mathcal{S}(\mathcal{K}_3,\mathcal{K}_1)\\ &\quad + \frac{3}{4}\eta(B_1)\eta(\mathcal{K}_4)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) - \frac{3}{2}n\cdot\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3)\eta(\mathcal{K}_4)\\ &\quad - \frac{3}{2}n\cdot\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4)\eta(\mathcal{K}_3).\end{aligned} \tag{6.10}$$

If we assume $(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\cdot\mathcal{S})(\mathcal{K}_3,\mathcal{K}_4) = (\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\cdot\widetilde{\mathcal{S}})(\mathcal{K}_3,\mathcal{K}_4)$, then from equation (6.10), we have

$$\begin{aligned}&-\frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_4) + \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\mathcal{K}_4) - \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\mathcal{K}_4)\\ &\quad + \frac{3}{4}\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_4) - \frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_4)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) + \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_4)\mathcal{S}(\mathcal{K}_3,\mathcal{K}_1)\\ &\quad - \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\mathcal{S}(\mathcal{K}_3,\mathcal{K}_1) + \frac{3}{4}\eta(\mathcal{K}_1)\eta(\mathcal{K}_4)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) - \frac{3}{2}n\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3)\eta(\mathcal{K}_4)\\ &\quad - \frac{3}{2}n\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4)\eta(\mathcal{K}_3) = 0,\end{aligned} \tag{6.11}$$

where

$$(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\cdot\mathcal{S})(\mathcal{K}_3,\mathcal{K}_4) = -\mathcal{S}(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4) - \mathcal{S}(\mathcal{K}_3,\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4). \tag{6.12}$$

Now, replacing $\mathcal{K}_4$ by ξ in the equation (6.11), we have

$$\begin{aligned}&-\frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_2,\xi) + \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\xi) - \frac{3}{4}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\xi) + \frac{3}{4}\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{S}(\mathcal{K}_2,\xi)\\ &\quad - \frac{1}{2}g(\mathcal{K}_1,\xi)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3) + \frac{1}{2}g(\mathcal{K}_2,\xi)\mathcal{S}(\mathcal{K}_3,\mathcal{K}_1) - \frac{3}{4}\eta(\mathcal{K}_2)\eta(\xi)\mathcal{S}(\mathcal{K}_3,\mathcal{K}_1) + \frac{3}{4}\eta(\mathcal{K}_1)\eta(\xi)\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3)\\ &\quad - \frac{3}{2}n\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3)\eta(\xi) - \frac{3}{2}n\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\xi)\eta(\mathcal{K}_3) = 0.\end{aligned} \tag{6.13}$$

Now, using equations (2.1), (2.10) and (6.5) in equation (6.13), we have

$$-3ng(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)-\frac{3}{2}ng(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)+3ng(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)+\frac{3}{2}ng(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)=0, \quad (6.14)$$

i.e.

$$\frac{9}{2}n[\eta(\mathcal{K}_1)g(\mathcal{K}_2,\mathcal{K}_3)-\eta(\mathcal{K}_2)g(\mathcal{K}_1,\mathcal{K}_3)]=0, \quad (6.15)$$

which is not possible. Hence we have the following:

Corollary 6.1. *In a Ricci semi-symmetric Kenmotsu manifold admitting connection* $\widetilde{\nabla}$

$$(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\cdot\mathcal{S})(\mathcal{K}_3,\mathcal{K}_4)\neq(\widetilde{\mathfrak{R}}(\mathcal{K}_1,\mathcal{K}_2)\cdot\widetilde{\mathcal{S}})(\mathcal{K}_3,\mathcal{K}_4). \quad (6.16)$$

Using equations (5.12) and (6.5) in the equation (5.10), we have

$$2(3n+\Theta)g(\mathcal{K}_1,\mathcal{K}_2)+g(\phi\mathcal{K}_1,\phi\mathcal{K}_2)+3n\eta(\mathcal{K}_1)\eta(\mathcal{K}_2)=0. \quad (6.17)$$

On taking $\mathcal{K}_1=\mathcal{K}_2=\xi$ and using (2.1) in (6.17), we have

$$\Theta=-\frac{9n}{2}<0. \quad (6.18)$$

Thus, we have the following:

Theorem 6.2. *A Ricci semi-symmetric Kenmotsu manifold admitting connection* $\widetilde{\nabla}$*, the Ricci soliton of data* (g,ξ,Θ) *is always shrinking.*

7. Locally ϕ-Symmetric Kenmotsu Manifolds

Definition 7.1. A Kenmotsu manifolds $\mathfrak{M}$ admitting connection $\widetilde{\nabla}$ is called locally ϕ-symmetric [24] if

$$\phi^2((\widetilde{\nabla}_{\mathcal{K}_4}\widetilde{\mathfrak{R}})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3)=0$$

$\forall$ $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4$ are orthogonal to ξ.

Taking covariant differentiation of $\mathfrak{R}$ with respect to $\mathcal{K}_4$, we have

$$\begin{aligned}(\widetilde{\nabla}_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3&=\widetilde{\nabla}_{\mathcal{K}_4}\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3-\mathfrak{R}(\widetilde{\nabla}_{\mathcal{K}_4}\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3\\&\quad-\mathfrak{R}(\mathcal{K}_1,\widetilde{\nabla}_{\mathcal{K}_4}\mathcal{K}_2)\mathcal{K}_3-\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)(\widetilde{\nabla}_{\mathcal{K}_4}\mathcal{K}_3). \quad (7.1)\end{aligned}$$

Now using equations (2.10) and (3.1) in equation (7.1), we have

$$\begin{aligned}(\widetilde{\nabla}_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3&=(\nabla_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3+\frac{1}{2}[2\eta(\mathcal{K}_4)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3-\eta(\mathcal{K}_1)\mathfrak{R}(\mathcal{K}_4,\mathcal{K}_2)\mathcal{K}_3\\&\quad-\eta(\mathcal{K}_2)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_4)\mathcal{K}_3-\eta(\mathcal{K}_3)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4+g(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)\mathcal{K}_4\\&\quad-g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)\mathcal{K}_4]. \quad (7.2)\end{aligned}$$

Applying covariant differentiation on (4.1) with respect to $\mathcal{K}_4$, we have

$$\begin{aligned}(\widetilde{\nabla}_{\mathcal{K}_4}\widetilde{\mathfrak{R}})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3&=(\widetilde{\nabla}_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3+\frac{1}{2}[(\widetilde{\nabla}_{\mathcal{K}_4}g)(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_2-(\widetilde{\nabla}_{\mathcal{K}_4}g)(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1]\\&\quad+\frac{3}{4}[(\widetilde{\nabla}_{\mathcal{K}_4}\eta)(\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{K}_1+(\widetilde{\nabla}_{\mathcal{K}_4}\eta)(\mathcal{K}_3)\eta(\mathcal{K}_2)\mathcal{K}_1-(\widetilde{\nabla}_{\mathcal{K}_4}\eta)(\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{K}_2\\&\quad-(\widetilde{\nabla}_{\mathcal{K}_4}\eta)(\mathcal{K}_3)\eta(\mathcal{K}_1)\mathcal{K}_2]. \quad (7.3)\end{aligned}$$

Using equations (2.6), (3.3), (3.5) and (7.2), we have

$$(\widetilde{\nabla}_{\mathcal{K}_4}\widetilde{\mathfrak{R}})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = (\nabla_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 + \frac{1}{2}[2\eta(\mathcal{K}_4)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 - \eta(\mathcal{K}_1)\mathfrak{R}(\mathcal{K}_4,\mathcal{K}_2)\mathcal{K}_3$$
$$- \eta(\mathcal{K}_2)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_4)\mathcal{K}_3 - \eta(\mathcal{K}_3)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4 + g(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_2)\mathcal{K}_4$$
$$- g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_1)\mathcal{K}_4] + \frac{1}{2}g(\mathcal{K}_1,\mathcal{K}_3)\eta(\mathcal{K}_4)\mathcal{K}_2 - \frac{1}{2}g(\mathcal{K}_2,\mathcal{K}_3)\eta(\mathcal{K}_4)\mathcal{K}_1$$
$$- g(\mathcal{K}_4,\mathcal{K}_1)\eta(\mathcal{K}_3)\mathcal{K}_2 + g(\mathcal{K}_4,\mathcal{K}_2)\eta(\mathcal{K}_3)\mathcal{K}_1 - g(\mathcal{K}_4,\mathcal{K}_3)\eta(\mathcal{K}_1)\mathcal{K}_2$$
$$+ g(\mathcal{K}_4,\mathcal{K}_3)\eta(\mathcal{K}_2)\mathcal{K}_1 - \frac{3}{2}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\eta(\mathcal{K}_4)\mathcal{K}_1 + \frac{3}{2}\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\eta(\mathcal{K}_4)\mathcal{K}_1. \tag{7.4}$$

Now applying ϕ^2 on both sides of equation (7.4) and using equation (2.2), we have

$$\phi^2((\widetilde{\nabla}_{\mathcal{K}_4}\widetilde{\mathfrak{R}})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3) = \phi^2((\nabla_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3) + \frac{1}{2}[-2\eta(\mathcal{K}_4)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3$$
$$+ 2\eta(\mathcal{K}_4)\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3)\xi + \eta(\mathcal{K}_1)\mathfrak{R}(\mathcal{K}_4,\mathcal{K}_2)\mathcal{K}_3$$
$$- \eta(\mathcal{K}_1)\eta(\mathfrak{R}(\mathcal{K}_4,\mathcal{K}_2)\mathcal{K}_3)\xi + \eta(\mathcal{K}_2)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_4)\mathcal{K}_3$$
$$- \eta(\mathcal{K}_2)\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_4)\mathcal{K}_3)\xi + \eta(\mathcal{K}_3)\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\xi$$
$$- \eta(\mathcal{K}_3)\eta(\mathfrak{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4)\xi - \eta(\mathcal{K}_2)g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_4$$
$$+ 2\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)g(\mathcal{K}_1,\mathcal{K}_3)\xi + \eta(\mathcal{K}_1)g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_4$$
$$- 2\eta(\mathcal{K}_1)\eta(\mathcal{K}_4)g(\mathcal{K}_2,\mathcal{K}_3)\xi - \eta(\mathcal{K}_4)g(\mathcal{K}_1,\mathcal{K}_3)\mathcal{K}_2$$
$$+ \eta(\mathcal{K}_4)g(\mathcal{K}_2,\mathcal{K}_3)\mathcal{K}_1 + 2\eta(\mathcal{K}_3)g(\mathcal{K}_1,\mathcal{K}_4)\mathcal{K}_2$$
$$- 2\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)g(\mathcal{K}_1,\mathcal{K}_4)\xi - 2\eta(\mathcal{K}_3)g(\mathcal{K}_2,\mathcal{K}_4)\mathcal{K}_1$$
$$+ 2\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)g(\mathcal{K}_2,\mathcal{K}_4)\xi + 2\eta(\mathcal{K}_1)g(\mathcal{K}_4,\mathcal{K}_3)\mathcal{K}_2$$
$$- 2\eta(\mathcal{K}_2)g(\mathcal{K}_4,\mathcal{K}_3)\mathcal{K}_1 + 3\eta(\mathcal{K}_2)\eta(\mathcal{K}_3)\eta(\mathcal{K}_4)\mathcal{K}_1$$
$$- 3\eta(\mathcal{K}_1)\eta(\mathcal{K}_3)\eta(\mathcal{K}_4)\mathcal{K}_2]. \tag{7.5}$$

Taking $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ and $\mathcal{K}_4$ orthogonal to ξ, then from equation (7.5), we have

$$\phi^2((\widetilde{\nabla}_{\mathcal{K}_4}\widetilde{\mathfrak{R}})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3) = \phi^2((\nabla_{\mathcal{K}_4}\mathfrak{R})(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3). \tag{7.6}$$

Theorem 7.1. *The necessary and sufficient condition for a Kenmotsu manifold to be locally ϕ-symmetric with respect to connection $\widetilde{\nabla}$ is that the manifold is also locally ϕ-symmetric with respect to the connection ∇.*

8. Example of a Three-Dimensional Kenmotsu Manifold

Let three-dimensional manifold $\mathfrak{M}^3 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 : t_3 > 0\}$, where (t_1, t_2, t_3) are the standard co-ordinates in $\mathbb{R}^3$. The vector fields [12]

$$\varsigma_1 = t_3\frac{\partial}{\partial t_1}, \quad \varsigma_2 = t_3\frac{\partial}{\partial t_2}, \quad \varsigma_3 = -t_3\frac{\partial}{\partial t_3}$$

are linearly independent at each point of $\mathfrak{M}$. Let g be the Riemannian metric defined by

$$\left.\begin{aligned} g(\varsigma_1,\varsigma_2)=g(\varsigma_2,\varsigma_3)=g(\varsigma_3,\varsigma_1)=0,\\ g(\varsigma_1,\varsigma_1)=g(\varsigma_2,\varsigma_2)=g(\varsigma_3,\varsigma_3)=1,\end{aligned}\right\} \tag{8.1}$$

where

$$g=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Let η be the 1-form defined by $\eta(\mathcal{K}_1)=g(\mathcal{K}_1,\varsigma_3)$ for any $\mathcal{K}_1\in\mathfrak{X}(\mathfrak{M})$. Let ϕ be the $(1,1)$-tensor field defined by

$$(\phi\varsigma_1)=-\varsigma_2,\quad (\phi\varsigma_2)=\varsigma_1,\quad (\phi\varsigma_3)=0. \tag{8.2}$$

Now for $\mathcal{K}_1=\mathcal{K}_1^1\varsigma_1+\mathcal{K}_1^2\varsigma_2+\mathcal{K}_1^3\varsigma_3$ and $\xi=\varsigma_3$, using linearity of ϕ and g, we have

$$\eta(\varsigma_3)=\eta(\xi)=1,\quad \phi^2(\mathcal{K}_1)=-\mathcal{K}_1+\eta(\mathcal{K}_1)\varsigma_3=-(\mathcal{K}_1^1\varsigma_1+\mathcal{K}_1^2\varsigma_2) \tag{8.3}$$

where $\mathcal{K}_1^1,\mathcal{K}_1^2,\mathcal{K}_1^3$ are the scalars and $\forall\ \mathcal{K}_1,\mathcal{K}_2\in\mathfrak{X}(\mathfrak{M})$. Thus for $\varsigma_3=\xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on $\mathfrak{M}$. Let ∇ be the Levi-Civita connection with respect to the metric g. Then, we have

$$\left.\begin{aligned} &[\varsigma_1,\varsigma_1]=0, && [\varsigma_1,\varsigma_2]=0, && [\varsigma_1,\varsigma_3]=\varsigma_1,\\ &[\varsigma_2,\varsigma_1]=0, && [\varsigma_2,\varsigma_2]=0, && [\varsigma_2,\varsigma_3]=\varsigma_2,\\ &[\varsigma_3,\varsigma_1]=-\varsigma_1, && [\varsigma_3,\varsigma_2]=-\varsigma_2, && [\varsigma_3,\varsigma_3]=0.\end{aligned}\right\} \tag{8.4}$$

Now using equation (2.3), we have

$$g(\mathcal{K}_1,\mathcal{K}_2)=\mathcal{K}_1^1\mathcal{K}_2^1+\mathcal{K}_1^2B_2^2+\mathcal{K}_1^3\mathcal{K}_2^3. \tag{8.5}$$

Let us consider ∇, a Levi-Civita connection admitting a Riemannian metric g. Using the Koszul formula

$$\begin{aligned} 2g(\nabla_{\mathcal{K}_1}\mathcal{K}_2,\mathcal{K}_3)&=\mathcal{K}_1g(\mathcal{K}_2,\mathcal{K}_3)+\mathcal{K}_2g(\mathcal{K}_3,\mathcal{K}_1)-\mathcal{K}_3g(\mathcal{K}_1,\mathcal{K}_2)\\ &\quad+g([\mathcal{K}_1,\mathcal{K}_2],\mathcal{K}_3)-g([\mathcal{K}_2,\mathcal{K}_3],\mathcal{K}_1)+g([\mathcal{K}_3,\mathcal{K}_1],\mathcal{K}_2).\end{aligned} \tag{8.6}$$

By virtue of (8.6), we have

$$\left.\begin{aligned} &\nabla_{\varsigma_1}\varsigma_1=0, && \nabla_{\varsigma_1}\varsigma_2=0, && \nabla_{\varsigma_1}\varsigma_3=\varsigma_1,\\ &\nabla_{\varsigma_2}\varsigma_1=0, && \nabla_{\varsigma_2}\varsigma_2=-\varsigma_3, && \nabla_{\varsigma_2}\varsigma_3=\varsigma_2,\\ &\nabla_{\varsigma_3}\varsigma_1=0, && \nabla_{\varsigma_3}\varsigma_2=0, && \nabla_{\varsigma_3}\varsigma_3=0.\end{aligned}\right\} \tag{8.7}$$

Again for $\mathcal{K}_1=\mathcal{K}_1^1\varsigma_1+\mathcal{K}_1^2\varsigma_2+\mathcal{K}_1^3\varsigma_3$ and $\xi=\varsigma_3$, we have

$$\frac{3}{2}\nabla_{\mathcal{K}_1}\xi=\frac{3}{2}[\mathcal{K}_1^1\varsigma_1+\mathcal{K}_1^2\varsigma_2], \tag{8.8}$$

i.e.

$$\nabla_{\mathcal{K}_1}\xi=\mathcal{K}_1^1\varsigma_1+\mathcal{K}_1^2\varsigma_2, \tag{8.9}$$

$$\mathcal{K}_1-\eta(\mathcal{K}_1)\xi=\mathcal{K}_1^1\varsigma_1+\mathcal{K}_1^2\varsigma_2, \tag{8.10}$$

where $\mathcal{K}_1^1,\mathcal{K}_1^2,\mathcal{K}_1^3$ are scalars. From equations (8.9) and (8.10) it follows that the manifold satisfies equation (2.5) for $\xi=\varsigma_3$. Thus manifold is a Kenmotsu manifold. In reference of equations (2.1), (3.1) and (8.7), we have the following:

$$\left.\begin{array}{lll}\widetilde{\nabla}_{\varsigma_1}\varsigma_1=0, & \widetilde{\nabla}_{\varsigma_1}\varsigma_2=0, & \widetilde{\nabla}_{\varsigma_1}\varsigma_3=\frac{3}{2}\varsigma_1\\ \widetilde{\nabla}_{\varsigma_2}\varsigma_1=0, & \widetilde{\nabla}_{\varsigma_2}\varsigma_2=0, & \widetilde{\nabla}_{\varsigma_2}\varsigma_3=\frac{3}{2}\varsigma_2\\ \widetilde{\nabla}_{\varsigma_3}\varsigma_1=-\frac{\varsigma_1}{2}, & \widetilde{\nabla}_{\varsigma_3}\varsigma_2=-\frac{1}{2}\varsigma_2, & \widetilde{\nabla}_{\varsigma_3}\varsigma_3=0.\end{array}\right\} \tag{8.11}$$

In equations (3.2) and (3.3), we have

$$\begin{aligned}\widetilde{\mathfrak{T}}(\varsigma_1,\varsigma_3)&=\eta(\varsigma_3)\varsigma_1-\eta(\varsigma_1)\varsigma_3\\ &=g(\varsigma_3,\varsigma_3)\varsigma_1-g(\varsigma_1,\varsigma_3)\varsigma_3\\ &=\varsigma_1\neq 0\end{aligned} \tag{8.12}$$

and

$$\begin{aligned}(\widetilde{\nabla}_{\varsigma_1}g)(\varsigma_1,\varsigma_3)&=\frac{1}{2}\{2\eta(\varsigma_1)g(\varsigma_1,\varsigma_3)-\eta(\varsigma_1)g(\varsigma_1,\varsigma_3)-\eta(\varsigma_3)g(\varsigma_1,\varsigma_1)\}\\ &=-\frac{1}{2}\neq 0.\end{aligned} \tag{8.13}$$

Thus it is clear from (3.1) that $\widetilde{\nabla}$ is a semi-symmetric non-metric connection. Now

$$\begin{aligned}\widetilde{\nabla}_{\mathcal{K}_1}\xi&=\widetilde{\nabla}_{\mathcal{K}_1^1\varsigma_1+\mathcal{K}_1^2\varsigma_2+\mathcal{K}_1^3\varsigma_3}\varsigma_3\\ &=\mathcal{K}_1^1\widetilde{\nabla}_{\varsigma_1}\varsigma_3+\mathcal{K}_1^2\widetilde{\nabla}_{\varsigma_2}\varsigma_3+\mathcal{K}_1^3\widetilde{\nabla}_{\varsigma_3}\varsigma_3\\ &=\frac{3}{2}(\mathcal{K}_1^1\varsigma_1+\mathcal{K}_1^2\varsigma_2).\end{aligned} \tag{8.14}$$

By virtue of (8.8) and 8.12, we have verified the equations (3.6) and (3.7). The $\mathfrak{R}(\varsigma_i,\varsigma_j)\varsigma_k$; $i,j,k=1,2,3$ of connection ∇ can be estimated by using (3.11), (8.4) and (8.7), we have

$$\left.\begin{array}{lll}\mathfrak{R}(\varsigma_1,\varsigma_2)\varsigma_1=0, & \mathfrak{R}(\varsigma_1,\varsigma_2)\varsigma_2=0, & \mathfrak{R}(\varsigma_1,\varsigma_2)\varsigma_3=0,\\ \mathfrak{R}(\varsigma_1,\varsigma_3)\varsigma_1=0, & \mathfrak{R}(\varsigma_1,\varsigma_3)\varsigma_2=0, & \mathfrak{R}(\varsigma_1,\varsigma_3)\varsigma_3=-\varsigma_1,\\ \mathfrak{R}(\varsigma_2,\varsigma_3)\varsigma_1=0, & \mathfrak{R}(\varsigma_2,\varsigma_3)_{\varsigma}2=0, & \mathfrak{R}(\varsigma_2,\varsigma_3)\varsigma_3=-\varsigma_2,\end{array}\right\} \tag{8.15}$$

along with $\mathfrak{R}(\varsigma_i,\varsigma_i)\varsigma_i=0$; $\forall\ i=1,2,3$. By above discussions it has been verified equations (2.7), (2.8), (2.10) and (2.12) hold.

Analogously, we can estimate the $\widetilde{\mathfrak{R}}(\varsigma_i,\varsigma_j)\varsigma_k$; $i,j,k=1,2,3$ of connection $\widetilde{\nabla}$ by using equations (3.10), (8.4) and (8.11), we have

$$\left.\begin{array}{lll}\widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_2)\varsigma_1=0, & \widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_2)\varsigma_2=0, & \widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_2)\varsigma_3=0,\\ \widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_3)\varsigma_1=0, & \widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_3)\varsigma_2=0, & \widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_3)\varsigma_3=-\frac{3}{4}\varsigma_1,\\ \widetilde{\mathfrak{R}}(\varsigma_2,\varsigma_3)\varsigma_1=0, & \widetilde{\mathfrak{R}}(\varsigma_2,\varsigma_3)\varsigma_2=0, & \widetilde{\mathfrak{R}}(\varsigma_2,\varsigma_3)\varsigma_3=-\frac{3}{4}\varsigma_2,\end{array}\right\} \tag{8.16}$$

along with $\widetilde{\mathfrak{R}}(\varsigma_i,\varsigma_i)\varsigma_i=0$; $\forall\ i=1,2,3$.

By virtue of (8.15) and (8.16), we have verified equations (4.1), (4.5), (4.6), (4.7) and (4.8). The Ricci tensors $\mathcal{S}(\varsigma_j,\varsigma_k)$; $j,k=1,2,3$ of connection ∇ can be estimated by using (8.15) as under

$$\mathcal{S}(\varsigma_j,\varsigma_k)=\sum_{i=1}^{3}g(\mathfrak{R}(\varsigma_i,\varsigma_j)\varsigma_k,\varsigma_i).$$

It is as under:

$$\left.\begin{aligned}&\mathcal{S}(\varsigma_1,\varsigma_1)=0, \quad \mathcal{S}(\varsigma_2,\varsigma_2)=0, \quad \mathcal{S}(\varsigma_3,\varsigma_3)=-2,\\ &\mathcal{S}(\varsigma_1,\varsigma_2)=0, \quad \mathcal{S}(\varsigma_1,\varsigma_3)=0, \quad \mathcal{S}(\varsigma_2,\varsigma_3)=0.\end{aligned}\right\} \tag{8.17}$$

In view of equation (8.17), we can easily verify equation (2.12).

Also in view of equation (8.17) we have verified the following:

$$\left.\begin{aligned}&(\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_1,\phi\varsigma_2)=0, \quad (\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_2,\phi\varsigma_3)=0, \quad (\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_1,\phi\varsigma_1)=0,\\ &(\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_1,\phi\varsigma_3)=0, \quad (\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_3,\phi\varsigma_1)=0, \quad (\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_2,\phi\varsigma_2)=0,\\ &(\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_2,\phi\varsigma_1)=0, \quad (\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_3,\phi\varsigma_2)=0, \quad (\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\varsigma_3,\phi\varsigma_3)=0.\end{aligned}\right\} \tag{8.18}$$

Thus we note that

$$(\nabla_{\mathcal{K}_1}\mathcal{S})(\phi\mathcal{K}_2,\phi\mathcal{K}_3)=0. \tag{8.19}$$

$\forall\ \mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3 \in \mathfrak{X}(\mathfrak{M})$. Hence the Ricci tensor is η-parallel. In view of equation (8.18) we can easily verify the equation (2.16).

The $\widetilde{\mathcal{S}}(\varsigma_j,\varsigma_k)$; $j,k=1,2,3$ of $\widetilde{\nabla}$ estimated by using (8.16) as under

$$\widetilde{\mathcal{S}}(\varsigma_j,\varsigma_k)=\sum_{i=1}^{3} g(\widetilde{\mathfrak{R}}(\varsigma_i,\varsigma_j)\varsigma_k,\varsigma_i).$$

It follows as under:

$$\left.\begin{aligned}&\widetilde{\mathcal{S}}(\varsigma_1,\varsigma_1)=0, \quad \widetilde{\mathcal{S}}(\varsigma_2,\varsigma_2)=0, \quad \widetilde{\mathcal{S}}(\varsigma_3,\varsigma_3)=-\frac{3}{2},\\ &\widetilde{\mathcal{S}}(\varsigma_1,\varsigma_2)=0, \quad \widetilde{\mathcal{S}}(\varsigma_1,\varsigma_3)=0, \quad \widetilde{\mathcal{S}}(\varsigma_2,\varsigma_3)=0.\end{aligned}\right\} \tag{8.20}$$

In view of equation (8.20), we can say that the example validate the equations (4.2) and (4.9).

Hence, we can say that given example is suitable for verification.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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