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Research Article

# Construction and Convergence of H-S Combined Mean Method for Multiple Polynomial Zeros

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**Abstract.** In this article, we have constructed an iterative method of third order for solving polynomial equations with multiple polynomial zeros. We have combined two well known third order methods one is Halley and another is Super-Halley for this construction purpose. This constructed method is basically the mean of the methods Halley and Super-Halley, so we name the method as H-S Combined Mean Method. We have proposed some local convergence theorems of this H-S Combined Mean Method to establish the computation of a polynomial with known multiple zeros. For the establishment of this local convergence theorem, the key role is performed by a function (Real valued) termed as the function of initial conditions. Function of initial conditions $I$ is a mapping from the set $D$ into the set $X$, where $D$ (subset of $X$) is the domain of the H-S Combined Mean iterative scheme. Here the initial conditions uses the information only at the initial point and are given in the form $I(w_0)$ which belongs to $J$, where $J$ is an in interval on the positive real line which also contains zero and $w_0$ is the starting point. We have used the notion of gauge function which also plays very important role in establishing the convergence theorem. Here we have used two types of initial conditions over an arbitrary normed field and established local convergence theorems of the constructed H-S Combined Mean Method. The error estimations are also found in our convergence analysis. For simple zero, the method as well as the results hold good.

**Keywords.** Local convergence, Halley method, Super-Halley method, Initial conditions, Multiple zeros, Normed field

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## 1. Introduction

In the literature of iterative method for solving non-linear equations, Halley and Super-Halley method are among the efficient methods in solving non-linear equation along with Newton's and Chebyshev method. Osada ([8]), Neta ([1]), Chun and Neta ([2]), Ren and Argyros ([4]) and many others have studied iterative method for solving an equation of non-linear type having multiple roots.

Halley iterative method for multiple zeros ([3], [5], [7]) is defined as following:

$$H(w)=\begin{cases} w-\left(\frac{p+1}{2p}\frac{g'(w)}{g(w)}-\frac{1}{2}\frac{g''(w)}{g'(w)}\right)^{-1}, & \text{if } g(w)\neq 0,\\ w, & \text{otherwise.}\end{cases} \tag{1.1}$$

The domain the function $H$ is $D_H$, defined as following:

$$D_H=\left\{w\in F: g(w)\neq 0 \Rightarrow g'(w)\neq 0 \text{ and } \frac{p+1}{2p}\frac{g'(w)}{g(w)}-\frac{1}{2}\frac{g''(w)}{g'(w)}\neq 0\right\}. \tag{1.2}$$

Super-Halley iterative method for multiple zeros [6] is defined as following:

$$S(w)=\begin{cases} w-\frac{g(w)}{2g'(w)}\left(p+\frac{1}{1-\frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}}\right), & \text{if } g'(w)\neq 0,\\ w, & \text{otherwise.}\end{cases} \tag{1.3}$$

The domain the function $S$ is $D_S$, defined as following:

$$D_S=\left\{w\in F: g'(w)\neq 0\Rightarrow 1-\frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\neq 0\right\}. \tag{1.4}$$

Here, we have combined the above two methods to construct the H-S Combined Mean Method.

Proinov [9, 10], and later, Ivanov [11] have introduced convergence theorems for the Picard iterative scheme given as:

$$w_{m+1}=T(w_m), \quad m=0,1,2,\dots, \tag{1.5}$$

where $T:D\to F$ is the function of iteration defined in a metric space $F$ and $D\subset F$.

Here, we investigate convergence of the H-S Combined Mean Method for polynomial zeros which are multiple in nature with the help of the same initial conditions as in the Proinov [[10], [9]] and Ivanov [11].

In this paper, Section 2 is devoted to the preliminaries, necessary in establishing our results. Construction of our H-S Combined Mean Method is presented in Section 3. We have devoted Section 4 in establishing two types of local convergence analysis of the proposed H-S Combined Mean Method.

## 2. Preliminaries

In this paper, $J$ will be treated as an interval in the real line containing zero. $S_l(u)$ is the polynomial defined as

$$S_l(u)=1+u+u^2+\ldots+u^{l-1}. \tag{2.1}$$

If $l=0$, then here we take $S_l(u)=0$. Here, we will use that $0^0$ equal to 1.

**Definition 2.1** ([10]). A function $\varphi : J \to R_+$ is called quasi-homogeneous of order $r \geq 0$ on $J$ if there exists a non decreasing function $\Psi : J \to R_+$ such that

$$\varphi(u) = u^r \Psi(u), \quad \text{for all } u \in J.$$

Following are some properties of the above function.

(P1) A function $g$ is quasi-homogeneous function of degree $r = 0$ on $J$ if and only if $g$ is non-decreasing on $J$.

(P2) If $f$ and $g$ are quasi-homogeneous functions of degree $r \geq 0$ and $s \geq 0$ on $J$, then $fg$ is quasi-homogeneous of degree $r + s$ on $J$.

(P3) If two functions $f$ and $g$ are quasi-homogeneous of degree $r \geq 0$ on $J$, then $f + g$ is also quasi-homogeneous of degree $r$ on $J$.

We will use these properties in proving Lemmas and Theorems in the later section.

**Definition 2.2** ([9]). A function $\varphi : J \to R_+$ is called gauge function of order $r \geq 1$ on $J$ if it satisfies the following conditions:

(i) $\varphi$ is quasi-homogeneous function of degree $r$ on $J$.

(ii) $\varphi(u) \leq u$, for all $u \in J$.

A gauge function $\varphi$ of order $r$ on $J$ is said to be a strict gauge function if the last inequality is strict whenever $u \in J \setminus \{0\}$.

**Lemma 2.1** ([9]). *If $\varphi : J \to R_+$ is a quasi-homogeneous function of degree $r \geq 1$ on an interval $J$ and $R \in J \setminus \{0\}$ is a fixed point of $\varphi$, then $\varphi$ is a gauge function of order $r$ on* $[0, R]$. *Moreover, if $r > 0$, then $\varphi$ is a strict gauge function on* $[0, R)$.

**Definition 2.3** ([10]). Let $T : D \subset X \to X$ be a map on an arbitrary set $X$. A function $I : D \to R_+$ is said to be a function of initial conditions of $T$ (with gauge function $\varphi$ on $J$) if there exists a function $\varphi : J \to J$ such that

$$I(T(w)) \leq \varphi(I(w)) \text{ with all } w \in D \text{ with } Tw \in D \text{ and } I(w) \in J. \tag{2.2}$$

**Definition 2.4** ([10]). Let $T : D \subset X \to X$ be a map on a arbitrary set $X$ and $I : D \to R_+$ be a function of initial conditions of $T$ with gauge function on $J$. Then, a point $w \in D$ is said to be an initial point of $T$ if $I(w) \in J$ and all of the iterates $T^m w$ $(m = 0, 1, 2, \dots)$ are well defined and belong to $D$.

**Definition 2.5** ([9]). Let $T : D \subset X \to X$ be an operator in a normed space $(X, |\cdot|)$, and let $I : D \to R_+$ be a function of initial conditions of $T$ with gauge function on $J$. Then $T$ is said to be an iterated contraction with respect to $I$ at a point $\zeta \in D$ (with control function $\vartheta$) if $I(\zeta) \in J$ and

$$|Tw - \zeta| \leq \vartheta(I(w))|w - \zeta| \quad \text{for all } w \in D \text{ with } I(w) \in J, \tag{2.3}$$

where $\vartheta : J \to [0, 1)$ is a non-decreasing function.

We will use the following two theorems of Ivanov [11] to establish our result.

**Theorem 2.1** ([11]). *Let $T : D \subset X \to X$ be an iteration function, $\zeta \in X$ and $I : D \to R_+$ defined by* (4.1). *Suppose $\phi : J \to R_+$ is a quasi-homogeneous function of degree $p \geq 0$ and for each $w \in X$ with $I(w) \in J$, the following two conditions are satisfied:*

(i) *$w$ belongs to the set $D$;*

(ii) $|Tw - \zeta| \leq \phi(I(w))|w - \zeta|$.

*Let also $w_0 \in X$ be an initial guess such that*

$$I(w_0) \in J \text{ and } \phi(I(w_0)) < 1, \tag{2.4}$$

*then the following statements hold.*

(i) *Then the Picard iteration* (1.5) *is well defined and converges to $\zeta$ with order $r = p + 1$.*

(ii) *For all $m \geq 0$, we have the following error estimates:*

$$|w_{m+1} - \zeta| \leq \mu^{r^m} |w_m - \zeta| \text{ and } |w_m - \zeta| \leq \mu^{s_m(r)} |w_0 - \zeta|,$$

*where $\mu = \phi(I(w_0))$.*

(iii) *The Picard iteration* (1.5) *converges to $\zeta$ with Q-order $r = p + 1$ and with the following error estimates:*

$$|w_{m+1} - \zeta| \leq (Rd)^{1-r} |w_m - \zeta|^r, \quad \text{for all } m \geq 0,$$

*where $R$ is the minimal solution of the equation $\phi(u) = 1$ in the interval $J \setminus \{0\}$.*

**Theorem 2.2** ([11]). *Let $T : D \subset X \to X$ be an iteration function, $\zeta \in X$ and $I : D \subset X \to R_+$ defined by* (4.21). *Suppose $\vartheta : J \to R_+$ is a nonzero quasi-homogeneous function of degree $p \geq 0$ and for each $w \in X$ with $I(w) \in J$, the following two conditions are satisfied:*

(i) *$w$ belongs to the set $D$;*

(ii) $|Tx - \zeta| \leq \vartheta(I(w))|w - \zeta|$.

*Let also, $w_0 \in X$ be an initial guess such that*

$$I(w_0) \in J \text{ and } \vartheta(I(w_0)) \leq \psi(I(w_0)), \tag{2.5}$$

*where $\psi$ is defined by*

$$\psi(u) = 1 - u(1 + \vartheta(u)).$$

*Then, Picard iteration* (1.5) *is well defined and converges to $\zeta$ with the following error estimates:*

$$|w_{m+1} - \zeta| \leq \theta \mu^{r^m} |w_m - \zeta| \text{ and } |w_m - \zeta| \leq \theta^m \mu^{s_m(r)} |w_0 - \zeta|, \quad \text{for all } m \geq 0, \tag{2.6}$$

*where $\mu = \frac{\vartheta(I(w_0))}{\psi(I(w_0))}$ and $\theta = \psi(I(w_0))$. In addition, if the second inequality in* (2.5) *is strict, then the order of convergence of Picard iteration* (1.5) *is at least $r = p + 1$.*

## 3. Recurrence Relation for the Method

Here, we have derived a relation of the H-S Combined Mean Method combining the two third order iterative method namely Halley and Super-Halley method. For $g(w)$ and $g'(w) \neq 0$, we define the H-S Combined Mean Method as follows:

$$T(w) = \frac{1}{2}H(w) + \frac{1}{2}S(w) = w - \frac{1}{2}\left(\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)}\right)^{-1} - \frac{g(w)}{4g'(w)}\left(p + \frac{1}{1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}}\right).$$

Thus our H-S Combined Mean Method has the following form

$$T(w) = \begin{cases} w - \frac{1}{2}\left(\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)}\right)^{-1} - \frac{g(w)}{4g'(w)}\left(p + \frac{1}{1-\frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}}\right), & \text{if } g(w) \text{ and } g'(w) \neq 0, \\ w, & \text{otherwise.} \end{cases} \tag{3.1}$$

The domain of the H-S Combined Mean iteration function $T$ (3.1) is the set $D$, which is defined below:

$$D = \left\{ w \in F : g(w) \neq 0 \text{ and } g'(w) \neq 0 \Rightarrow 1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)} \neq 0, \frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} \neq 0 \right\}. \tag{3.2}$$

## 4. Local Convergence of Combined Mean Method

Let assume that $g \in F[w]$ be a polynomial which having degree $q(\geq 2)$, such that all the zeros of $g$ are in $F$, and also let $\zeta \in F$ be a zero of the polynomial $g$, multiplicity being $p$.
Here $(F, |\cdot|)$ denotes a field having a norm and $F[w]$ is the ring of polynomial on the field $F$.
Here, we examine the convergence of H-S Combined Mean Method (3.1) with the help of function of initial conditions $I$, which is a map from $F$ to $R_+$ and is defined as follows:

$$I(w) = I_g(w) = \frac{|(w-\zeta)|}{d}, \tag{4.1}$$

where $d$ represents the distance from the zero $\zeta$ to the closest zero of $g$ other than $\zeta$; if $\zeta$ is a only zero of $g$ then we set $I(w)$ as zero.

**Lemma 4.1.** *Let $g \in F[w]$ be a $q(\geq 2)$ degree polynomial having all zeros in $F$, where $F$ is a field. If $\zeta_1, \dots, \zeta_s$, are the all zeros of $g$, multiplicity of the zeros being $p_1, \dots, p_s$, respectively. Then*

(i) *If $w \in F$ be such that for those $w$, $g(w) \neq 0$, then for any one of $i = 1, \dots, s$, we have the following:*

$$\frac{g'(w)}{g(w)} = \frac{p_i + \gamma_i}{w - \zeta_i}, \quad \text{where } \gamma_i = (w - \zeta_i)\sum_{j \neq i} \frac{p_j}{w - \zeta_j}.$$

(ii) *If $w \in F$ is not a zero of $g$ and $g'$, then for any $i = 1, \dots, s$, we have*

$$\frac{g''(w)}{g'(w)} = \frac{(p_i + \gamma_i)^2 - (p_i + \delta_i)}{(w - \zeta_i)(p_i + \gamma_i)}, \quad \text{where } \delta_i = (w - \zeta_i)^2 \sum_{j \neq i} \frac{p_j}{(w - \zeta_j)^2}.$$

*Proof.* (i) From

$$\frac{g'(w)}{g(w)} = \sum_{j=1}^{s} \frac{p_j}{w - \zeta_j},$$

we have

$$\begin{aligned}\frac{g'(w)}{g(w)} &= \sum_{j=1}^{s} \frac{p_j}{w-\zeta_j} \\ &= \frac{p_i}{w-\zeta_i} + \sum_{j\neq i} \frac{p_j}{w-\zeta_j} \\ &= \frac{p_i+\gamma_i}{w-\zeta_i}, \quad \text{where } \gamma_i = (w-\zeta_i)\sum_{j\neq i}\frac{p_j}{w-\zeta_j}.\end{aligned}$$

Which proves the first part of the lemma.

(ii) Using the above identity and the following two identities

$$\frac{g''(w)}{g'(w)} = \frac{g'(w)}{g(w)} - \frac{g(w)}{g'(w)}\sum_{j=1}^{s}\frac{p_j}{(w-\zeta_j)^2} \quad \text{and} \quad \sum_{j=1}^{s}\frac{p_j}{(w-\zeta_j)^2} = \frac{p_i+\delta_i}{(w-\zeta_i)^2},$$

we get

$$\frac{g''(w)}{g'(w)} = \frac{(p_i+\gamma_i)^2-(p_i+\delta_i)}{(w-\zeta_i)(p_i+\gamma_i)}, \quad \text{where } \delta_i = (w-\zeta_i)^2\sum_{j\neq i}\frac{p_j}{(w-\zeta_j)^2}. \qquad \square$$

**Lemma 4.2.** *Let $w,\ \zeta \in F$ and $\zeta_1,\dots,\zeta_s \in F$ be the list of all zeros of $g$ which are other than $\zeta$, then for any of $i = 1,\dots,s$, the inequality listed below is accurate.*

$$|w-\zeta_j| \geq (1-I(w))d, \tag{4.2}$$

*where $I : F \to R_+$ is defined by* (4.1).

*Proof.* According to the definition of $d$ we have $d \leq |\zeta-\zeta_j|$ for all $j = 1,\dots,s$.
So, using above and triangle inequality we have the following

$$|w-\zeta_j| = |\zeta-\zeta_j+w-\zeta| \geq |\zeta-\zeta_j| - |w-\zeta| \geq (1-I(w))d. \qquad \square$$

## 4.1 First Kind of Local Convergence Theorem

Here, $F[w]$ is the ring of polynomials over the field $F$. Let $g$ be a polynomial of degree $q(\geq 2)$, which is in $F[w]$. In this section of the article we will establish the convergence of the H-S Combined Mean Method (3.1) using the function of initial condition $I : F \to R_+$ which is defined in (4.1).

Next, we define the functions $\phi_{sn}, \phi_{sd}$ and $\phi_h$,

$$\phi_{sn}(u) = (q-p)\left(\frac{2(q-p)u}{1-u}+q\right)u^2, \tag{4.3}$$

$$\phi_{sd}(u) = \frac{2(p-2pu+(2p-q)u^2)(p-qu)}{1-u}, \tag{4.4}$$

$$\phi_h(u) = \frac{q(q-p)u^2}{2p^2-2p(p+q)u+q(3p-q)u^2}, \tag{4.5}$$

where $q \geq p$ and $p \geq 1$.

Clearly, $\phi_{sn}$ is positive in the clo-open interval $[0,1)$. Easily we can show that $\phi_{sn}$ quasi-homogeneous on the clo-open interval $[0,1)$ of degree 2. The second function $\phi_{sd}$ is decreasing as well as positive on the clo-open interval $[0,\tau_0)$, where $\tau_0$ is defined by

$$\tau_0 = \begin{cases} \frac{p}{q}, & \text{if } q \geq 2p, \\ \frac{p}{p+\sqrt{p(q-p)}}, & \text{if } q < 2p. \end{cases} \tag{4.6}$$

Easily we can show that on the interval $\left[0, \frac{2p}{q+p+\sqrt{(5q-p)(q-p)}}\right)$ the function $\phi_h$ is a second degree quasi-homogeneous function. So, we can now define the function $\phi_s : [0, \tau_0) \to R_+$ defined by

$$\phi_s(u) = \frac{\phi_{sn}}{\phi_{sd}} = \frac{(q-p)(q+(q-2p)u)u^2}{2(p-qu)(p-2pu+(2p-q)u^2)}. \tag{4.7}$$

From the properties (P1) and (P2), we can easily say that on the interval $[0, \tau_0)$, $\phi_s$ is quasi-homogeneous of degree 2.

Now, we define a function $\phi : \left[0, \frac{2p}{q+p+\sqrt{(5q-p)(q-p)}}\right) \to R_+$ defined as follows

$$\phi(u) = \frac{\phi_s(u)}{2} + \frac{\phi_h(u)}{2}. \tag{4.8}$$

As $\phi_s(u)$ and $\phi_h(u)$ are both second degree quasi homogeneous function, so by property (P3), $\phi$ is also quasi-homogeneous function of the same degree 2 in the clo-open interval $\left[0, \frac{2p}{q+p+\sqrt{(5q-p)(q-p)}}\right)$.

**Lemma 4.3.** *Suppose that $g(w) \in F[w]$ be a $q(\geq 2)$ degree polynomial which splits over $F$, and let $\zeta \in F$ be a multiple zero of $g(w)$, multiplicity being $p$. Let $w \in F$ satisfies the following*

$$I(w) < \tau_1 = \frac{2p}{q+p+\sqrt{(5q-p)(q-p)}}, \tag{4.9}$$

*where $I$ is defined by* (4.1) *and $\tau_1$ is defined in* (4.9). *Then the following two statements (i) and (ii) are true.*

(i) *$w$ is in $D$, the domain of the H-S Combined mean method and is defined in* (3.2).

(ii) *$|Tw-\zeta| \leq \phi(I(w))|w-\zeta|$, where $\phi$ defined in* (4.8).

*Proof.* Let $w \in F$ satisfy the inequality (4.9). If any of $p = q$ or $w = \zeta$ or both are true, then $Tw = \zeta$ and therefore both the statements of the lemma holds. So we assume that $p \neq q$ and $w \neq \zeta$. Let $\zeta_1, \dots, \zeta_s$ be the list of all distinct zeros of $g$ with multiplicities $p_1, \dots, p_s$, respectively. Let $\zeta = \zeta_i$, $p = p_i$, $\gamma = \gamma_i$ and $\delta = \delta_i$ for some $1 \leq i \leq s$, where $\gamma_i$ and $\delta_i$ defined in Lemma 4.1.

To prove the first part of the lemma we have to show that $g'(w) \neq 0$ and $g'(w) \neq 0$ implies $1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)} \neq 0$ and $\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} \neq 0$. From Lemma 4.2 and equation (4.9), we get

$$|w-\zeta_j| \geq (1-I(w))d > 0, \quad \text{as } \tau_1 < 1 \tag{4.10}$$

for each $j \neq i$. Above assures $g(w) \neq 0$. Then, Lemma 4.1 gives the following

$$\frac{g'(w)}{g(w)} = \frac{p+\gamma}{w-\zeta}, \quad \text{where } \gamma = (w-\zeta)\sum_{j\neq i}\frac{p_j}{w-\zeta_j}. \tag{4.11}$$

Using the triangle inequality and equation (4.10), we have the following

$$|\gamma| \leq |w-\zeta|\sum_{j\neq i}\frac{p_j}{|w-\zeta_j|} \leq \frac{|w-\zeta|}{(1-I(w))d}\sum_{j\neq i}p_j = \frac{(q-p)I(w)}{1-I(w)}. \tag{4.12}$$

Using the triangle inequality, equation (4.12) and as $I(w) < \tau_1 \le \frac{p}{q}$, we get the following

$$|p+\gamma| \ge p - |\gamma| \ge p - \frac{(q-p)I(w)}{1-I(w)} = \frac{p-qI(w)}{1-I(w)} > 0. \tag{4.13}$$

Hence, $p+\gamma \neq 0$. This implies $g'(w) \neq 0$.

Then, from the Lemma 4.1, we have the following

$$\frac{g''(w)}{g'(w)} = \frac{(p+\gamma)^2 - (p+\delta)}{(w-\zeta)(p+\gamma)}, \quad \text{where } \delta = (w-\zeta)^2 \sum_{j\neq i} \frac{p_j}{(w-\zeta_j)^2}. \tag{4.14}$$

Now, from (4.11) and (4.14), we have

$$1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)} = 1 - \frac{(w-\zeta)}{p+\gamma}\frac{(p+\gamma)^2-(p+\delta)}{(w-\zeta)(p+\gamma)} = \frac{p+\delta}{(p+\gamma)^2}. \tag{4.15}$$

Therefore, by triangle inequality, (4.10) and $I(w) < \tau_1$, we have the following estimate

$$|\delta| \le \frac{(q-p)I(w)^2}{(1-I(w))^2} \text{ and } |p+\delta| \ge p - |\delta| \ge \frac{\phi_{sd}(I(w))}{2(p-qI(w))(1-I(w))} > 0. \tag{4.16}$$

From above, we conclude that

$$\left|1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\right| > 0.$$

Now it remains to prove that $\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} \neq 0$,

$$\begin{aligned}\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} &= \frac{1}{2}\left(\frac{p+\gamma}{p} + \frac{p+\delta}{p+\gamma}\right)\frac{1}{w-\zeta} \\ &= \left(1 + \frac{\gamma^2+p\delta}{2p(p+\gamma)}\right)\frac{1}{w-\zeta} \\ &= \frac{1+\sigma_h}{w-\zeta},\end{aligned}$$

where

$$\sigma_h = \frac{\gamma^2+p\delta}{2p(p+\gamma)}.$$

Now,

$$|\sigma_h| = \left|\frac{\gamma^2+p\delta}{2p(p+\gamma)}\right| \le \frac{|\gamma|^2+p|\delta|}{2p|(p+\gamma)|} \le \frac{q(q-p)I(w)^2}{2p(1-I(w))(p-qI(w))}.$$

So if we prove that $1+\sigma_h \neq 0$ our purpose will be completed.

Now,

$$|1+\sigma_h| \ge 1 - |\sigma_h| \ge 1 - \frac{1}{2p}\frac{q(q-p)I(w)^2}{(1-I(w))(p-qI(w))} = \frac{q(3p-q)I(w)^2 - 2p(p+q)I(w) + 2p^2}{2p(1-I(w))(p-qI(w))} > 0.$$

This shows that $1+\sigma_h \neq 0$.

Therefore, we can say that $w \in D$ which proves (i).

From the construction of the H-S Combined Mean Method, we have the following

$$Tw - \zeta = w - \zeta - \frac{1}{2}\left(\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)}\right)^{-1} - \frac{g(w)}{4g'(w)}\left(p + \frac{1}{1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}}\right)$$

$$
\begin{aligned}
&= w-\zeta-\frac{1}{2}\frac{w-\zeta}{1+\sigma_h}-\frac{(w-\zeta)}{4(p+\gamma)}\left[p+\frac{(p+\gamma)^2}{p+\delta}\right]\\
&= \frac{w-\zeta}{2}\frac{\sigma_h}{1+\sigma_h}+\frac{(w-\zeta)}{2}\left[1-\frac{2p^2+p\delta+2p\gamma+\gamma^2}{2(p+\gamma)(p+\delta)}\right]\\
&= \frac{w-\zeta}{2}\frac{\sigma_h}{1+\sigma_h}+\frac{(w-\zeta)}{2}\left[\frac{2\gamma\delta+p\delta-\gamma^2}{2(p+\gamma)(p+\delta)}\right]\\
&= \sigma(w-\zeta),
\end{aligned}
$$

where

$$
\sigma=\frac{1}{2}\left(\left[\frac{\sigma_h}{1+\sigma_h}\right]+\left[\frac{2\gamma\delta+p\delta-\gamma^2}{2(p+\gamma)(p+\delta)}\right]\right). \tag{4.17}
$$

Using (4.12), (4.13) and (4.16), we now estimate $|\sigma|$ and is as follows.

$$
\begin{aligned}
|\sigma| &\le \frac{1}{2}\left(\left|\left[\frac{\sigma_h}{1+\sigma_h}\right]\right|+\left|\left[\frac{2\gamma\delta+p\delta-\gamma^2}{2(p+\gamma)(p+\delta)}\right]\right|\right)\\
&\le \frac{1}{2}\left(\left[\frac{|\sigma_h|}{1-|\sigma_h|}\right]+\left[\frac{2|\gamma||\delta|+p|\delta|+|\gamma|^2}{2|(p+\gamma)||(p+\delta)|}\right]\right)\\
&\le \frac{1}{2}\frac{q(q-p)I(w)^2}{q(3p-q)I(w)^2-2p(p+q)I(w)+2p^2}+\frac{2\frac{(q-p)I(w)}{1-I(w)}\frac{(q-p)I(w)^2}{(1-I(w))^2}+p\frac{(q-p)I(w)^2}{(1-I(w))^2}+\left(\frac{(q-p)I(w)}{1-I(w)}\right)^2}{4\frac{p-qI(w)}{1-I(w)}\frac{\phi_{sd}(I(w))}{2(p-qI(w))(1-I(w))}}\\
&= \frac{1}{2}\left[\phi_h(I(w))+\frac{\phi_{sn}(I(w))}{\phi_{sd}(I(w))}\right]\\
&= \frac{\phi_h(I(w))}{2}+\frac{\phi_s(I(w))}{2}\\
&= \phi(I(w))
\end{aligned}
$$

which proves (ii). □

**Theorem 4.1.** *Let $g\in F[w]$ be a polynomial of degree $q\ge 2$ that splits over $F$, and let $\zeta\in F$ be a zero of $g$ such that the multiplicity of $\zeta$ is $p$. Let $w_0\in F$ satisfies the following initial condition*

$$
I(w_0)<\tau_1 \text{ and } \phi(I(w_0))<1, \tag{4.18}
$$

*where $I:F\to R_+$ is defined in* (4.1) *and $\phi$ is defined in* (4.8)*. Then, the following three statements are true.*

(i) *Iterative sequence* (3.1) *of the H-S Combined Mean Method is defined and converges to $\zeta$ having order of convergence* 3.

(ii) *Error estimates are as follows*

$$
|w_{m+1}-\zeta|\le\mu^{3^m}|w_m-\zeta| \text{ and } |w_m-\zeta|\le\mu^{(3^m-1)/2}|w_0-\zeta|, \quad \text{for all } m\ge 0, \tag{4.19}
$$

*where $\mu=\phi(I(w_0))$.*

(iii) *A posteriori error estimate given below*

$$
|w_{m+1}-\zeta|<\frac{1}{(Ud)^2}|w_m-\zeta|^3, \quad \text{for all } m\ge 0, \tag{4.20}
$$

*where $U\in(0,\tau_1)$ is the unique solution of $\phi(t)=1$ in $(0,\tau)$.*

*Proof.* Lemma 4.3 and Theorem 2.1 gives the proof. □

## 4.2 Second Kind of Local Convergence theorem

Let assume that $g \in F[w]$ be a polynomial which having degree $q(\geq 2)$, such that all the zeros of $g$ are in $F$, and also let $\zeta \in F$ be a zero of the polynomial $g$, multiplicity being $p$.

Here $(F, |\cdot|)$ denotes a field having a norm and $F[w]$ is the ring of polynomial on the field $F$. Here, we examine the convergence of H-S Combined Mean Method (3.1) with the help of function of initial conditions $I$, which is a map from $F$ to $R_+$ and is defined as follows:

$$I(w) = I_g(w) = \frac{|(w-\zeta)|}{\rho(w)}, \tag{4.21}$$

here $\rho(w)$ represents the distance from $w$ to the closest zero of $g$ other than $\zeta$; if $\zeta$ is a only zero of $g$ then we set $I(w)$ as zero.

Now, we define two real functions $\vartheta_s$ and $\vartheta_h$, for $q > p \geq 1$, by

$$\vartheta_s(u) = \frac{(q-p)(q+2(q-p)u)u^2}{2(p-(q-p)u^2)(p-(q-p)u)} \tag{4.22}$$

and

$$\vartheta_h(u) = \frac{q(q-p)u^2}{2p^2 - 2p(q-p)u - q(q-p)u^2}. \tag{4.23}$$

Clearly, the functions $\vartheta_s$ is quasi-homogeneous functions of degree 2 on $\left[0, \sqrt{\frac{p}{q-p}}\right]$ and $\vartheta_h$ is a quasi-homogeneous functions of degree 2 on $\left[0, \frac{2p}{q+\sqrt{3q^2-6pq+4p^2}}\right]$.

Now, we can define a function $\vartheta : \left[0, \tau_2 = \frac{2p}{q+\sqrt{3q^2-6pq+4p^2)}}\right) \to R_+$ defined by

$$\vartheta(u) = \frac{\vartheta_s(u)}{2} + \frac{\vartheta_h(u)}{2}. \tag{4.24}$$

As both the functions $\vartheta_s(u)$ and $\vartheta_h(u)$ are quasi-homogeneous, therefore by property (P3) we can say that $\vartheta$ is also quasi-homogeneous of exact same degree 2 in the interval $[0, \tau_2)$.

**Lemma 4.4.** *Let $g \in F[w]$ be a polynomial of degree $q(\geq 2)$ which splits over $F$, and let $\zeta \in F$ be a zero of $g$ with multiplicity $p$. Let $w \in F$ be such that*

$$I(w) < \tau_2, \tag{4.25}$$

*where the function $I$ is defined by* (4.21)*. Then:*

(i) *$w$ is in $D$, the domain of the H-S Combined Mean Method and is defined in* (3.2)*.*

(ii) *$|Tw - \zeta| \leq \vartheta(I(w))|w-\zeta|$, where $\vartheta$ is defined in* (4.24)*.*

*Proof.* Let $w \in F$ satisfy the inequality (4.25). If any of $p = q$ or $w = \zeta$ or both are true, then $Tw = \zeta$ and therefore both the statements of the lemma holds. So, we assume that $p \neq q$ and $w \neq \zeta$. Let $\zeta_1, \dots, \zeta_s$ be the list of all distinct zeros of $g$ with multiplicities $p_1, \dots, p_s$, respectively. Let $\zeta = \zeta_i$, $p = p_i$, $\gamma = \gamma_i$ and $\delta = \delta_i$ for some $1 \leq i \leq s$, where $\gamma_i$ and $\delta_i$ defined in Lemma 4.1.

To prove the first part of the lemma we have to show that $g(w) \neq 0$ and $g'(w) \neq 0$ implies $1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)} \neq 0$ and $\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} \neq 0$. Clearly, we can write the following

$$|w - \zeta_j| \geq \rho(w) > 0, \tag{4.26}$$

for each $j \neq i$. This assures that $g(w) \neq 0$. Then, Lemma 4.1 gives the following

$$\frac{g'(w)}{g(w)} = \frac{p+\gamma}{w-\zeta}, \quad \text{where } \gamma = (w-\zeta)\sum_{j\neq i}\frac{p_j}{w-\zeta_j}. \tag{4.27}$$

Using the triangle inequality and (4.26), we have the following:

$$|\gamma| \leq |w-\zeta|\sum_{j\neq i}\frac{p_j}{|w-\zeta_j|} \leq \frac{|w-\zeta|}{\rho(w)}\sum_{j\neq i} p_j = (q-p)I(w). \tag{4.28}$$

Using the triangle inequality, equation (4.28) and $I(w) < \tau_2$, we have the following:

$$|p+\gamma| \geq p - |\gamma| \geq p - (q-p)I(w) > 0. \tag{4.29}$$

Hence, $p + \gamma \neq 0$. This implies $g'(w) \neq 0$.

Then from the Lemma 4.1, we have the following

$$\frac{g''(w)}{g'(w)} = \frac{(p+\gamma)^2 - (p+\delta)}{(w-\zeta)(p+\gamma)}, \quad \text{where } \delta = (w-\zeta)^2\sum_{j\neq i}\frac{p_j}{(w-\zeta_j)^2}. \tag{4.30}$$

Now from (4.27) and (4.30), we have

$$1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)} = 1 - \frac{(w-\zeta)}{p+\gamma}\frac{(p+\gamma)^2-(p+\delta)}{(w-\zeta)(p+\gamma)} = \frac{p+\delta}{(p+\gamma)^2}. \tag{4.31}$$

Now, by triangle inequality, (4.26) and $I(w) < \tau_2$, we have the following estimate

$$|\delta| \leq (q-p)I(w)^2 \text{ and } |p+\delta| \geq p - |\delta| \geq p - (q-p)I(w)^2 \geq 0. \tag{4.32}$$

From above, we conclude that

$$\left|1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\right| > 0.$$

Now it remains to prove that $\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} \neq 0$,

$$\begin{aligned}\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} &= \frac{1}{2}\left(\frac{p+\gamma}{p} + \frac{p+\delta}{p+\gamma}\right)\frac{1}{w-\zeta}\\ &= \left(1 + \frac{\gamma^2 + p\delta}{2p(p+\gamma)}\right)\frac{1}{w-\zeta}\\ &= \frac{1+\sigma_h}{w-\zeta},\end{aligned}$$

where

$$\sigma_h = \frac{\gamma^2 + p\delta}{2p(p+\gamma)}.$$

Now,

$$|\sigma_h| = \left|\frac{\gamma^2+p\delta}{2p(p+\gamma)}\right| \leq \frac{|\gamma|^2 + p|\delta|}{2p|(p+\gamma)|} \leq \frac{q(q-p)I(w)^2}{2p(p-(q-p)I(w))}.$$

So if we prove that $1 + \sigma_h \neq 0$ our purpose will be completed.

Now,

$$|1+\sigma_h| \geq 1-|\sigma_h| \geq 1-\frac{q(q-p)I(w)^2}{2p(p-(q-p)I(w))} = \frac{q(3p-q)I(w)^2-2p(p+q)I(w)+2p^2}{2p(1-I(w))(p-qI(w))} > 0.$$

This shows that $1+\sigma_h \neq 0$.

Therefore, we can say that $w \in D$ which proves (i).

From the construction of the Combined mean method, we have the following

$$\begin{aligned}
Tw-\zeta &= w-\zeta-\frac{1}{2}\left(\frac{p+1}{2p}\frac{g'(w)}{g(w)}-\frac{1}{2}\frac{g''(w)}{g'(w)}\right)^{-1}-\frac{g(w)}{4g'(w)}\left(p+\frac{1}{1-\frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}}\right) \\
&= w-\zeta-\frac{1}{2}\frac{w-\zeta}{1+\sigma_h}-\frac{(w-\zeta)}{4(p+\gamma)}\left[p+\frac{(p+\gamma)^2}{p+\delta}\right] \\
&= \frac{w-\zeta}{2}\frac{\sigma_h}{1+\sigma_h}+\frac{(w-\zeta)}{2}\left[1-\frac{2p^2+p\delta+2p\gamma+\gamma^2}{2(p+\gamma)(p+\delta)}\right] \\
&= \frac{w-\zeta}{2}\frac{\sigma_h}{1+\sigma_h}+\frac{(w-\zeta)}{2}\left[\frac{2\gamma\delta+p\delta-\gamma^2}{2(p+\gamma)(p+\delta)}\right] \\
&= \sigma(w-\zeta),
\end{aligned}$$

where

$$\sigma = \frac{1}{2}\left(\left[\frac{\sigma_h}{1+\sigma_h}\right]+\left[\frac{2\gamma\delta+p\delta-\gamma^2}{2(p+\gamma)(p+\delta)}\right]\right). \tag{4.33}$$

We now use the estimates (4.28), (4.29) and (4.32) to estimate $|\sigma|$ and is given as following

$$\begin{aligned}
|\sigma| &\leq \frac{1}{2}\left(\left|\left[\frac{\sigma_h}{1+\sigma_h}\right]\right|+\left|\left[\frac{2\gamma\delta+p\delta-\gamma^2}{2(p+\gamma)(p+\delta)}\right]\right|\right) \\
&\leq \frac{1}{2}\left(\left[\frac{|\sigma_h|}{1-|\sigma_h|}\right]+\left[\frac{2|\gamma||\delta|+p|\delta|+|\gamma|^2}{2|(p+\gamma)||(p+\delta)|}\right]\right) \\
&\leq \frac{q(q-p)I(w)^2}{4p^2-4p(q-p)I(w)-2q(q-p)I(w)^2} \\
&\quad +\frac{2(q-p)I(w)(q-p)I(w)^2+p(q-p)I(w)^2+((q-p)I(w))^2}{4(p-(q-p)I(w))(p-(q-p)I(w)^2)} \\
&= \frac{q(q-p)I(w)^2}{4p^2-4p(q-p)I(w)-2q(q-p)I(w)^2} \\
&\quad +\frac{(q-p)(q+2(q-p)I(w))I(w)^2}{4(p-(q-p)I(w))(p-(q-p)I(w)^2)} \\
&= \frac{\vartheta_h(I(w))}{2}+\frac{\vartheta_s(I(w))}{2} \\
&= \vartheta(I(w)).
\end{aligned}$$

Hence, the proof of the lemma is completed. □

Next, we state the convergence theorem second type.

**Theorem 4.2.** *Let $g \in F[w]$ be a polynomial of degree $q \geq 2$ which splits over $F$, and let $\zeta \in F$ be a zero of $g$ such that the multiplicity of $\zeta$ is $p$. Let $w_0 \in F$ satisfies the following initial conditions*

$$I(w_0) < \tau_2 \text{ and } \vartheta(I(w_0)) \leq \psi(I(w_0)), \tag{4.34}$$

*where the function $I$ is defined in* (4.21) *and the function $\psi$ is defined below as*

$$\psi(u) = 1 - u(1 + \vartheta(u)). \tag{4.35}$$

*Then, the H-S Combined Mean Method is defined and converges to $\zeta$ having the following error estimates*

$$|w_{m+1} - \zeta| \leq \theta\mu^{3^m}|w_m - \zeta| \text{ and } |w_{m+1} - \zeta| \leq \theta^m \mu^{(3^m-1)/2}|w_0 - \zeta|, \quad \text{for all } m \geq 0, \tag{4.36}$$

*where $\theta = \psi(I(w_0))$ and $\mu = \frac{\vartheta(I(w_0))}{\psi(I(w_0))}$.*

*Proof.* Lemma 4.4 and Theorem 2.2 guarantees the proof. □

## 5. Conclusion

In the first part of this study, we have combined the Halley and Super-Halley methods to create a method for solving nonlinear equations. Secondly, we have demonstrated the method's local convergence for multiple polynomial zero of a polynomial $g$ over any normed field $F$.

### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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