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Research Article

Some Applications of Soft ∂-Closed Sets in Soft Closure Spaces

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Abstract. This study aims to present some applications of the notion of soft ∂-closed sets in soft closure spaces, which not only generalizes classical soft closed sets but also establishes a connection with soft g-closed sets. We discuss their basic characterizations of these sets and examine their implications in soft closure spaces. Furthermore, we apply these sets to introduce the notion of $S\partial$-continuous and $S\partial$-closed maps and present their various properties with some supported examples. Moreover, we propose two separation properties, which utilize the notion of $S\partial$-closed sets and explore their characteristics.

Keywords. Soft set, $S\partial$-closed set, soft closure space, $S\partial$-continuous maps, $S\partial$-$T_{\frac{1}{2}}$ and $S\partial$-$T^*_{\frac{1}{2}}$ spaces

Mathematics Subject Classification (2020). 54A05, 54B05

1. Introduction and Preliminaries

Soft sets(briefly, S-sets), first introduced by Molodtsov [18] in 1999, have since gained significant attention in recent years due to their ability to model various types of uncertainty and vagueness. Since then, many works have been published on S-sets and their applications in different fields as in [1–3, 5, 10]. Shabir and Naz [20] presented the notion of soft topological spaces. Gowri *et al*. [11] proposed the structure of Čech soft closure space. Recently, Ekram and Majeed [8] defined and studied the notion of soft closure spaces (briefly, SC-spaces) as a generalization of classical closure spaces proposed by Čech and Katětov [6].

Closed sets play a crucial role in understanding the topological properties of a space. Levine [15] proposed the concept of generalized closed sets(briefly, g-closed sets) in general topology. However, the current literature on SC-spaces has mainly focused on the study of soft closed sets(briefly, S-closed sets) and soft g-closed sets (briefly, Sg-closed sets). Kannan [13] proposed the notion of Sg-closed sets in soft topology. The notions of g-closed sets and g-continuous maps have been extended to closure spaces by Boonpok and Khampakdee [4], as well as they are defined and studied the notion of ∂-closed sets in closure spaces. Then Gowri and Jegadeesan [12] extended this notion to Čech soft closure space defined on S-sets. Recently, Ekram and Majeed studied the concept of Sg-closed sets and S-continuous maps in SC-spaces as in [9, 16].

In the percent paper, we discuss and study the notion of soft ∂-closed sets in SC-spaces, that generalize traditional soft closed sets and bridge the gap between them and soft g-closed sets. We discuss more characterizations and properties of them and their implications in the context of SC-spaces. Further, we apply these sets to introduce the notion of $S\partial$-continuous and $S\partial$-closed maps. Some of their properties with many illustrate examples are presented. Finally, we provide and discuss some new separability properties, called $S\partial$-$T_{\frac{1}{2}}$ and $S\partial$-$T^*_{\frac{1}{2}}$ that utilize the notion of soft ∂-closed sets. This study provides a deeper understanding of the topological properties of SC-spaces and contributes to the ongoing exploration of this area of study.

In all the study, U refers to an initial universe set, M is a set of parameters, 2^U is the power set of U, and $(U,\widetilde{c},M)$ refers to the SC-space on U. In the next, we recall some concepts and results about S-sets, for more details see [7, 14, 17–19].

An S-set $H_M = (H,M)$ on U is a mapping $H : M \to 2^U$ that is, H_M can be defined as a set of ordered pairs $H_M = \{(m,h(m)) : m \in M,\ H(m) \in 2^U\}$. The collection of all S-sets on U symbolized by $SS(U)$.

For two S-subsets H_M and G_M in U, we have:

- If $H(m) = \emptyset$ (resp. $H(m) = U$) for every $m \in M$, then H_M is called a null (resp. universal) S-set and symbolized by $\widetilde{\emptyset}$ (resp. $\widetilde{U}$).
- The relative complement H^c_M of H_M where, $H^c : M \to 2^U$ is a mapping given by $H^c(m) = U - H(m)$ for every $m \in M$. Clearly, $(H^c_M)^c = H_M$.
- H_M is an S-subset of G_M denoted by $H_M \sqsubseteq G_M$ if $H(m) \subseteq G(m)$ for each $m \in M$.
- The union (resp. intersection) of H_M and G_M is an S-set K_M (resp. F_M) given by $K(m) = H(m) \cup G(m)$ (resp. $F(m) = H(m) \cap G(m)$) for each $m \in M$ and symbolized by $H_M \sqcup G_M$ (resp. $H_M \sqcap G_M$).
- The difference between H_M, G_M denoted by $H_M - G_M$, is an S-set given as $F(m) = H(m) - G(m)$ for each $m \in M$.

For $H_M \in SS(U)$, $Y \subseteq U$ and $u \in U$, we have:

- If $H(m) = \{u\}$ and $H(m') = \emptyset$ for every $m' \in M - \{m\}$, then H_M is called an S-point in U_M symbolized by u_m. We write $u_m \widetilde{\in} H_M$ if for the element $m \in M$, $u \in H(m)$. The family of all S-points in U_M is denoted by $SP(U)$.

- $u \in H_M$ if $u \in H(m)$ for every $m \in M$, and $u \notin H_M$ if $u \notin H(m)$ for some $m \in M$.
- If $H(m) = \{u\}$ for every $m \in M$, then H_M is called an S-singleton point denoted by u_M.
- $\widetilde{Y} = (Y, M)$ refers to the S-set on U for which $Y(m) = Y$ for all $m \in M$, is called stable.

Definition 1.1 ([14]). Let $SS(U)$ and $SS(V)$ be two families of all S-sets on U, V respectively and let $u : U \to V$ and $p : M \to E$ be two maps, then the map $f_{up} : SS(U) \to SS(V)$ is called a soft map(briefly, S-map). For $H_M \in SS(U)$, $f_{up}(H_M)$ is the S-set on V given by $f_{up}(H_M)(e) = \cup\{u(H(m)) : m \in p^{-1}(e)\}$ if $p^{-1}(e) \neq \emptyset$ and $f_{up}(H_M)(e) = \widetilde{\emptyset}$ otherwise for each $e \in E$, and for $G_E \in SS(V)$, $f^{-1}_{up}(G_E)$ is the S-set on U given by $f^{-1}_{up}(G_E)(m) = u^{-1}(G(p(m)))$ for each $m \in M$.

Definition 1.2 ([8]). A map $\widetilde{c} : SS(U) \to SS(U)$ is called a soft closure operator (briefly, SC-operator) on U, if for any F_M and $G_M \in SS(U)$ the next conditions hold:

(i) $\widetilde{c}(\widetilde{\emptyset}) = \widetilde{\emptyset}$.

(ii) $F_M \sqsubseteq \widetilde{c}(F_M)$.

(iii) $F_M \sqsubseteq G_M \Rightarrow \widetilde{c}(F_M) \sqsubseteq \widetilde{c}(G_M)$.

The triple $(U, \widetilde{c}, M)$ is called a soft closure space (briefly, SC-space). An S-set F_M in $(U, \widetilde{c}, M)$ is said to be a soft closed set (briefly, S-closed set) if $F_M = \widetilde{c}(F_M)$. The complement of any S-closed set in U is called a soft open set (briefly, S-open set). The set of all S-closed sets (resp. S-open sets) in $(U, \widetilde{c}, M)$ is denoted by $SCS(U)$ (resp. $SOS(U)$).

Definition 1.3 ([8]). For S-set F_M in an SC-space $(U, \widetilde{c}, M)$. The interior of F_M denoted by $int(F_M)$ and given by $int(F_M) = (\widetilde{c}(F^c_M))^c$.

Definition 1.4 ([8]). Let $(U, \widetilde{c}, M)$ be an SC-space and let $Y \subseteq U$, then $\widetilde{c}_Y : SS(Y) \to SS(Y)$ given by $\widetilde{c}_Y(F_M) = \widetilde{Y} \sqcap \widetilde{c}(F_M)$ is the relative SC-operator on Y. $(Y, \widetilde{c}_Y, M)$ is called a soft closure subspace (briefly, SC-subspace) of SC-space $(U, \widetilde{c}, M)$. If F_T is an S-open set in $(U, \widetilde{c}, M)$ and $(Y, \widetilde{c}_Y, M)$ is an SC-subspace of $(U, \widetilde{c}, M)$, then $\widetilde{Y} \sqcap F_M$ is an S-open set in $(Y, \widetilde{c}_Y, M)$.

Proposition 1.5 ([8]). *Let $\{F_{iM} : i \in J\}$ be a class of S-sets in $(U, \widetilde{c}, M)$, then:*

(i) $\sqcup_{i\in J}\widetilde{c}(F_{iM}) \sqsubseteq \widetilde{c}(\sqcup_{i\in J}(F_{iM}))$.

(ii) $\widetilde{c}(\sqcap_{i\in J}F_{iM}) \sqsubseteq \sqcap_{i\in J}\widetilde{c}(F_{iM})$.

Definition 1.6 ([9]). An S-map $f_{up} : (U, \widetilde{c}, M) \to (V, \widetilde{u}, E)$ is called:

(i) S-continuous if $f_{up}(\widetilde{c}(H_M)) \sqcap \widetilde{u}(f_{up}(H_M))$ for any S-set H_M in $(U, \widetilde{c}, M)$.

(ii) S-closed(resp. S-open) if $f_{up}(H_M)$ is S-closed(resp. S-open) in $(V, \widetilde{u}, E)$ for any S-closed (resp. S-open) set H_M in $(U, \widetilde{c}, M)$.

Definition 1.7 ([16]). An S-set F_M in $(U, \widetilde{c}, M)$ is called a soft generalized closed set (briefly, Sg-closed set), if $\widetilde{c}(F_M) \sqsubseteq G_M$ whenever $F_M \sqcap G_M$ and G_M is an S-open set in $(U, \widetilde{c}, M)$. The relative complement of an Sg-closed set F_M is denoted by and called a soft generalized open set (briefly, Sg-open set).

Remark 1.8 ([16])**.** Every S-closed(S-open) set in $(U,\widetilde{c},M)$ is Sg-closed(Sg-open).

Definition 1.9 ([16])**.** An SC-space $(U,\widetilde{c},M)$ is called $ST_{\frac{1}{2}}$ if any Sg-closed set is S-closed.

2. On Soft ∂-Closed Sets in Soft Closure Spaces

The notion of $S\partial$-closed sets located between the S-closed sets and the Sg-closed sets. In this context, we discuss their basic characterizations.

Definition 2.1. An S-set F_M in an SC-space $(U,\widetilde{c},M)$ is called:

(i) Soft ∂-closed (briefly, $S\partial$-closed set) if $\widetilde{c}(F_M) \sqsubseteq H_M$ whenever $F_M \sqsubseteq H_M$ and H_M is an Sg-open set in $(U,\widetilde{c},M)$. The set of all $S\partial$-closed set denoted by $S\partial C(U)$.

(ii) Soft ∂-open (briefly, $S\partial$-open set) if it is relative complement is an $S\partial$-closed set in $(U,\widetilde{c},M)$. The set of all $S\partial$-open set denoted by $S\partial O(U)$.

Example 2.2. Let $U = \{a,b\}$ and $M = \{m_1,m_2\}$. Then

$$SS(U) = \{F_{1M},F_{2M},F_{3M},F_{4M},F_{5M},F_{6M},F_{7M},F_{8M},F_{9M},F_{10M},F_{11M},F_{12M},F_{13M}, F_{14M},F_{15M},F_{16M}\},$$

where

$$F_{1M} = \{(m_1,\{a\})\},\ \ F_{2M} = \{(m_1,\{b\})\},\ \ F_{3M} = \{(m_1,U)\},$$
$$F_{4M} = \{(m_2,\{a\})\},\ \ F_{5M} = \{(m_2,\{b\})\},\ \ F_{6M} = \{(m_2,U)\},$$
$$F_{7M} = \{(m_1,\{a\}),(m_2,\{a\})\},\ \ F_{8M} = \{(m_1,\{a\}),(m_2,\{b\})\},$$
$$F_{9M} = \{(m_1,\{b\}),(m_2,\{a\})\},\ \ F_{10M} = \{(m_1,\{b\}),(m_2,\{b\})\},$$
$$F_{11M} = \{(m_1,\{a\}),(m_2,U)\},\ \ F_{12M} = \{(m_1,\{b\}),(m_2,U)\},$$
$$F_{13M} = \{(m_1,U),(m_2,\{a\})\},\ \ F_{14M} = \{(m_1,U),(m_2,\{b\})\},$$
$$F_{15M} = \widetilde{U},\ \ F_{16M} = \widetilde{\emptyset}.$$

We define the SC-operator $\widetilde{c} : SS(U) \to SS(U)$ by:

$$\widetilde{c}(\widetilde{\emptyset}) = \widetilde{\emptyset},\ \ \widetilde{c}(F_{1M}) = \{(m_1,\{a\})\},\ \ \widetilde{c}(F_{2M}) = \widetilde{c}(F_{3M}) = \{(m_1,U)\},$$
$$\widetilde{c}(F_{4M}) = \{(m_2,\{a\})\},\ \ \widetilde{c}(F_{5M}) = \widetilde{c}(F_{6M}) = \{(m_2,U)\},\ \ \widetilde{c}(F_{7M}) = \{(m_1,\{a\}),(m_2,\{a\})\},$$
$$\widetilde{c}(F_{8M}) = \widetilde{c}(F_{11M}) = \{(m_1,\{a\}),(m_2,U)\},\ \ \widetilde{c}(F_{9M}) = \widetilde{c}(F_{13M}) = \{(m_1,U),(m_2,\{a\})\},$$
$$\widetilde{c}(F_{10M}) = \widetilde{c}(F_{12M}) = \widetilde{c}(F_{14M}) = \widetilde{c}(\widetilde{U}) = \widetilde{U}.$$

Then $(U,\widetilde{c},M)$ is an SC-space. The $S\partial$-closed sets are:

$$\{\widetilde{\emptyset},\widetilde{U},F_{1M},F_{3M},F_{4M},F_{6M},F_{7M},F_{11M},F_{13M}\}.$$

The set of all $S\partial$-open sets are:

$$\{\widetilde{\emptyset},\widetilde{U},F_{2M},F_{3M},F_{5M},F_{6M},F_{10M},F_{12M},F_{14M}\}.$$

The S-set $F_{2M} = \{(m_1,\{b\})\}$ is neither $S\partial$-closed set nor Sg-closed set in $(U,\widetilde{c},M)$ because, there is an S-open set $F_{10M} = \{(m_1,\{b\}),(m_2,\{b\})\}$ which is also an Sg-open set containing F_{2M} but $\widetilde{c}(F_{2M}) = \{(m_1,U)\} \not\sqsubseteq F_{10M}$.

Remark 2.3. Clearly, every S-closed(S-open) set in SC-space $(U,\widetilde{c},M)$ is $S\partial$-closed($S\partial$-open) but not conversely.

Example 2.4. Let $U=\{x,y\}$ and $M=\{m_1,m_2\}$. We define the SC-operator $\widetilde{c}$ by:

$$\widetilde{c}(\widetilde{\emptyset})=\widetilde{\emptyset},\ \widetilde{c}(\{(m_1,\{x\})\})=\{(m_1,\{x\})\},\widetilde{c}(\{(m_1,\{y\})\})=\widetilde{c}(\{(m_1,U)\})=\{(m_1,U)\}\ \text{ and }\ \widetilde{c}(F_M)=\widetilde{U}$$

for all other $F_M\in SS(U)$. Then $(U,\widetilde{c},M)$ is an SC-space on U and we have, the S-set $H_M=\{(m_1,\{x\}),(m_2,\{y\})\}$ is an $S\partial$-closed set but not S-closed set.

Remark 2.5. Clearly, every $S\partial$-closed ($S\partial$-open) set in SC-space $(U,\widetilde{c},M)$ is an Sg-closed (Sg-open) set but not conversely.

Example 2.6. Let $U=\{a,b\}$ and $M=\{m_1,m_2\}$. We define the SC-operator $\widetilde{c}$ by:

$$\widetilde{c}(\widetilde{\emptyset})=\widetilde{\emptyset},\ \ \widetilde{c}(F_M)=\widetilde{U},\quad \text{for all the other } S\text{-sets } F_M\in SS(U).$$

Then $(U,\widetilde{c},M)$ is an SC-space on U. Let $F_M=\{(m_1,\{a\})\}\in SS(U)$, we have F_M is an Sg-closed set in $(U,\widetilde{c},M)$ but not $S\partial$-closed set. Indeed $F_M=\{(m_1,\{a\})\}$ is an Sg-open set containing itself but $\widetilde{c}(F_M)=\widetilde{U}\not\sqsubseteq F_M$.

From the above remarks and examples, we can summarize the next relation.

Corollary 2.7. *For an SC-space $(U,\widetilde{c},M)$, the next implications hold but not conversely.*

S-closed sets → $S\partial$-closed sets → Sg-closed sets.

Corollary 2.8. *Let $(U,\widetilde{c},M)$ be an SC-space, we have:*

(1) *The union (resp. intersection) of two $S\partial$-closed (resp. $S\partial$-open) sets need not to be $S\partial$-closed (resp. $S\partial$-open) set.*

(2) *The intersection (resp. union) of two $S\partial$-closed (resp. $S\partial$-open) sets is not $S\partial$-closed (resp. $S\partial$-open) set.*

Proof. We can verify this corollary by using the next examples. □

Example 2.9. Let $U=\{x,y,z\}$ and $M=\{m_1,m_2\}$. Define the SC-operator $\widetilde{c}$ by:

$$\widetilde{c}(\widetilde{\emptyset})=\widetilde{\emptyset},\ \ \widetilde{c}(\{(m_1,\{x\})\})=\{(m_1,\{x\})\},\ \ \widetilde{c}(\{(m_2,\{y\})\})=\{(m_2,\{y\})\},\ \ \widetilde{c}(\{(m_1,\{z\})\})=\{(m_1,\{z\})\},$$

and $\widetilde{c}(F_M)=\widetilde{U}$ for other S-sets F_M on U. Then $(U,\widetilde{c},M)$ is an SC-space. Now for two $S\partial$-closed sets $F_M=\{(m_1,\{x\})\}$ and $G_M=\{(m_2,\{y\})\}$ in $(U,\widetilde{c},M)$, we have $F_M\sqcup G_M=\{(m_1,\{x\})\},(m_2,\{y\})\}$ which is not $S\partial$-closed set in $(U,\widetilde{c},M)$. Indeed, there is an Sg-open set say, $H_M=\{(m_1,\{x,y\}),(m_2,U)\}$ in $(U,\widetilde{c},M)$ with $F_M\sqcup G_M\sqsubseteq H_M$ but $\widetilde{c}(F_M\sqcup G_M)=\widetilde{U}\not\sqsubseteq H_M$. We can verify the other case by taking the soft complement to the S-sets in first case.

Example 2.10. Let $U=\{x,y,z\}$ and $M=\{m_1,m_2\}$. Define the SC-operator $\widetilde{c}$ by:

$$\widetilde{c}(\emptyset)=\widetilde{\emptyset},\ \ \widetilde{c}(\{(m_1,\{x\})\})=\widetilde{c}(\{(m_2,\{x\})\})=\widetilde{c}(\{(m_1,\{x\}),(m_2,\{x\})\})=\{(m_1,\{x\}),(m_2,\{x\})\},$$

$$\widetilde{c}(\{(m_1,\{z\})\})=\widetilde{c}(\{(m_1,\{z\}),(m_2,\{x\})\})=\{(m_1,\{z\}),(m_2,\{x\})\},$$

$$\widetilde{c}(\{(m_2,\{y\})\})=\widetilde{c}(\{(m_1,\{x,z\})\})=\widetilde{c}(\{(m_2,\{x,y\})\})=\widetilde{c}(\{(m_1,\{x\}),(m_2,\{y\})\})$$

$$=\widetilde{c}(\{(m_1,\{z\}),(m_2,\{y\})\})=\widetilde{c}(\{(m_1,\{x,z\}),(m_2,\{x,y\})\})=\{(m_1,\{x,z\}),(m_2,\{x,y\})\},$$

and $\widetilde{c}(F_M) = \widetilde{U}$ for other S-sets F_M on U. Then $(U,\widetilde{c},M)$ is an SC-space. Now for two $S\partial$-closed sets $F_M = \{(m_1,\{x\}),(m_2,\{x\})\}$ and $G_M = \{(m_1,\{z\}),(m_2,\{x\})\}$ in $(U,\widetilde{c},M)$, we have $F_M \sqcap G_M = \{(m_2,\{x\})\}$ which is not $S\partial$-closed set in $(U,\widetilde{c},M)$. Indeed, there is an Sg-open set say, $H_M = \{(m_2,\{x,z\})\}$ in $(U,\widetilde{c},M)$ with $(F_M \sqcap G_M) \sqsubseteq H_M$ but $\widetilde{c}(F_M \sqcap G_M) = \{(m_1,\{x\}),(m_2,\{x\})\} \not\sqsubseteq H_M$. One can verify the other case by taking the soft complement to the S-sets in first case.

Theorem 2.11. *For an S-subset H_M in $(U,\widetilde{c},M)$, the next items are equivalent:*
(1) *H_M is an $S\partial$-open set.*
(2) *$F_M \sqsubseteq (\widetilde{c}(H_M^c))^c$ whenever F_M is an Sg-closed set in $(U,\widetilde{c},M)$ with $F_M \sqsubseteq H_M$.*

Proof. (1)⇒(2). Let H_M be an $S\partial$-open set and F_M be an Sg-closed set in $(U,\widetilde{c},M)$ with $F_M \sqsubseteq H_M$, then $H_M^c \sqsubseteq F_M^c$. But H_M^c is $S\partial$-closed and F_M^c is Sg-open. It follows that $\widetilde{c}(H_M^c) \sqsubseteq F_M^c$ and so, $F_M \sqsubseteq (\widetilde{c}(H_M^c))^c$.

(2)⇒(1). Let G_M be an Sg-open set in $(U,\widetilde{c},M)$ with $H_M^c \sqsubseteq G_M$. Then $G_M^c \sqsubseteq H_M$. Since G_M^c is Sg-closed, $G_M^c \sqsubseteq (\widetilde{c}(H_M^c))^c$. Hence $\widetilde{c}(H_M^c) \sqsubseteq G_M$. Thus, H_M^c is $S\partial$-closed and so, H_M is $S\partial$-open. □

Proposition 2.12. *For an S-set F_M in an SC-space $(U,\widetilde{c},M)$. If F_M is both Sg-open and $S\partial$-closed, then F_M is S-closed.*

Proof. It is obvious. □

Proposition 2.13. *Let $(Y,\widetilde{c}_Y,M)$ be an closed SC-subspace of $(U,\widetilde{c},M)$. If H_M is an $S\partial$-closed set in $(Y,\widetilde{c}_Y,M)$, then H_M is an $S\partial$-closed set in $(U,\widetilde{c},M)$.*

Proof. Let H_M be an $S\partial$-closed set in $(Y,\widetilde{c}_Y,M)$ and F_M be an Sg-open set in $(U,\widetilde{c},M)$ such that $H_M \sqsubseteq F_M$. Then, $H_M \sqsubseteq \widetilde{Y} \sqcap F_M$. From Definition 1.4, $\widetilde{Y} \sqcap F_M$ is an Sg-open set in $(Y,\widetilde{c}_Y,M)$. Since, H_M is an $S\partial$-closed set, we have $\widetilde{c}_Y(H_M) \sqsubseteq \widetilde{Y} \sqcap F_M$ this implies that $\widetilde{Y} \sqcap \widetilde{c}(H_M) \sqsubseteq \widetilde{Y} \sqcap F_M$. Since $\widetilde{Y}$ is S-closed set on U, we have $\widetilde{c}(\widetilde{Y}) \sqcap \widetilde{c}(H_M) \sqsubseteq \widetilde{Y} \sqcap F_M$. from Proposition 1.5, we get $\widetilde{c}(\widetilde{Y} \sqcap H_M) \sqsubseteq \widetilde{Y} \sqcap F_M$ implies that $\widetilde{c}(H_M) \sqsubseteq \widetilde{Y} \sqcap F_M \sqsubseteq F_M$. Hence H_M be an $S\partial$-closed set in $(U,\widetilde{c},M)$. □

Definition 2.14. An SC-operator $\widetilde{c}$ is called idempotent if $\widetilde{c}(\widetilde{c}(F_M)) = \widetilde{c}(F_M)$ for any $F_M \in SS(U)$.

Theorem 2.15. *Let $(U,\widetilde{c},M)$ be an SC-space and $\widetilde{c}$ be idempotent. If H_M is an $S\partial$-closed set in $(U,\widetilde{c},M)$ with $H_M \sqsubseteq F_M \sqsubseteq \widetilde{c}(H_M)$, then F_M is an $S\partial$-closed set in $(U,\widetilde{c},M)$.*

Proof. Let G_M be an Sg-open set in $(U,\widetilde{c},M)$ such that $F_M \sqsubseteq G_M$, then $H_M \sqsubseteq G_M$. Since H_M is $S\partial$-closed, $\widetilde{c}(H_M) \sqsubseteq G_M$. As $\widetilde{c}$ is idempotent, we have $\widetilde{c}(F_M) \sqsubseteq \widetilde{c}(\widetilde{c}(H_M)) = \widetilde{c}(H_M) \sqsubseteq G_M$. Therefore, F_M is an $S\partial$-closed set in $(U,\widetilde{c},M)$. □

Theorem 2.16. *If H_M is an $S\partial$-closed set in $(U,\widetilde{c},M)$, then $\widetilde{c}(H_M) - H_M$ contains only null Sg-closed set.*

Proof. Let H_M be an $S\partial$-closed set and F_M be an Sg-closed set in $(U,\widetilde{c},M)$ such that $F_M \sqsubseteq \widetilde{c}(H_M) - H_M$, then $F_M \sqsubseteq \widetilde{c}(H_M)$ and $F_M \sqsubseteq H_M^c$ implies $H_M \sqcap F_M^c$. Since H_M is $S\partial$-closed set and F_M^c is an Sg-open set, then $\widetilde{c}(H_M) \sqsubseteq F_M^c$ implies $F_M \sqsubseteq (\widetilde{c}(H_M))^c$. Thus, $F_M \sqsubseteq \widetilde{c}(H_M) \sqcap \{(\widetilde{c}(H_M))\}^c = \widetilde{\emptyset}$ and so, $F_M = \widetilde{\emptyset}$. The proof is complete. □

Theorem 2.17. *For an S-set H_M in $(U,\widetilde{c},M)$, the next items are equivalent:*

(1) *H_M is an $S\partial$-closed set.*

(2) *$F_M \sqsubseteq int(H_M)$ whenever $F_M \sqsubseteq H_M$ and F_M is an S-closed set.*

Proof. (1)⇒(2). Let H_M be an $S\partial$-open set and F_M be an S-closed set with $F_M \sqsubseteq H_M$, we have $H_M^c \sqsubseteq F_M^c$, where H_M^c is $S\partial$-closed set and F_M^c is S-open which is an Sg-open set in $(U,\widetilde{c},M)$. From Definition 2.1, we have $\widetilde{c}(H_M^c) \sqsubseteq F_M^c$. By taking the complement and from Definition 1.3, we get $F_M \sqsubseteq int(H_M)$.

(2)⇒(1). By a similar way to that in the converse part of Theorem 2.11. □

Theorem 2.18. *Let $(U,\widetilde{c},M)$ be SC-space. If H_M is an open $S\partial$-closed set and F_M is an S-closed set $(U,\widetilde{c},M)$, then $H_M \sqcap F_M$ is an $S\partial$-closed set.*

Proof. Suppose that H_M be an open and $S\partial$-closed set, then $\widetilde{c}(H_M) \sqsubseteq H_M$. But $H_M \sqsubseteq \widetilde{c}(H_M)$ and so, $H_M = \widetilde{c}(H_M)$. Thus, H_M is an S-closed set. Since F_M is an S-closed set, we have $H_M \sqcap F_M$ is an S-closed set. From Remark 2.3, the proof is complete. □

Corollary 2.19. *An $S\partial$-closed set H_M is an S-closed set in $(U,\widetilde{c},M)$ if and only if $\widetilde{c}(H_M) - H_M$ is an S-closed set.*

Proof. "⇒". Let H_M be an S-closed set, we have $\widetilde{c}(H_M) = H_M$ and so, $\widetilde{c}(H_M) - H_M = \widetilde{\emptyset}$ which is an S-closed set.

"⟸". To show that H_M is S-closed. Let H_M be an $S\partial$-closed set, then by Theorem 2.16, we have $\widetilde{c}(H_M) - H_M$ contains only null S-closed set. By hypothesis, $\widetilde{c}(H_M) - H_M$ is an S-closed set and so, $\widetilde{c}(H_M) - H_M = \widetilde{\emptyset}$. Hence the result holds. □

3. Soft ∂-Continuous and Closed Mappings

In this section, we introduce a new class of S-maps namely, $S\partial$-continuous maps in SC-spaces. These maps are lying between the class of S-continuous maps and the class of Sg-continuous maps. We also define the concept of $S\partial$-closed maps and discuss some of its properties.

Definition 3.1. Let $(U,\widetilde{c},M)$, $(V,\widetilde{v},E)$ be SC-spaces. An S-map $f_{up} : (U,\widetilde{c},M) \to (V,\widetilde{v},E)$ is called:

(i) Soft ∂-continuous (briefly, $S\partial$-continuous) if $f_{up}^{-1}(H_E)$ is an $S\partial$-closed set in $(U,\widetilde{c},M)$ for each S-closed set H_E in $(V,\widetilde{v},E)$.

(ii) Soft g-continuous (briefly, Sg-continuous) if $f_{up}^{-1}(H_E)$ is an Sg-closed set in $(U,\widetilde{c},M)$ for each S-closed set H_E in $(V,\widetilde{v},E)$ [13].

Remark 3.2. For any S-map $f_{up} : (U,\widetilde{c},M) \to (V,\widetilde{v},E)$, the next implications hold but not conversely as shown by the next examples.

f_{up} is S-continuous → f_{up} is $S\partial$-continuous → f_{up} is Sg-continuous.

Example 3.3. Let $U = \{a,b,c\}, V = \{x,y,z\}$ and $M = \{m_1,m_2\}$, $E = \{e_1,e_2\}$. Define an SC-operator $\widetilde{c}$ on U as $\widetilde{c}(\widetilde{\emptyset}) = \widetilde{\emptyset}, \widetilde{c}(\{(m_1,\{b\})\}) = \{(m_1,\{b\})\}$ and $\widetilde{c}(F_M) = \widetilde{U}$ for all other $F_M \in SS(U)$. Define an SC-operator $\widetilde{v}$ on V as $\widetilde{v}(\widetilde{\emptyset}) = \widetilde{\emptyset}$, $\widetilde{v}(\{(e_1,\{x\})\}) = \{(e_1,\{x,y\})\}$, $\widetilde{v}(\{(e_1,\{y\})\}) = \{(e_1,\{y\})\}$, $\widetilde{v}(\{(e_2,\{z\})\}) = \{(e_1,\{z\}),(e_2,\{z\})\}$, and $\widetilde{v}(G_E) = \widetilde{V}$ for all other $G_E \in SS(V)$. Clearly, $(U,\widetilde{c},M),(V,\widetilde{v},E)$ are SC-

spaces. Now let $f_{up}:(U,\widetilde{c},M)\to(V,\widetilde{v},E)$ be an S-map, where u,p are maps defined as $u(a)=x$, $u(b)=y$, $u(c)=z$ and $p(m_1)=e_1$, $p(m_2)=e_2$. Then f_{up} is $S\partial$-continuous but it is not S-continuous. Indeed, for the S-set $\{(m_2,\{c\})\}$ we have, $f_{up}(\widetilde{c}(\{(m_2,\{c\})\}))=\widetilde{V}\not\sqsubseteq\widetilde{v}(f_{up}(\{(m_2,\{c\})\})=\{(e_1,\{z\}),(e_2,\{z\})\}$.

Example 3.4. Let $U=\{x,y\}=V$ and $M=\{m_1,m_2\}=E$. Define an SC-operator $\widetilde{c}$ on U as $\widetilde{c}(\widetilde{\emptyset})=\widetilde{\emptyset}$ and $\widetilde{c}(F_M)=\widetilde{U}$ for all other $F_M\in SS(U)$. Define an SC-operator $\widetilde{v}$ on V as $\widetilde{v}(\widetilde{\emptyset})=\widetilde{\emptyset}$, $\widetilde{v}(\{(m_1,\{x\})\})=\{(m_1,\{x\})\}$, $\widetilde{v}(\{(m_1,\{y\})\})=\{(m_1,\{y\})\}$, and $\widetilde{v}(G_E)=\widetilde{V}$ for all other $G_E\in SS(V)$. Clearly, $(U,\widetilde{c},M),(V,\widetilde{v},E)$ are SC-spaces. Let $f_{up}:(U,\widetilde{c},M)\to(V,\widetilde{v},E)$ be an S-map, where u,p are maps defined as $u(x)=x$, $u(y)=y$ and $p(m_1)=m_1$, $p(m_2)=m_2$. Then f_{up} is Sg-continuous, but it is not $S\partial$-continuous, because for the S-closed set $\{(m_1,\{x\})\}$ in $(V,\widetilde{v},E)$, we have $f_{up}^{-1}(\{(m_1,\{x\})\})=\{(m_1,\{x\})\}$ is not $S\partial$-closed set in $(U,\widetilde{c},M)$. Since $\{(m_1,\{x\})\}$ is an Sg-open set containing itself but $\widetilde{c}(\{(m_1,\{x\})\})=\widetilde{U}\not\sqsubseteq\{(m_1,\{x\})\}$.

Theorem 3.5. *An S-map $f_{up}:(U,\widetilde{c},M)\to(V,\widetilde{v},E)$ is $S\partial$-continuous if and only if $f_{up}^{-1}(H_E)$ is an $S\partial$-open set in $(U,\widetilde{c},M)$ for any S-open set H_E in $(V,\widetilde{v},E)$.*

Proof. It is obvious. □

Proposition 3.6. *For the SC-spaces $(U,\widetilde{c},M)$, $(V,\widetilde{v},E)$, and $(W,\widetilde{u},K)$. If $f_{up}:(U,\widetilde{c},M)\to(V,\widetilde{v},E)$ is $S\partial$-continuous and $f_{vq}:(V,\widetilde{v},E)\to(W,\widetilde{u},K)$ is S-continuous, then $f_{vq}\circ f_{up}:(U,\widetilde{c},M)\to(W,\widetilde{u},K)$ is $S\partial$-continuous.*

Proof. Let H_K be an S-closed set in $(W,\widetilde{u},K)$. From definition of the composition $(f_{vq}^{-1}\circ f_{up}^{-1})(H_K)=f_{up}^{-1}(f_{vq}^{-1}(H_K))$. Since f_{vq} is S-continuous, we have $f_{vq}^{-1}(H_K)$ is S-closed set in $(V,\widetilde{v},E)$. Again f_{up} is $S\partial$-continuous, we get $f_{up}^{-1}(f_{vq}^{-1}(H_K))$ is an $S\partial$-closed set in $(U,\widetilde{c},M)$. Therefore, $f_{vq}\circ f_{up}:(U,\widetilde{c},M)\to(W,\widetilde{u},K)$ is $S\partial$-continuous. □

Definition 3.7. For two SC-spaces $(U,\widetilde{c},M)$, $(V,\widetilde{v},E)$. An S-map $f_{up}:(U,\widetilde{c},M)\to(V,\widetilde{v},E)$ is called $S\partial$-closed if $f_{up}(H_M)$ is an $S\partial$-closed set in $(V,\widetilde{v},E)$ for any S-closed in $(U,\widetilde{c},M)$.

Remark 3.8. Every S-closed map can be categorized as an $S\partial$-closed map. However, the converse does not hold true, as illustrated by the following example.

Example 3.9. Let $U=\{x,y\}$, $V=\{a,b\}$ and $M=\{t_1,t_2\}$, $E=\{e_1,e_2\}$. Define an SC-operator $\widetilde{c}$ on U by: $\widetilde{c}(\widetilde{\emptyset})=\widetilde{\emptyset}$, $\widetilde{c}(\{(m_1,\{y\})\})=\{(m_1,\{y\})\}$ and $\widetilde{c}(F_M)=\widetilde{U}$ for all other $F_M\in SS(U)$. Define an SC-operator $\widetilde{u}$ on V as $\widetilde{v}(\widetilde{\emptyset})=\widetilde{\emptyset}$, and $\widetilde{v}(G_E)=\widetilde{V}$ for all other $G_M\in SS(V)$. Then $(U,\widetilde{c},M),(V,\widetilde{v},E)$ are SC-spaces. Now let $f_{up}:(U,\widetilde{c},M)\to(V,\widetilde{v},E)$ be an S-map, where u, p are maps defined as $u(x)=a$, $u(y)=b$ and $p(m_1)=e_1$, $p(m_2)=e_2$. Clearly, f_{up} is $S\partial$-closed but it is not S-closed. Indeed, for the S-closed set $\{(m_1,\{y\})\}$ in $(U,\widetilde{c},M)$ we have, f_{up} $(\{(m_1,\{y\})\})=\{(e_1,\{b\})\}$ is not S-closed set in $(V,\widetilde{v},E)$.

Theorem 3.10. *An S-map $f_{up}:(U,\widetilde{c},M)\to(V,\widetilde{v},E)$ $S\partial$-closed if and only if for any S-set F_E in $(V,\widetilde{v},E)$ and for any S-open set G_M in $(U,\widetilde{c},M)$ with $f_{vq}^{-1}(F_E)\sqsubseteq G_M$, there is an $S\partial$-open set H_E in $(V,\widetilde{v},E)$ such that $F_E\sqsubseteq H_E$ and $f_{vq}^{-1}(H_E)\sqsubseteq G_M$.*

Proof. Suppose that f_{up} is $S\partial$-closed. Let F_E is an S-set in $(V,\widetilde{v},E)$ and G_M is an S-open set in $(U,\widetilde{c},M)$ with $f_{vq}^{-1}(F_E)\sqsubseteq G_M$, then $f_{up}(G_M^c)$ is an $S\partial$-closed set in $(V,\widetilde{v},E)$. Now take

$H_E = [f_{up}(G^c_M)]^c$ which is an $S\partial$-open set, then $f^{-1}_{up}(H_E) = f^{-1}_{up}([f_{up}(G^c_M)]^c) = [f^{-1}_{up}(f_{up}(G^c_M))]^c \sqsubseteq f^{-1}_{up}(f_{up}(G_M)) = G_M$. The result holds.

Conversely, let F_M be an S-closed in $(U,\tilde{c},M)$, then $f^{-1}_{up}([f_{up}(F_M)]^c) \sqsubseteq F^c_M$ which is S-open. From hypothesis, there is an $S\partial$-open set in H_E in $(V,\tilde{v},E)$ such that $(f_{up}(F_M))^c \sqsubseteq H_E$ and $f^{-1}_{up}(H_E) \sqsubseteq F^c_M$. Thus, $F_M \sqsubseteq (f^{-1}_{up}(H_E))^c$ and so, $H^c_E \sqsubseteq f_{up}(F_M) \sqsubseteq f_{up}((f^{-1}_{up}(H_E))^c) \sqsubseteq H^c_E$, this implies that $f_{up}(F_M) = H^c_E$. This means that $f_{up}(F_M)$ is an $S\partial$-closed set. Hence the proof is complete. □

Based on the previous results, one can to verify the subsequent proposition.

Proposition 3.11. *For the SC-spaces* $(U,\tilde{c},M)$, $(V,\tilde{v},E)$, *and* $(W,\tilde{u},K)$. *Let* $f_{up}:(U,\tilde{c},M) \to (V,\tilde{v},E)$ *and* $f_{vq}:(V,\tilde{v},E) \to (W,\tilde{u},K)$ *be S-maps, we have:*

(i) *If* f_{up} *is* $S\partial$*-closed and* f_{vq} *is* S*-closed, then* $f_{vq}\circ f_{up}$ *is* $S\partial$*-closed.*

(ii) *If* $f_{vq}\circ f_{up}$ *is* $S\partial$*-closed and* f_{up} *is* S*-continuous and onto, then* f_{vq} *is* S*-closed.*

(iii) *If* $f_{vq}\circ f_{up}$ *is* S*-closed and* f_{vq} *is* $S\partial$*-continuous and one-to-one, then* f_{up} *is* $S\partial$*-closed.*

4. Two New Types of Soft Separation Axioms

Here, we apply the notion of $S\partial$-closed sets to introduce two types of separation properties called, soft $\partial T_{\frac{1}{2}}$ and soft $\partial T^*_{\frac{1}{2}}$ spaces and investigate some of their properties.

Definition 4.1. An SC-space $(U,\tilde{c},M)$ is said to be:

(i) Soft $\partial T_{\frac{1}{2}}$ (briefly, $S\partial$-$T_{\frac{1}{2}}$) if any $S\partial$-closed set is S-closed in $(U,\tilde{c},M)$.

(ii) Soft $\partial T^*_{\frac{1}{2}}$ (briefly, S∂-$T^*_{\frac{1}{2}}$) if any Sg-closed set is $S\partial$-closed in $(U,\tilde{c},M)$.

Example 4.2. Consider Example 2.2, the S-closed sets in $(U,\tilde{c},M)$ are:

$$\{\tilde{\emptyset},\tilde{U},F_{1M},F_{3M},F_{4M},F_{6M},F_{7M},F_{11M},F_{13M}\}$$

which are equal to the $S\partial$-closed sets and the Sg-closed sets. So that the SC-space $(U,\tilde{c},M)$ is an $S\partial$-$T_{\frac{1}{2}}$ and $S\partial$-$T^*_{\frac{1}{2}}$ space. because, every Sg-closed set in $(U,\tilde{c},M)$ is $S\partial$-closed and every $S\partial$-closed set is S-closed.

Remark 4.3. Clearly, every $ST_{\frac{1}{2}}$-space $(U,\tilde{c},M)$ is an $S\partial$-$T_{\frac{1}{2}}$ space but not conversely. This fact can be shown by the next example.

Example 4.4. Let $U = \{a,b\}$ and $M = \{m_1,m_2\}$. Define an SC-operator $\tilde{c}$ on U by: $\tilde{c}(\tilde{\emptyset}) = \tilde{\emptyset}$ and $\tilde{c}(F_M) = \tilde{U}$ for all other $F_T \in SS(U)$. Then $(U,\tilde{c},M)$ is an SC-space. one can verify that $(U,\tilde{c},M)$ is an $S\partial$-$T_{\frac{1}{2}}$ space but it is not $ST_{\frac{1}{2}}$ because, $\{(m_1,\{a\})\}$ is an Sg-closed set in $(U,\tilde{c},M)$ but it is not S-closed set.

Remark 4.5. Clearly, every $ST_{\frac{1}{2}}$-space $(U,\tilde{c},M)$ is an $S\partial$-$T^*_{\frac{1}{2}}$ space but not conversely. This fact can be shown by the next example.

Example 4.6. Consider Example 2.2. Define the SC-operator $\tilde{c}: SS(U) \to SS(U)$ by:

$$\tilde{c}(\tilde{\emptyset}) = \tilde{\emptyset}, \quad \tilde{c}(F_{1M}) = \tilde{c}(F_{5M}) = \{(m_1,\{a\}),(m_2,\{b\})\}, \quad \tilde{c}(F_{2M}) = \{(m_1,U)\},$$

$$\tilde{c}(F_{3M}) = \tilde{c}(F_{9M}) = \tilde{c}(F_{13M}) = \{(m_1,U),(m_2,\{a\})\}, \quad \tilde{c}(F_{4M}) = \{(m_2,\{a\})\},$$

$\widetilde{c}(F_{6M}) = \widetilde{c}(F_{8M}) = \widetilde{c}(F_{11M}) = \{(m_1,\{a\}),(m_2,U)\}$,

$\widetilde{c}(F_{7M}) = \{(m_1,\{a\}),(m_2,\{a\})\}$, $\widetilde{c}(F_{10M}) = \{(m_1,U),(m_2,\{b\})\}$,

and

$\widetilde{c}(F_{12M}) = \widetilde{c}(F_{14M}) = \widetilde{c}(\widetilde{U}) = \widetilde{U}$.

Then $(U,\widetilde{c},M)$ is an SC-space. One can verify $(U,\widetilde{c},M)$ is an $S\partial$-$T^*_{\frac{1}{2}}$ space but not $ST_{\frac{1}{2}}$. Indeed, for $F_M = \{(m_2,U)\} \in SS(U)$, we have F_M is an Sg-closed set in $(U,\widetilde{c},M)$ but it is not S-closed set.

Proposition 4.7. *An SC-space $(U,\widetilde{c},M)$ is $ST_{\frac{1}{2}}$ if and only if it is both $S\partial$-$T_{\frac{1}{2}}$ and $S\partial$-$T^*_{\frac{1}{2}}$ space.*

Proof. It is obvious. □

Remark 4.8. The notions of $S\partial$-$T_{\frac{1}{2}}$ and $S\partial$-$T^*_{\frac{1}{2}}$ spaces are independent.

Example 4.9. From Example 4.6, we have $(U,\widetilde{c},M)$ is an $S\partial$-$T^*_{\frac{1}{2}}$ space but it is not $S\partial$-$T_{\frac{1}{2}}$. Clearly, the S-set $F_M = \{(m_2,U)\}$ is an $S\partial$-closed set in $(U,\widetilde{c},M)$ but it is not S-closed.

Example 4.10. From Example 4.4, one can verify that $(U,\widetilde{c},M)$ is an $S\partial$-$T_{\frac{1}{2}}$ space but it is not $S\partial$-$T^*_{\frac{1}{2}}$. Indeed, for the S-set $F_M = \{(m_1,\{b\})\}$, we have F_M is an Sg-closed set in $(U,\widetilde{c},M)$ but it is not $S\partial$-closed.

Proposition 4.11. *For an SC-space $(U,\widetilde{c},M)$. If $(U,\widetilde{c},M)$ is $S\partial$-$T_{\frac{1}{2}}$, then the closed SC-subspace $(Y,\widetilde{c}_Y,M)$ of $(U,\widetilde{c},M)$ is $S\partial$-$T_{\frac{1}{2}}$.*

Proof. It follows from Proposition 2.13 and Definition 1.4. □

Proposition 4.12. *For two SC-spaces $(U,\widetilde{c},M)$ and $(V,\widetilde{v},E)$ such that $(U,\widetilde{c},M)$ is an $S\partial$-$T^*_{\frac{1}{2}}$ space. If $f_{up}:(U,\widetilde{c},M)\to(V,\widetilde{v},E)$ is Sg-continuous, then f_{up} is $S\partial$-continuous.*

Proof. Let H_M be an S-closed set in $(V,\widetilde{v},E)$. Clearly, f_{up} is Sg-continuous, we have $f_{up}^{-1}(H_M)$ is an Sg-closed set in $(U,\widetilde{c},M)$ which is an $S\partial$-$T^*_{\frac{1}{2}}$ space, Thus $f_{up}^{-1}(H_M)$ is an $S\partial$-closed set in $(U,\widetilde{c},M)$. Therefore, f_{up} is $S\partial$-continuous. □

Proposition 4.13. *For the SC-spaces $(U,\widetilde{c},M)$, $(V,\widetilde{v},E)$, and $(W,\widetilde{u},K)$ such that $(V,\widetilde{v},E)$ is $ST_{\frac{1}{2}}$. Then if $f_{up}:(U,\widetilde{c},M)\to(V,\widetilde{v},E)$ is Sg-continuous and $f_{vq}:(V,\widetilde{v},E)\to(W,\widetilde{u},K)$ is $S\partial$-continuous, then $f_{vq}\circ f_{up}:(U,\widetilde{c},M)\to(W,\widetilde{u},K)$ is $S\partial$-continuous.*

Proof. Let H_M be an S-closed set in $(W,\widetilde{u},K)$. Since f_{vq} is Sg-continuous, we have $f_{vq}^{-1}(H_M)$ is an Sg-closed set in $(V,\widetilde{v},E)$. By hypothesis $(V,\widetilde{v},E)$ is $ST_{\frac{1}{2}}$, we have $f_{vq}^{-1}(H_M)$ is an S-closed set in $(V,\widetilde{v},E)$. Since f_{up} is Sg-continuous, then $(f_{up}^{-1}(f_{vq}^{-1}(H_M)) = (f_{vq}\circ f_{up})^{-1}(H_M)$ is an $S\partial$-closed set in $(U,\widetilde{c},M)$. This completes the proof. □

Proposition 4.14. *For the SC-spaces $(U,\widetilde{c},M)$, $(V,\widetilde{v},E)$, and $(W,\widetilde{u},K)$ such that $(V,\widetilde{v},E)$ is $S\partial$-$ST_{\frac{1}{2}}$, then if $f_{up}:(U,\widetilde{c},M)\to(V,\widetilde{v},E)$ and $f_{vq}:(V,\widetilde{v},E)\to(W,\widetilde{u},K)$ are $S\partial$-continuous, then $f_{vq}\circ f_{up}:(U,\widetilde{c},M)\to(W,\widetilde{u},K)$ is also $S\partial$-continuous.*

Proof. By similar way of that in the above proposition. □

5. Conclusion

In this paper, we defined and studied the concept of soft $S\partial$-closed sets in soft closure spaces, which are lying between S-closed sets and Sg-closed sets. We investigated the basic properties for them and examined their implications in the context of soft closure spaces. Further, we introduced the concepts of $S\partial$-continuous and $S\partial$-closed maps and present their various properties with the help of supported examples. Moreover, we defined and studied two separation properties, namely $S\partial$-$T_{\frac{1}{2}}$ and $S\partial$-$T^*_{\frac{1}{2}}$, which utilize the notion of $S\partial$-closed sets. Our study provides a deeper understanding of the topological properties of soft closure spaces and contributes significantly to the ongoing exploration of this field of research.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] H. Aktaş and N. Çağman, Soft sets and soft groups, *Information Sciences* **177**(13) (2007), 2726 – 2735, DOI: 10.1016/j.ins.2006.12.008.

[2] J. Al-Mufarrij and S. Saleh, New results on soft generalized topological spaces, *Journal of Mathematics and Computer Science* **32**(1) (2024), 43 – 53, DOI: 10.22436/jmcs.032.01.04.

[3] T. M. Al-Shami, New soft structure: Infra soft topological spaces, *Mathematical Problems in Engineering* **2021** (2021), ID 3361604, DOI: 10.1155/2021/3361604.

[4] C. Boonpok and J. Khampakdee, Between closed sets and generalized closed sets in closure spaces, *Acta Mathematica Universitatis Ostraviensis* **16**(1) (2008), 3 – 14, http://dml.cz/dmlcz/137494.

[5] N. Çağman and S. Enginoğlu, Soft set theory and *uni-int* decision making, *European Journal of Operational Research* **207**(2) (2010), 848 – 855, DOI: 10.1016/j.ejor.2010.05.004.

[6] E. Čech, M. Katětov, J. Novák and A. Alois, *Topological Papers of Eduard Čech*, Academia, Prague (1968), 436 – 472.

[7] S. Das and S. K. Samanta, Soft metric, *Annals of Fuzzy Mathematics and Informatics* **6**(1) (2013), 77 – 94, URL: http://www.afmi.or.kr/papers/2013/Vol-06_No-01/AFMI-6-1(1--226)/AFMI-6-1(77--94)-J-120715R1.pdf.

[8] S. T. Ekram and R. N. Majeed, Soft closure spaces, *Journal of Physics: Conference Series* **1591** (2020), 012076, DOI: 10.1088/1742-6596/1591/1/012076.

[9] S. T. Ekram and R. N. Majeed, Soft continuous mappings in soft closure spaces, *Iraqi Journal of Science* **62** (2021), 2676 – 2684, DOI: 10.24996/ijs.2021.62.8.20.

[10] D. N. Georgiou and A. C. Mergaritis, Soft set theory and topology, *Applied General Topology* **15**(1) (2014), 93 – 109, DOI: 10.4995/agt.2014.2268.

[11] R. Gowri and G. Jegadeesan, On soft Čech closure spaces, *International Journal of Mathematical Trends and Technology* **9** (2014), 122 – 127, DOI: 10.14445/22315373/IJMTT-V9P513.

[12] R. Gowri and G. Jegadeesan, Soft ∂-closed sets in soft Čech closure space, *Advances in Theoretical and Applied Mathematics* **11**(2) (2016), 105 – 114.

[13] K. Kannan, Soft generalized closed sets in soft topological spaces, *Journal of Theoretical and Applied Information Technology* **37**(1) (2012), 17 – 21, http://www.jatit.org/volumes/Vol37No1/2Vol37No1.pdf.

[14] A. Kharal and B. Ahmad, Mapping on soft classes, *New Mathematics and Natural Computation* **7**(3) (2011), 471 – 481, DOI: 10.1142/S1793005711002025.

[15] N. Levine, Generalized closed sets in topology, *Rendiconti del Circolo Matematico di Palermo* **19** (1970), 89 – 96, DOI: 10.1007/BF02843888.

[16] R. N. Majeed and S. T. Ekram, g-Closed soft sets in soft closure spaces, *Journal of Physics: Conference Series* **1879** (2021), 022128, DOI: 10.1088/1742-6596/1879/2/022128.

[17] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, *Computers & Mathematics with Applications* **45**(4-5) (2003), 555 – 562, DOI: 10.1016/S0898-1221(03)00016-6.

[18] D. Molodtsov, Soft set theory – First results, *Computers & Mathematics with Applications* **37**(4-5) (1999), 19 – 31, DOI: 10.1016/S0898-1221(99)00056-5.

[19] S. Saleh, Some new properties on soft topological spaces, *Annals of Fuzzy Mathematics and Informatics* **17**(3) (2019), 303 – 312, DOI: 10.30948/afmi.2019.17.3.303.

[20] M. Shabir and M. Naz, On soft topological spaces, *Computers & Mathematics with Applications* **61**(7) (2011), 1786 – 1799, DOI: 10.1016/j.camwa.2011.02.006.