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On a Ahlfors-Denjoy Type Result

Research Article

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Abstract. The present paper is concerned with the general problem of extending the classical theory of analytic functions of a Complex variable. The asymptotic behavior of power series defined on a Banach algebra with multiplicative functional or with a Gelfand theory is analyzed here and some lower estimates for the order of power series defined on this Banach algebras are given here.

Keywords. Ahlfors-Denjoy's Theorem; Asymptotic elements; Power series; Banach algebras; Power series' order

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1. Introduction

Let $\mathbb{B}$ be an Banach algebra with a unit element. A power series defined on $\mathbb{B}$ is a function $F : \mathbb{B} \to \mathbb{B}$ such that

$$F(w) = \sum_{n=0}^{\infty} a_n w^n, \quad w \in \mathbb{B}, \tag{1.1}$$

with $a_n \in \mathbb{B}$ and $\lim\limits_{n\to\infty} \|a_n\|^{\frac{1}{n}} = 0$.

Let $M(r,F)$ denote the supreme of its norm on the closed ball with center in 0 and radio $r > 0$, that is

$$M(r,F) = \sup_{\|w\|\le r} \|F(w)\|.$$

The concept of order for entire functions of complex variable [5], can be carried without changes to the case of a power series defined on a Banach algebra [2]. Therefore, the power series F is of finite order, if exist $\mu > 0$ such that

$$M(r,F) \underset{\text{as}}{\leq} e^{r^{\mu}}.$$

The infimum of μ is called *order* of F and it will be denoted by $\rho(F)$. So for $\epsilon > 0$, there is a $R > 0$ which satisfies for $r > R$

$$M(r,F) \underset{\text{as}}{\leq} e^{r^{\rho(F)+\epsilon}}.$$

Furthermore, there is a sequence $\{r_n\}$ and $r_n \to \infty$ such that

$$e^{r_n^{\rho(F)-\epsilon}} \underset{\text{n}}{\leq} M(r_n,F).$$

Thus,

$$\rho(F) - \epsilon \underset{\text{n}}{\leq} \frac{\ln\ln M(r,F)}{\ln r} \underset{\text{as}}{\leq} \rho(F) + \epsilon,$$

hence

$$\rho(F) = \limsup_{r\to\infty} \frac{\ln\ln M(r,F)}{\ln r}\ . \tag{1.2}$$

2. Asymptotic Elements of a Power Series

In 1976 with the purpose of studying the solution of transcendental equations on commutative Banach algebras, E.A. Gorin and C. Sanchez introduced in [4] the concept of asymptotic curve for L-entire functions in the sense of Lorch [6]. However, this concept does not itself constitute an extension of the classic concept of asymptotic value of an entire function of a complex variable, as it was shown by A. Bezanilla [1]. The concept of asymptotic element for a L-entire function on a commutative Banach algebra with unit element who was introduced by A. Bezanilla [1], can be carried to the case of power series defined on a Banach algebra not necessarily commutative.

Definition 2.1. Let $\mathbb{B}$ be a Banach algebra with unit element and let F be a power series defined on $\mathbb{B}$. Then $\alpha \in \mathbb{B}$ is an asymptotic element of F, if a continuous application $\gamma : [0,1[\to \mathbb{B}$ exists and

(i) $\gamma(t) \in \mathbb{B}^{-1}$ for all $t \in [0,1[$

(ii) $\gamma(t)^{-1} \to 0$ if $t \to 1$

(iii) $\lim_{t\to 1} \|F(\gamma(t)) - \alpha\| = 0$

3. Banach Algebra with Multiplicative Functional

In this section, the asymptotic behavior of the power series defined on Banach algebras with identity and with multiplicative functional is studied. C.K. Fong and A. Sołtysiak proved a result which guarantees the existence of a non-zero functional multiplicative on a non-commutative Banach algebra [7].

Let $\mathbb{B}$ be Banach algebra with identity and a non-zero multiplicative functionals. Let $M_{\mathbb{B}}$ denote the set of all multiplicative functional of $\mathbb{B}$. It is known that for all $\varphi \in M_{\mathbb{B}}$, $\varphi(1_{\mathbb{B}}) = 1$ and $\|\varphi\| = 1$.

Let F be a power series on $\mathbb{B}$ as in (1.1). For each $\varphi \in M_{\mathbb{B}}$, we define the entire function of a complex variable

$$f_{\varphi}(z) = \widehat{F(z1_{\mathbb{B}})}(\varphi) = \sum_{n=0}^{\infty} \varphi(a_n) z^n, \quad z \in \mathbb{C}, \tag{3.1}$$

where $\widehat{F(z1_{\mathbb{B}})}$ denotes the Gelfand's transform of $F(z1_{\mathbb{B}})$. Let $\mathbb{B}_F$ denote the entire functions associated with F as in (3.1), that is,

$$\mathbb{B}_F = \left\{ f_{\varphi} : \varphi \in M_{\mathbb{B}} \right\}.$$

Proposition 3.1. *Let $\mathbb{B}$ be a Banach algebra with identity and non-zero multiplicative functionals. Let F be a power series defined on $\mathbb{B}$ with finite order $\rho(F)$. If $\mathbb{B}_F$ is the set of entire functions associated with F, then*

$$\sup_{f_{\varphi} \in \mathbb{B}_F} \rho(f_{\varphi}) \le \rho(F).$$

Proof. Let $f_{\varphi} \in \mathbb{B}_F$ and $w \in \mathbb{B}$, then

$$\begin{aligned} f_{\varphi}(\varphi(w)) &= \widehat{F(w)}(\varphi) \\ &= \sum_{n=0}^{\infty} \varphi(a_n) [\varphi(w)]^n \\ &= \varphi\left(\sum_{n=0}^{\infty} a_n w^n\right) \\ &= \left(\varphi \circ F\right)(w). \end{aligned} \tag{3.2}$$

Then, for all $w \in \mathbb{B}$

$$\left|\left(\varphi \circ F\right)(w)\right| \le \|\varphi\| \|F(w)\|.$$

Therefore,

$$M(r, f_{\varphi}) \le M(r, F),$$

and applying the expression (1.2) we obtain the result. □

4. Banach Algebra with a Gelfand Theory

For an arbitrary, not necessarily commutative Banach algebra $\mathbb{B}$, let $\Lambda_{\mathbb{B}}$ denote the set of maximal modular left ideals of $\mathbb{B}$. The pair $(\vartheta, \mathbb{U})$ is a Gelfand theory for $\mathbb{B}$, if $\mathbb{U}$ is a C^*–algebra; $\vartheta : \mathbb{B} \to \mathbb{U}$ is a homomorphism; there exists a bijection between $\Lambda_{\mathbb{B}}$ and $\Lambda_{\mathbb{U}}$; and for each $L \in \Lambda_{\mathbb{U}}$ the application $\vartheta_L : \mathbb{B} / \vartheta^{-1}(L) \to \mathbb{U} / L$, induced by ϑ has dense range (see [3]).

If a Banach algebra $\mathbb{B}$ has multiplicative functionals, we can find lowers estimates for the order of a power series defined on $\mathbb{B}$ using the entire functions of a complex variable associated with them, as those given for the L-entire functions in Proposition 6.1 and Proposition 6.3 of [1]. Here, we will use similar ideas to find lower estimates for the order of a power series defined on a Banach algebra $\mathbb{B}$ with a Gelfand theory $(\vartheta,\mathbb{U})$.

Let $(\vartheta,\mathbb{U})$ be a Gelfand theory for the Banach algebra $\mathbb{B}$ and let

$$F(w)=\sum_{n=0}^{\infty} a_n w^n, \quad w\in\mathbb{B}.$$

Using the homomorphism ϑ, we can associate to the power series F a function defined on $\mathbb{U}$, namely,

$$F_\vartheta(u)=\sum_{n=0}^{\infty}\vartheta(a_n)u^n, \quad u\in\mathbb{U}. \tag{4.1}$$

Since $\|\vartheta(a_n)\|\le\|a_n\|$, then F_ϑ is a power series on $\mathbb{U}$.

Therefore, each power series F defined on $\mathbb{B}$, is associated with a family of power series defined on $\mathbb{U}$, namely, $\{F_\vartheta : (\vartheta,\mathbb{U})$ is a Gelfand theory for $\mathbb{B}\}$.

We will denote by G the set of Gelfand theories for $\mathbb{B}$ and by $\widehat{G}$ the subset of G,

$$\widehat{G}=\{(\vartheta,\mathbb{U})\in G : \vartheta \text{ is onto}\}.$$

If $(\vartheta,\mathbb{U})\in G$ and F_ϑ is a power series associated by (4.1) to the power series F, then

$$\begin{aligned} F_\vartheta(\vartheta(w)) &= \sum_{n=0}^{\infty}\vartheta(a_n)\vartheta(w)^n \\ &= \sum_{n=0}^{\infty}\vartheta(a_n w^n) \\ &= \vartheta\left(\sum_{n=0}^{\infty} a_n w^n\right) \\ &= \vartheta\circ F(w) \end{aligned} \tag{4.2}$$

so,

$$\|F_\vartheta(\vartheta(w))\|\le\|F(w)\|.$$

Therefore, if $(\vartheta,\mathbb{U})\in\widehat{G}$,

$$M(r,F_\vartheta)\le M(r,F)$$

and

$$\rho(F_\vartheta)\le\rho(F).$$

From all this and the expression (1.2), the following proposition is obtained.

Theorem 4.1. *Let $\mathbb{B}$ be a Banach algebra such that $\widehat{G}\neq\emptyset$ and let F be a power series on $\mathbb{B}$ of order $\rho(F)<\infty$. If $\{F_\vartheta : (\vartheta,\mathbb{U})\in G\}$ is the power series family associated with F, then*

$$\sup_{(\vartheta,\mathbb{U})\in\widehat{G}}\rho(F_\vartheta)\le\rho(F).$$

5. A Theorem Type Ahlfors-Denjoy

Theorem 5.1. *Let $\mathbb{B}$ be a Banach algebra with a Gelfand theory $(\vartheta,\mathbb{U})$ and let F be a power series on $\mathbb{B}$ with $\rho(F) > \infty$. If $\alpha \in \mathbb{B}$ is an asymptotic element of F, then $\vartheta(\alpha)$ is an asymptotic element of the power series F_ϑ associated with F.*

Proof. Since ϑ is a homomorphism, if $w \in \mathbb{B}^{-1}$ then $\vartheta(w) \in \mathbb{U}^{-1}$, in addition, if $\gamma : [0,1[\to \mathbb{B}^{-1}$ is the asymptotic path on which F tends to the asymptotic element α,

$$\begin{aligned}\left\|F_\vartheta(\vartheta(\gamma(t))) - \vartheta(\alpha)\right\| &\le \left\|\vartheta\left(F(\gamma(t)) - \alpha\right)\right\| \\ &\le \left\|F(\gamma(t)) - \alpha\right\|.\end{aligned}$$

So,

$$\lim_{t\to 1}\left\|F_\vartheta(\zeta(\gamma(t))) - \vartheta(\alpha)\right\| = 0,$$

that is, $\vartheta(\alpha)$ is an asymptotic element of the power series F_ϑ, which is reached on the asymptotic path $\zeta \circ \gamma : [0,1[\to \mathbb{U}^{-1}$. □

When we impose new restrictions to the Gelfand theory, we can find a result of type Ahlfors-Denjoy. For a Banach algebra $\mathbb{B}$ with a Gelfand theory $(\vartheta,\mathbb{U})$, let $\mathbb{U}_H$ denote the set of modular ideals of $\mathbb{U}$ that they are hyperplanes and let $\widehat{G_H} = \{(\vartheta,\mathbb{U}) \in \widehat{G} : \mathbb{U}_H \neq \emptyset\}$. If $\mathbb{U}_H \neq \emptyset$ then the maximal ideals space $M_\mathbb{U}$ of $\mathbb{U}$ is not empty (see [8]).

Proposition 5.2. *Let $\mathbb{B}$ be a Banach algebra so that $\widehat{G_H} \neq \emptyset$. Let F be a power series with finite order defined on $\mathbb{B}$. Assume that $\{\alpha_j : j \in J\}$ is the collection of all the asymptotic elements different of F, then*

$$\rho(F) \ge \sup_{(\vartheta,\mathbb{U})\in\widehat{G_H}}\left\{\frac{1}{2}\sup_{\varphi\in M_\mathbb{U}}\operatorname{card}\left\{\varphi\circ\vartheta\left(\alpha_j\right) : j \in J\right\}\right\}.$$

Proof. By Proposition 2, $\left\{\vartheta\left(\alpha_j\right) : j \in J\right\}$ is a collection of all the asymptotic elements of the power series F_ϑ associated to F. Since $M_\mathbb{U} \neq \emptyset$ for everything $(\vartheta,\mathbb{U}) \in \widehat{G_H}$, then

$$\rho(F_\vartheta) \ge \frac{1}{2}\sup_{\varphi\in M_\mathbb{U}}\operatorname{card}\left\{\varphi\circ\vartheta\left(\alpha_j\right) : j \in J\right\}$$

(Proposition 7.3.1 of [1]). On the other hand, since $\widehat{G_H} \subseteq \widehat{G}$, ϑ is an onto homomorphism, so

$$\rho(F) \ge \sup_{(\vartheta,\mathbb{U})\in\widehat{G_H}}\rho(F_\vartheta),$$

Therefore,

$$\rho(F) \ge \sup_{(\vartheta,\mathbb{U})\in\widehat{G_H}}\left\{\frac{1}{2}\sup_{\varphi\in M_\mathbb{U}}\operatorname{card}\left\{\varphi\circ\vartheta\left(\alpha_j\right) : j \in J\right\}\right\}.$$ □

6. Conclusion

There is another important aspect of the asymptotic behavior of analytic functions of a complex variable that can be extended to power series defined on a Banach algebra, namely the extension of the Phragmen-Lindelöf's function (see [5]). Our main goal now is to study the possible practical applications that such extensions may have.

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