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Research Article

Lacunary Statistical Convergence Sequence in Neutrosophic Metric Space

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Abstract. Researchers describe the theory of *Lacunary Strongly Convergence* (*LSC*) and its application to sequences of sets in *Neutrosophic Metric Spaces* (*NMS*). We derive a conceptual connection between these ideas. In addition, we have defined certain required and adequate criteria to ensure the similarity for *Statistical Convergence* (*StC*) and *Lacunary Statistical Convergence* (*LStC*) sets for the sequence of *NMS*. We develop certain findings along with the idea of enhanced Cesaro summability in *NMS*.

Keywords. Lacunary sequence, Neutrosophic normed linear space, Wijsman convergence, Cesaro summability, Sequence of sets

Mathematics Subject Classification (2020). 11B39, 41A10, 41A25, 41A36, 40A30, 40G15

1. Introduction

Fuzzy sets were initially described by Zadeh [18]. Various article's publishing has far-reaching consequences throughout scientific disciplines. The concept has real-world relevance, yet it does not offer satisfactory answers for various issues. These difficulties inspire creative investigations. Atanassov [1] looked studied intuitionistic fuzzy sets and found that they work well in this kind of scenario. Park [13] established the concept of *NMS*. Saadati and Park [14] also provide a definition for Neutrosophic normed space *NNS*. Following the research of [5, 8, 9, 11], the field of (*NNS*) has advanced significantly. The concept of *StC* for a series of real numbers was initially introduced by Fast [2]. Researchers have since began to work on distance spaces built over *NMS*.

Jeyaraman *et al.* [10], and Sowndararajan *et al.* [15] proposed the Neutrosophic Metric Spaces concept and outlined several fixed-point solutions.

The term "lacunary sequence" refers to an ascending integer sequences where $\psi = \{\iota_\mathfrak{r}\}$ with $\iota_0 = 0$ and $\mathfrak{v}_\mathfrak{r} = \iota_\mathfrak{r} - \iota_{\mathfrak{r}-1} \to \infty$ as $\mathfrak{r} \to \infty$ and ratio $\frac{\iota_\mathfrak{r}}{\iota_{\mathfrak{r}-1}}$ are both represented simply $\rho_\mathfrak{r}$. Across this work the periods specified by ψ shall become indicated by $\mathfrak{I}_\mathfrak{r} = (\iota_{\mathfrak{r}-1}, \iota_\mathfrak{r}]$. Fridy and Orhan [3] developed the idea of *LStC* via *Lacunary sequence* (*Ls*). Major article's publishing had a significant impact across all disciplines of science. There are several *LStC* [3,6].

Nuray and Rhoades [12] investigated Wijsman Statistical Convergence (*WStC*), while Ulusu and Nuray [16] presented Wijsman Lacunary Statistical Convergence (*WLStC*) of sequences of collections also evaluated its link to *WStC* [17].

To begin, let's review a few important terms from earlier in this section.

Following descriptions provided by George and Veeramani [5], *NMS* were introduced by Park [13] as a logical generalisation of fuzzy metric space.

Let Ω be a non-empty set, $\odot$ is a continuous φ-norm, $\oplus$ is a continuous φ-conorm and ς, ϱ, ν are fuzzy sets on $\Omega^2 \times (0,\infty)$. Then, the six-tuple $(\Omega, \varsigma, \varrho, \nu, \odot, \oplus)$ is commonly referred by the term *Neutrosophic Metric Space* (*NMS*) ensuring the following requirements for each $\mathfrak{x}, \mathfrak{y}, \xi \in \Omega$ and for every $s, \varphi > 0$:

(i) $\varsigma(\lambda,\varpi,\varphi) + \varrho(\lambda,\varpi,\varphi) + \nu(\lambda,\varpi,\varphi) \le 3$,

(ii) $\varsigma(\lambda,\varpi,\varphi) > 0$,

(iii) $\varsigma(\lambda,\varpi,\varphi) = 1$ if and only if $\lambda = \varpi$,

(iv) $\varsigma(\lambda,\varpi,\varphi) = \varsigma(\mathfrak{y},\mathfrak{x},\varphi)$,

(v) $\varsigma(\lambda,\varpi,\varphi) \odot \varsigma(\mathfrak{y},\xi,s) \le \varsigma(\lambda,\xi,\varphi+s)$,

(vi) $\varsigma(\lambda,\varpi,\cdot) : (0,\infty) \to [0,1]$ is continuous;

(vii) $\varrho(\lambda,\varpi,\varphi) > 0$,

(viii) $\varrho(\lambda,\varpi,\varphi) = 0$ if and only if $\lambda = \varpi$,

(ix) $\varrho(\lambda,\varpi,\varphi) = \varrho(\mathfrak{y},\mathfrak{x},\varphi)$,

(x) $\varrho(\lambda,\varpi,\varphi) \oplus \varrho(\mathfrak{y},\xi,s) \ge \varrho(\lambda,\xi,\varphi+s)$,

(xi) $\varrho(\lambda,y,\cdot) : (0,\infty) \to [0,1]$ is continuous,

(xii) $\nu(\lambda,\varpi,\varphi) > 0$,

(xiii) $\nu(\lambda,\varpi,\varphi) = 0$ if and only if $\lambda = \varpi$,

(xiv) $\nu(\lambda,\varpi,\varphi) = \nu(\mathfrak{y},\mathfrak{x},\varphi)$

(xv) $\nu(\lambda,\varpi,\varphi) \oplus \nu(\mathfrak{y},\xi,s) \ge \nu(\lambda,\xi,\varphi+s)$,

(xvi) $\nu(\lambda,y,\cdot) : (0,\infty) \to [0,1]$ is continuous.

For an instance (ς,ϱ,ν) is called *Neutrosophic Metric* (*NM*) on Ω.

The current paper's objectives are to put forward the ideas of *LStC* and lacunary strongly convergence for sequences of sets in *NMS* and to derive some significant conclusions about these ideas. Additionally, we developed new descriptions of the lacunary statistically Cauchy (*LStCa*) sequences of sets in *NMS* concept.

Furthermore, various included ties among SC as well as *LStC* were also constructed for sequence of sets under *NMS*.

2. Main Results

Definition 2.1. Consider the *NMS* $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ and ψ the *Ls*. A sequence $\{\mathfrak{V}_\iota\}$ of nonempty closed ubsets of Ω is called as *WLStC* to $\mathfrak{V}$ with regards to *NM* (ς,ϱ,ν), if for every $\varepsilon\in(0,1)$, for each $\lambda\in\Omega$ and for all $\mathfrak{u}>0$,

$$\Delta_\psi\left(\left\{\begin{array}{l}\iota\in\mathbb{N}:|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|\le 1-\varepsilon\\ \text{or } |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|\ge\varepsilon\\ \text{and } |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|\ge\varepsilon\end{array}\right\}\right)=0, \tag{2.1}$$

or equivalently

$$\Delta_\psi\left(\left\{\begin{array}{l}\iota\in\mathbb{N}:|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|> 1-\varepsilon,\\ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon\\ \text{and } |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon\end{array}\right\}\right)=1. \tag{2.2}$$

We write $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim\mathfrak{V}_\iota=\mathfrak{V}$ or $\mathfrak{V}_\iota\xrightarrow{(\varsigma,\varrho,\nu)}\mathfrak{V}(\mathfrak{W}\mathfrak{T}_\psi)$. The collection of every single set sequences that converge to the Wijsman $\mathfrak{T}_\psi$ with respect to the $NM(\varsigma,\varrho,\nu)$ is denoted as $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}$.

Lemma 2.2. *Suppose* $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ *is an* NMS *and* ψ *is a* Ls. *For each* $\varepsilon>0$ *and* $\mathfrak{u}>0$, *similarity between the following claims:*

(a) $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim\mathfrak{V}_\iota=\mathfrak{V}$;

(b) $\Delta_\psi(\{\iota\in\mathbb{N}:|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|\le 1-\varepsilon\})=0$,
$\Delta_\psi(\{\iota\in\mathbb{N}:|\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|\ge\varepsilon\})=0$,
$\Delta_\psi(\{\iota\in\mathbb{N}:|\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|\ge\varepsilon\})=0$;

(c) $\Delta_\psi\left(\left\{\begin{array}{l}\iota\in\mathbb{N}:|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|> 1-\varepsilon,\\ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon\\ \textit{and } |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon\end{array}\right\}\right)=1$;

(d) $\Delta_\psi(\{\iota\in\mathbb{N}:|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|> 1-\varepsilon\})=1$,
$\Delta_\psi(\{\iota\in\mathbb{N}:|\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon\})=1$,
$\Delta_\psi(\{\iota\in\mathbb{N}:|\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon\})=1$;

(e) $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|=1$,
$\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim|\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|=0$, *and*
$\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim|\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|=0$.

Theorem 2.3. *If* $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim\mathfrak{V}_\iota=\mathfrak{V}$, *then* $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}$*-limit is distinct.*

Proof. Let's pretend Ω has two different sets, $\mathfrak{V}_1$ and $\mathfrak{V}_2$ so that

$$\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim\mathfrak{V}_\iota=\mathfrak{V}_1 \ \text{ and } \ \mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim\mathfrak{V}_\iota=\mathfrak{V}_2.$$

Given $\varepsilon > 0$, take $\tau > 0$ such that $(1-\tau)\odot(1-\tau) > 1-\varepsilon$ and $\tau \oplus \tau < \varepsilon$.
Consequently, for every $\mathfrak{u} > 0$, take into account the sets below as:

$$\mathfrak{J}_{\varsigma,1}(\tau,\mathfrak{u}) = \left\{\iota \in \mathbb{N} : \left|\varsigma\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \varsigma\left(\lambda, \mathfrak{V}_1, \frac{\mathfrak{u}}{2}\right)\right| \le 1-\tau\right\},$$
$$\mathfrak{J}_{\varsigma,2}(\tau,\mathfrak{u}) = \left\{\iota \in \mathbb{N} : \left|\varsigma\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \varsigma\left(\lambda, \mathfrak{V}_2, \frac{\mathfrak{u}}{2}\right)\right| \le 1-\tau\right\},$$
$$\mathfrak{J}_{\varrho,1}(\tau,\mathfrak{u}) = \left\{\iota \in \mathbb{N} : \left|\varrho\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \varrho\left(\lambda, \mathfrak{V}_1, \frac{\mathfrak{u}}{2}\right)\right| \ge \tau\right\},$$
$$\mathfrak{J}_{\varrho,2}(\tau,\mathfrak{u}) = \left\{\iota \in \mathbb{N} : \left|\varrho\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \varrho\left(\lambda, \mathfrak{V}_2, \frac{\mathfrak{u}}{2}\right)\right| \ge \tau\right\},$$
$$\mathfrak{J}_{\nu,1}(\tau,\mathfrak{u}) = \left\{\iota \in \mathbb{N} : \left|\nu\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \nu\left(\lambda, \mathfrak{V}_1, \frac{\mathfrak{u}}{2}\right)\right| \ge \tau\right\},$$
$$\mathfrak{J}_{\nu,2}(\tau,\mathfrak{u}) = \left\{\iota \in \mathbb{N} : \left|\nu\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \nu\left(\lambda, \mathfrak{V}_2, \frac{\mathfrak{u}}{2}\right)\right| \ge \tau\right\}.$$

Since $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)} - \lim \mathfrak{V}_\iota = \mathfrak{V}_1$, by Lemma 2.2 we have

$$\Delta_\psi(\mathfrak{J}_{\varsigma,1}(\tau,\mathfrak{u})) = \Delta_\psi(\mathfrak{J}_{\varrho,1}(\tau,\mathfrak{u})) = \Delta_\psi(\mathfrak{J}_{\nu,1}(\tau,\mathfrak{u})) = 0, \quad \text{for all } \mathfrak{u} > 0.$$

Furthermore, utilizing $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)} - \lim \mathfrak{V}_\iota = \mathfrak{V}_2$, we get

$$\Delta_\psi(\mathfrak{J}_{\varsigma,2}(\tau,\mathfrak{u})) = \Delta_\psi(\mathfrak{J}_{\varrho,2}(\tau,\mathfrak{u})) = \Delta_\psi(\mathfrak{J}_{\nu,2}(\tau,\mathfrak{u})) = 0, \quad \text{for all } \mathfrak{u} > 0.$$

Now, let

$$\mathfrak{J}_{\varsigma,\varrho,\nu}(\tau,\mathfrak{u}) = (\mathfrak{J}_{\varsigma,1}(\tau,\mathfrak{u}) \cup \mathfrak{J}_{\varsigma,2}(\tau,\mathfrak{u})) \cap (\mathfrak{J}_{\varrho,1}(\tau,\mathfrak{u}) \cup \mathfrak{J}_{\varrho,2}(\tau,\mathfrak{u})) \cap (\mathfrak{J}_{\nu,1}(\tau,\mathfrak{u}) \cup \mathfrak{J}_{\nu,2}(\tau,\mathfrak{u})).$$

Then, observe that $\Delta_\psi(\mathfrak{J}_{\varsigma,\varrho,\nu}(\tau,\mathfrak{u})) = 0$ which gives $\Delta_\psi(\mathbb{N}\backslash\mathfrak{J}_{\varsigma,\varrho,\nu}(\tau,\mathfrak{u})) = 1$.
If $\iota \in \mathbb{N}\backslash\mathfrak{J}_{\varsigma,\varrho,\nu}(\tau,\mathfrak{u})$, there are so three separate outcomes.

Case (i): $\iota \in \mathbb{N}\backslash(\mathfrak{J}_{\varsigma,1}(\tau,\mathfrak{u}) \cup \mathfrak{J}_{\varsigma,2}(\tau,\mathfrak{u}))$.

Case (ii): $\iota \in \mathbb{N}\backslash(\mathfrak{J}_{\varrho,1}(\tau,\mathfrak{u}) \cup \mathfrak{J}_{\varrho,2}(\tau,\mathfrak{u}))$.

Case (iii): $\iota \in \mathbb{N}\backslash(\mathfrak{J}_{\nu,1}(\tau,\mathfrak{u}) \cup \mathfrak{J}_{\nu,2}(\tau,\mathfrak{u}))$.

We first think that $\iota \in \mathbb{N}\backslash(\mathfrak{J}_{\varsigma,1}(\tau,\mathfrak{u}) \cup \mathfrak{J}_{\varsigma,2}(\tau,\mathfrak{u}))$. Then, we have

$$\begin{aligned}\varsigma(\mathfrak{V}_1 - \mathfrak{V}_2, \mathfrak{u}) &\ge \left|\varsigma\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \varsigma\left(\lambda, \mathfrak{V}_1, \frac{\mathfrak{u}}{2}\right)\right| \\ &\odot \left|\varsigma\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \varsigma\left(\lambda, \mathfrak{V}_2, \frac{\mathfrak{u}}{2}\right)\right| \\ &> (1-\tau)\odot(1-\tau).\end{aligned}$$

Since $(1-\tau)\odot(1-\tau) > 1-\varepsilon$, it follows that $\varsigma(\mathfrak{V}_1 - \mathfrak{V}_2, \mathfrak{u}) > 1-\varepsilon$.
Since $\varepsilon > 0$ is arbitrary, we obtain $\varsigma(\mathfrak{V}_1 - \mathfrak{V}_2, \mathfrak{u}) = 1$, for all $\mathfrak{u} > 0$, resulting in $\mathfrak{V}_1 = \mathfrak{V}_2$.
As an alternative, if $\iota \in \mathbb{N}\backslash(\mathfrak{J}_{\varrho,1}(\tau,\mathfrak{u}) \cup \mathfrak{J}_{\varrho,2}(\tau,\mathfrak{u}))$, we can write

$$\begin{aligned}\varrho(\mathfrak{V}_1 - \mathfrak{V}_2, \mathfrak{u}) &< \left|\varrho\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \varrho\left(\lambda, \mathfrak{V}_1, \frac{\mathfrak{u}}{2}\right)\right| \\ &\oplus \left|\varrho\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \varrho\left(\lambda, \mathfrak{V}_2, \frac{\mathfrak{u}}{2}\right)\right| \\ &< \tau \oplus \tau.\end{aligned}$$

Also, if $\iota \in \mathbb{N}\setminus(\mathfrak{J}_{\nu,1}(\tau,\mathfrak{u})\cup\mathfrak{J}_{\nu,2}(\tau,\mathfrak{u}))$, then we can write

$$\begin{aligned}\nu(\mathfrak{V}_1-\mathfrak{V}_2,\mathfrak{u}) &< \left|\nu\left(\lambda,\mathfrak{V}_\iota,\frac{\mathfrak{u}}{2}\right)-\nu\left(\lambda,\mathfrak{V}_1,\frac{\mathfrak{u}}{2}\right)\right| \\ &\quad \oplus \left|\nu\left(\lambda,\mathfrak{V}_\iota,\frac{\mathfrak{u}}{2}\right)-\nu\left(\lambda,\mathfrak{V}_2,\frac{\mathfrak{u}}{2}\right)\right| \\ &< \tau\oplus\tau.\end{aligned}$$

Presently using the fact that $\tau\oplus\tau<\varepsilon$, we are able to observe that $\varrho(\mathfrak{V}_1-\mathfrak{V}_2,\mathfrak{u})<\varepsilon$, $\nu(\mathfrak{V}_1-\mathfrak{V}_2,\mathfrak{u})<\varepsilon$. Since arbitrary $\varepsilon>0$, we obtain $\varrho(\mathfrak{V}_1-\mathfrak{V}_2,\mathfrak{u})=0, \nu(\mathfrak{V}_1-\mathfrak{V}_2,\mathfrak{u})=0$, for all $\mathfrak{u}>0$. It proves $\mathfrak{V}_1=\mathfrak{V}_2$. Finally, we deduce that, the limit of $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}$ is unique. □

Definition 2.4. Let us consider $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ be an *NMS*. A sequence of sets $\{\mathfrak{V}_\iota\}$ is known to be Wijsman convergent to $\mathfrak{V}$ with regards to $NM(\varsigma,\varrho,\nu)$, if for each $\varepsilon\in(0,1)$, for every $\lambda\in\Omega$ and for all $\mathfrak{u}>0$, there exists $\iota_0\in\mathbb{N}$ such that

$$|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|>1-\varepsilon,\quad |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon,$$
$$\text{and}\quad |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon,\quad \text{for all } \iota\ge\iota_0.$$

In this case, we write $(\varsigma,\varrho,\nu)_{\mathfrak{W}}-\lim\mathfrak{V}_\iota=\mathfrak{V}$.

Theorem 2.5. *Let* $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ *be an* NMS *and* ψ *be a* Ls. *If* $(\varsigma,\varrho,\nu)_{\mathfrak{W}}-\lim\mathfrak{V}_\iota=\mathfrak{V}$, *then* $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim\mathfrak{V}_\iota=\mathfrak{V}$.

Proof. Let $(\varsigma,\varrho,\nu)_{\mathfrak{W}}-\lim\mathfrak{V}_\iota=\mathfrak{V}$. Then, for every $\varepsilon>0$ and $\mathfrak{u}>0$, the quantity $\iota_0\in\mathbb{N}$ exists so that

$$|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|>1-\varepsilon, |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon,$$
$$\text{and}\quad |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon,\quad \text{for all } \iota\ge\iota_0.$$

Hence, the set

$$\left\{\begin{array}{l}\iota\in\mathbb{N}:|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|\le 1-\varepsilon\\ \text{or}\ \ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|\ge\varepsilon\\ \text{and}\ \ |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|\ge\varepsilon\end{array}\right\}$$

has limited number of terms. Since every finite subsets of $\mathbb{N}$ has consistency zero and thus

$$\Delta_\psi\left(\left\{\begin{array}{l}\iota\in\mathbb{N}:|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|\le 1-\varepsilon,\\ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|\ge\varepsilon\\ \text{or}\ \ |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|\ge\varepsilon\end{array}\right\}\right)=0,$$

that is, $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim\mathfrak{V}_\iota=\mathfrak{V}$. □

Definition 2.6. Let $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ be an *NMS*. A sequence $\{\mathfrak{V}_\iota\}$ of nonempty closed subsets of Ω is called as *WStC* to $\mathfrak{V}$ with regards to $NM(\varsigma,\varrho,\nu)$, if for every $\varepsilon\in(0,1)$, for each $\lambda\in\Omega$ and for all $\mathfrak{u}>0$,

$$\lim_{\mathfrak{n}\to\infty}\frac{1}{\mathfrak{n}}\left|\left\{\begin{array}{l}\iota\le\mathfrak{n}:|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|\le 1-\varepsilon\\ \text{or}\ \ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|\ge\varepsilon\\ \text{and}\ \ |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|\ge\varepsilon\end{array}\right\}\right|=0.$$

In this case, we write $\mathfrak{WT}^{(\varsigma,\varrho,\nu)} - \lim \mathfrak{V}_\iota = \mathfrak{V}$ or $\mathfrak{V}_\iota \xrightarrow{(\varsigma,\varrho,\nu)} \mathfrak{V}(\mathfrak{WT})$. We indicate the set of all *WStC* set sequences with regards to $NM(\varsigma,\varrho,\nu)$ by $\mathfrak{WT}^{(\varsigma,\varrho,\nu)}$.

Let $\mathfrak{WT}^{(\varsigma,\varrho,\nu)}$ and $\mathfrak{WT}_\psi^{(\varsigma,\varrho,\nu)}$ represnt the sets of *WStC* and *WLStC* sequences respectively in *NMS* $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$.

For $\lambda \in \Omega, \mathfrak{u} > 0$ and $\beta \in (0,1)$, the sphere centered at λ with radius β is initiated by

$$B^{\varpi}(\lambda,\beta,\mathfrak{u}) = \{\varpi \in \Omega : \varsigma(\lambda-\varpi,\mathfrak{u}) > 1-\beta, \varrho(\lambda-\varpi,\mathfrak{u}) < \beta \text{ and } \nu(\lambda-\varpi,\mathfrak{u}) < \beta\}.$$

Theorem 2.7. *For any* Ls $\psi, \mathfrak{WT}_\psi^{(\varsigma,\varrho,\nu)} \subseteq \mathfrak{WT}^{(\varsigma,\varrho,\nu)}$ *iff* $\limsup_{\mathfrak{r}} \rho_{\mathfrak{r}} < \infty$.

Proof. If $\limsup_{\mathfrak{r}} \rho_{\mathfrak{r}} < \infty$, there exists $\mathfrak{V} > 0$ such that $\rho_{\mathfrak{r}} < \mathfrak{V}$, for all $\mathfrak{r}$. Presume that $\{\mathfrak{V}_\iota\} \in \mathfrak{WT}_\psi^{(\varsigma,\varrho,\nu)}$ and $\mathfrak{WT}_\psi^{(\varsigma,\varrho,\nu)} - \lim \mathfrak{V}_\iota = \mathfrak{V}$. For $\tau \in (0,1)$ and $\mathfrak{u} > 0$, let

$$\mathfrak{N}_{\mathfrak{r}} = \left|\left\{\begin{array}{l} \iota \in \mathfrak{J}_{\mathfrak{r}} : |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varsigma(\lambda,\mathfrak{V},\mathfrak{u})| \le 1-\tau \\ \text{or } |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varrho(\lambda,\mathfrak{V},\mathfrak{u})| \ge \tau \\ \text{and } |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \nu(\lambda,\mathfrak{V},\mathfrak{u})| \ge \tau \end{array}\right\}\right|.$$

Then, for $\varepsilon > 0$, there exists $\mathfrak{r}_0 \in \mathbb{N}$ such that

$$\frac{\mathfrak{N}_{\mathfrak{r}}}{\mathfrak{v}_{\mathfrak{r}}} < \varepsilon, \quad \text{for all } \mathfrak{r} > \mathfrak{r}_0. \tag{2.3}$$

Consider $\mathfrak{J} = \max\{\mathfrak{N}_{\mathfrak{r}} : 1 \le \mathfrak{r} \le \mathfrak{r}_0\}$ and take $\mathfrak{n}$ so that $\iota_{\mathfrak{r}-1} < \mathfrak{n} \le \iota_{\mathfrak{r}}$. We get

$$\begin{aligned} \frac{1}{\mathfrak{n}}\left|\left\{\begin{array}{l} \iota \le \mathfrak{n} : |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varsigma(\lambda,\mathfrak{V},\mathfrak{u})| \le 1-\tau \\ \text{or } |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varrho(\lambda,\mathfrak{V},\mathfrak{u})| \ge \tau \\ \text{and } |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \nu(\lambda,\mathfrak{V},\mathfrak{u})| \ge \tau \end{array}\right\}\right| &\le \frac{1}{\iota_{\mathfrak{r}-1}}\left|\left\{\begin{array}{l} \iota \le \iota_{\mathfrak{r}} : |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varsigma(\lambda,\mathfrak{V},\mathfrak{u})| \le 1-\tau \\ \text{or } |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varrho(\lambda,\mathfrak{V},\mathfrak{u})| \ge \tau \\ \text{and } |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \nu(\lambda,\mathfrak{V},\mathfrak{u})| \ge \tau \end{array}\right\}\right| \\ &= \frac{1}{\iota_{\mathfrak{r}-1}}\{\mathfrak{N}_1 + \mathfrak{N}_2 + \ldots + \mathfrak{N}_{\mathfrak{r}_0} + \mathfrak{N}_{\mathfrak{r}_0+1} + \cdots + \mathfrak{N}_{\mathfrak{r}}\} \\ &\le \frac{\mathfrak{J}}{\iota_{\mathfrak{r}-1}}\mathfrak{r}_0 + \frac{1}{\iota_{\mathfrak{r}-1}}\left\{\mathfrak{v}_{\mathfrak{r}_0+1}\frac{\mathfrak{N}_{\mathfrak{r}_0+1}}{\mathfrak{v}_{\mathfrak{r}_0+1}} + \cdots + \mathfrak{v}_{\mathfrak{r}}\frac{\mathfrak{N}_{\mathfrak{r}}}{\mathfrak{v}_{\mathfrak{r}}}\right\} \\ &\le \frac{\mathfrak{r}_0\mathfrak{J}}{\iota_{\mathfrak{r}-1}} + \varepsilon\rho_{\mathfrak{r}} \le \frac{\mathfrak{r}_0\mathfrak{J}}{\iota_{\mathfrak{r}-1}} + \varepsilon\mathfrak{V}. \end{aligned}$$

To demonstrate the contrary, consider that $\limsup_{\mathfrak{r}} \rho_{\mathfrak{r}} = \infty$. Since ψ is a *Ls*, we can take a subsequence $\{\iota_{\mathfrak{r}(j)}\}$ of ψ such that $\rho_{\mathfrak{r}(j)} > j$.

Let $\{\iota\}(\neq \{0\}) \in \Omega$. We determine a sequence $\{\mathfrak{V}_\iota\}$ as follows:

$$\mathfrak{V}_\iota = \begin{cases} \{\iota\}, & \text{if } \iota_{\mathfrak{r}(j)-1} < \iota < 2\iota_{\mathfrak{r}(j)-1}, \text{ for some } j = 1,2,\ldots, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Since $\{\iota\}(\neq \{0\})$, we can pick $\tau \in (0,1)$ and $\mathfrak{u} > 0$ such that $\{\iota\} \notin B^{\varpi}(0,\tau,\mathfrak{u})$.

Now, for $j > 1$,

$$\frac{1}{\mathfrak{v}_{\mathfrak{r}(j)}}\left|\left\{\begin{array}{l} \iota \le \iota_{\mathfrak{q}(j)} : |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varsigma(\lambda,0,\mathfrak{u})| \le 1-\tau \\ \text{or } |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varrho(\lambda,0,\mathfrak{u})| \ge \tau \\ \text{and } |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \nu(\lambda,0,\mathfrak{u})| \ge \tau \end{array}\right\}\right| < \frac{1}{j-1}.$$

Thus $\{\mathfrak{V}_\iota\} \in \mathfrak{WT}_\psi^{(\varsigma,\varrho,\nu)}$. But $\{\mathfrak{V}_\iota\} \notin \mathfrak{WT}^{(\varsigma,\varrho,\nu)}$. For

$$\frac{1}{2\iota_{\mathfrak{q}(j)-1}}\left|\left\{\begin{array}{l}\iota \le 2\iota_{\mathfrak{q}(j)-1} : |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,0,\mathfrak{u})| \le 1-\tau \\ \text{or } \ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,0,\mathfrak{u})| \ge \tau \\ \text{and } \ |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,0,\mathfrak{u})| \ge \tau\end{array}\right\}\right| > \frac{1}{2}$$

and

$$\frac{1}{\iota_{\mathfrak{q}(j)}}\left|\left\{\begin{array}{l}\iota \le \iota_{\mathfrak{q}(j)} : |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,0,\mathfrak{u})| \le 1-\tau \\ \text{or } \ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,0,\mathfrak{u})| \ge \tau \\ \text{and } \ |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,0,\mathfrak{u})| \ge \tau\end{array}\right\}\right| > 1-\frac{2}{j},$$

which is a contradiction. □

Theorem 2.8. *For any* Ls $\psi, \mathfrak{WT}^{(\varsigma,\varrho,\nu)} \subseteq \mathfrak{WT}_\psi^{(\varsigma,\varrho,\nu)}$ *iff* $\liminf_{\mathfrak{q}} \rho_{\mathfrak{q}} > 1$.

Proof. Suppose $\liminf_{\mathfrak{q}} \rho_{\mathfrak{q}} > 1$, then there exists $\alpha > 0$ such that $\rho_{\mathfrak{q}} \ge 1+\alpha$ for moderately large $\mathfrak{q}$, which infers that $\frac{\mathfrak{v}_{\mathfrak{q}}}{\iota_{\mathfrak{q}}} \ge \frac{\alpha}{1+\alpha}$.

Let $\mathfrak{WT}^{(\varsigma,\varrho,\nu)}-\lim \mathfrak{V}_\iota = \mathfrak{V}$. Then, for various $\tau \in (0,1)$ and $\mathfrak{u} > 0$ and suitably large $\mathfrak{q}$, we find,

$$\frac{1}{\iota_{\mathfrak{q}}}\left|\left\{\begin{array}{l}\iota \le \iota_{\mathfrak{q}} : |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})| \le 1-\tau \\ \text{or } \ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})| \ge \tau \\ \text{and } \ |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})| \ge \tau\end{array}\right\}\right| \ge \frac{\alpha}{1+\alpha}\frac{1}{\mathfrak{v}_{\mathfrak{q}}}\left|\left\{\begin{array}{l}\iota \in \mathfrak{I}_{\mathfrak{q}} : |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})| \le 1-\tau \\ \text{or } \ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})| \ge \tau \\ \text{and } \ |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})| \ge \tau\end{array}\right\}\right|.$$

Thus $\mathfrak{WT}_\psi^{(\varsigma,\varrho,\nu)}-\lim \mathfrak{V}_\iota = \mathfrak{V}$.

Conversely, suppose that $\liminf_{\mathfrak{q}} \rho_{\mathfrak{q}} = 1$. Since ψ is a *Ls*, we can prefer a subsequence $\{\iota_{\mathfrak{q}(j)}\}$ of ψ such that

$$\frac{\iota_{\mathfrak{q}(j)}}{\iota_{\mathfrak{q}(j)}-1} < 1+\frac{1}{j} \quad \text{and} \quad \frac{\iota_{\mathfrak{q}(j)}-1}{\iota_{\mathfrak{q}(j-1)}} > j,$$

where $\mathfrak{q}(j) > \mathfrak{q}(j-1)+2$. Let $\{\iota\}\ (\neq \{0\}) \in \Omega$. We trace a sequence $\{\mathfrak{V}_\iota\}$ as ensues:

$$\mathfrak{V}_\iota = \begin{cases}\{\iota\}\, \iota \in \mathfrak{I}_{\mathfrak{q}(j)}, & \text{for some } \ j=1,2,\dots, \\ \{0\}, & \text{otherwise.}\end{cases}$$

We represent that $\{\mathfrak{V}_\iota\} \in \mathfrak{WT}^{(\varsigma,\varrho,\nu)}$. Let $\tau \in (0,1)$ and $\mathfrak{u} > 0$. Choose $\mathfrak{u}_1 > 0$ and $\tau_1 \in (0,1)$ such that $B(0,\tau_1,\mathfrak{u}_1) \subset B(0,\tau,\mathfrak{u})$ and $\{\iota\} \notin B(0,\tau_1,\mathfrak{u}_1)$. Also, for each $\mathfrak{n} \in \mathbb{N}$ we can find $j_{\mathfrak{n}} > 0$ so that $\iota_{\mathfrak{q}(j_{\mathfrak{n}})-1} < \mathfrak{n} \le \iota_{\mathfrak{q}(j_{\mathfrak{n}})}$. Then, for every $\mathfrak{n} \in \mathbb{N}$, we get

$$\frac{1}{\mathfrak{n}}\left|\left\{\begin{array}{l}\iota \le \mathfrak{n} : |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})| \le 1-\tau \\ \text{or } \ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})| \ge \tau \\ \text{and } \ |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})| \ge \tau\end{array}\right\}\right| \le \frac{\iota_{\mathfrak{q}(j_{\mathfrak{n}}-1)}+\mathfrak{v}_{\mathfrak{q}(j_{\mathfrak{n}})}}{\iota_{\mathfrak{q}(j_{\mathfrak{n}})-1}} \le \frac{2}{j_{\mathfrak{n}}}.$$

Thus $\mathfrak{WT}^{(\varsigma,\varrho,\nu)}-\lim \mathfrak{V}_\iota = \mathfrak{V}$.

Next we prove that $\{\mathfrak{V}_\iota\} \notin \mathfrak{WT}_\psi^{(\varsigma,\varrho,\nu)}$.

Since $\{\iota\}(\neq \{0\}) \in \Omega$, so we can choose $\mathfrak{u} > 0$ and $\tau \in (0,1)$ such that $\{\iota\} \notin B(0,\tau,\mathfrak{u})$.

Thus

$$\lim_{j\to\infty}\frac{1}{\mathfrak{v}_{\mathfrak{q}(j)}}\left|\left\{\begin{array}{l}\iota \in \mathfrak{I}_{\mathfrak{q}(j)} : |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,0,\mathfrak{u})| \le 1-\tau \\ \text{or } \ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,0,\mathfrak{u})| \ge \tau \\ \text{and } \ |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,0,\mathfrak{u})| \ge \tau\end{array}\right\}\right| = 1$$

and for $\mathfrak{q} \neq \mathfrak{q}_j$,

$$\lim_{\mathfrak{q}\to\infty} \frac{1}{\mathfrak{v}_\mathfrak{q}} \left| \left\{ \begin{array}{l} \iota \in \mathfrak{I}_\mathfrak{q} : |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varsigma(\lambda,\mathfrak{V},\mathfrak{u})| \leq 1-\tau \\ \text{or } |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varrho(\lambda,\mathfrak{V},\mathfrak{u})| \geq \tau \\ \text{and } |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \nu(\lambda,\mathfrak{V},\mathfrak{u})| \geq \tau \end{array} \right\} \right| = 1.$$

Hence $\{\mathfrak{V}_\iota\} \notin \mathfrak{WT}_\psi^{(\varsigma,\varrho,\nu)}$ a contradiction. □

Corollary 2.9. *Let* $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ *be an* NMS. *For any lacunary sequence* ψ*,*

$$\mathfrak{WT}^{(\varsigma,\varrho,\nu)} = \mathfrak{WT}_\psi^{(\varsigma,\varrho,\nu)} \quad \textit{iff} \quad 1 < \liminf_\mathfrak{q} \rho_\mathfrak{q} \leq \limsup_\mathfrak{q} \rho_\mathfrak{q} < \infty.$$

Definition 2.10. Let $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ be an *NMS* and ψ be a *Ls*. A sequence $\{\mathfrak{V}_\iota\}$ in Ω is known to be Wijsman lacunary summable to $\mathfrak{V}$ with regards to $NM(\varsigma,\varrho,\nu)$ if, for every $\lambda \in \Omega$ and $\mathfrak{u} > 0$,

$$\lim_{\mathfrak{q}\to\infty} \frac{1}{\mathfrak{v}_\mathfrak{q}} \sum_{\iota\in\mathfrak{I}_\mathfrak{q}} \varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) = \varsigma(\lambda,\mathfrak{V},\mathfrak{u}),$$

$$\lim_{\mathfrak{q}\to\infty} \frac{1}{\mathfrak{v}_\mathfrak{q}} \sum_{\iota\in\mathfrak{I}_\mathfrak{q}} \varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) = \varrho(\lambda,\mathfrak{V},\mathfrak{u}), \quad \text{and}$$

$$\lim_{\mathfrak{q}\to\infty} \frac{1}{\mathfrak{v}_\mathfrak{q}} \sum_{\iota\in\mathfrak{I}_\mathfrak{q}} \nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) = \nu(\lambda,\mathfrak{V},\mathfrak{u}).$$

Here, though we write $\mathfrak{WN}_\psi^{(\varsigma,\varrho,\nu)} - \lim \mathfrak{V}_\iota = \mathfrak{V}$ or $\mathfrak{V}_\iota \to \mathfrak{V}(\mathfrak{WN}_\psi^{(\varsigma,\varrho,\nu)})$.

Example 2.11. Let $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ be an *NMS* and $\mathfrak{V},\{\mathfrak{V}_\iota\}$ be any nonempty closed subsets of Ω. Assume $\Omega = \mathbb{R}$ and $\{\mathfrak{V}_\iota\}$ are sequence defined by

$$\mathfrak{V}_\iota = \begin{cases} \{\lambda \in \mathbb{R}, 2 \leq \lambda \leq \iota_\mathfrak{q} - \iota_{\mathfrak{q}-1}\}, & \text{if } \iota \geq 2 \text{ and } \iota \text{ is square integer}, \\ \{1\}, & \text{otherwise}. \end{cases}$$

This sequence is not $\mathfrak{WLS}$in *NMS* with regards to the $NM(\varsigma,\varrho,\nu)$. But, since

$$\lim_{\mathfrak{q}\to\infty} \frac{1}{\mathfrak{v}_\mathfrak{q}} \left| \left\{ \begin{array}{l} |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varsigma(\lambda,1,\mathfrak{u})| \leq 1-\varepsilon \\ \text{or } |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varrho(\lambda,1,\mathfrak{u})| \geq \varepsilon \\ \text{and } |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \nu(\lambda,1,\mathfrak{u})| \geq \varepsilon \end{array} \right\} \right| = 0,$$

this sequence is *WLStC* to the set $\mathfrak{V} = \{1\}$ in *NMS* with regards to the $NM(\varsigma,\varrho,\nu)$.

Definition 2.12. Let $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ be an *NMS* and ψ be a lacunary sequence*Ls*. A sequence $\{\mathfrak{V}_\iota\}$ in Ω is known to be Wijsman lacunary strongly convergent (*WLSC*) to $\mathfrak{V}$ with respect to the $NM(\varsigma,\varrho,\nu)$ if, for every $\varepsilon \in (0,1)$ and $\mathfrak{u} > 0$, there exist $\mathfrak{q}_0 \in \mathbb{N}$

$$\frac{1}{\mathfrak{v}_\mathfrak{q}} \sum_{\iota\in\mathfrak{I}_\mathfrak{q}} |\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varsigma(\lambda,\mathfrak{V},\mathfrak{u})| > 1-\varepsilon,$$

$$\frac{1}{\mathfrak{v}_\mathfrak{q}} \sum_{\iota\in\mathfrak{I}_\mathfrak{q}} |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \varrho(\lambda,\mathfrak{V},\mathfrak{u})| < \varepsilon, \quad \text{and}$$

$$\frac{1}{\mathfrak{v}_\mathfrak{q}} \sum_{\iota\in\mathfrak{I}_\mathfrak{q}} |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u}) - \nu(\lambda,\mathfrak{V},\mathfrak{u})| < \varepsilon,$$

for all $\mathfrak{q} > \mathfrak{q}_0$. We write $[\mathfrak{WN}_\psi^{(\varsigma,\varrho,\nu)}] - \lim \mathfrak{V}_\iota = \mathfrak{V}$ or $\mathfrak{V}_\iota \to \mathfrak{V}([\mathfrak{WN}_\psi^{(\varsigma,\varrho,\nu)}])$.

Definition 2.13. Let $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ be an *NMS* and ψ be a *Ls*. A sequence $\{\mathfrak{V}_\iota\}$ in Ω is strongly Cesaro summable to $\mathfrak{V}$ with respect to $NM(\varsigma,\varrho,\nu)$ if, for every $\varepsilon\in(0,1)$ and $\mathfrak{u}>0$, there exist $\mathfrak{n}_0\in\mathbb{N}$

$$\frac{1}{\mathfrak{n}}\sum_{\iota=1}^{\mathfrak{n}}|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|>1-\varepsilon,$$
$$\frac{1}{\mathfrak{n}}\sum_{\iota=1}^{\mathfrak{n}}|\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon,\quad\text{and}$$
$$\frac{1}{\mathfrak{n}}\sum_{\iota=1}^{\mathfrak{n}}|\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon,$$

for all $\mathfrak{n}>\mathfrak{n}_0$. In this case we write $|\mathfrak{W}\varsigma|^{(\varsigma,\varrho,\nu)}-\lim\mathfrak{V}_\iota=\mathfrak{V}$ or $\mathfrak{V}_\iota\to\mathfrak{V}\big(|\mathfrak{W}\varsigma|^{(\varsigma,\varrho,\nu)}\big)$.

Theorem 2.14. *Let* $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ *be an* NMS *and* ψ *be a* Ls. *Then,* $|\mathfrak{W}\varsigma|^{(\varsigma,\varrho,\nu)}\subseteq[\mathfrak{W}\mathfrak{N}_\psi^{(\varsigma,\varrho,\nu)}]$ *if* $\liminf_{\mathfrak{q}}\rho_{\mathfrak{q}}>1$.

Proof. Let $\liminf_{\mathfrak{q}}\rho_{\mathfrak{q}}>1$ and $\{\mathfrak{V}_\iota\}\in|\mathfrak{W}\varsigma|^{(\varsigma,\varrho,\nu)}$. Then, there exists $\Delta>0$ such that $\rho_{\mathfrak{q}}>1+\Delta$, for all $\mathfrak{q}\geq 1$. Then

$$\begin{aligned}&\frac{1}{\mathfrak{v}_{\mathfrak{q}}}\sum_{\iota\in\mathfrak{I}_{\mathfrak{q}}}|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|-1\\&\quad=\frac{1}{\mathfrak{v}_{\mathfrak{q}}}\sum_{\iota=1}^{\iota_{\mathfrak{q}}}|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|-\frac{1}{\mathfrak{v}_{\mathfrak{q}}}\sum_{\iota=1}^{\iota_{\mathfrak{q}-1}}|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|-1\\&\quad=\frac{\iota_{\mathfrak{q}}}{\mathfrak{v}_{\mathfrak{q}}}\left[\frac{1}{\iota_{\mathfrak{q}}}\sum_{\iota=1}^{\iota_{\mathfrak{q}}}|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|-1\right]-\frac{\iota_{\mathfrak{q}-1}}{\mathfrak{v}_{\mathfrak{q}}}\left[\frac{1}{\iota_{\mathfrak{q}-1}}\sum_{\iota=1}^{\iota_{\mathfrak{q}-1}}|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|-1\right].\end{aligned}$$

Since $\mathfrak{v}_{\mathfrak{q}}=\iota_{\mathfrak{q}}-\iota_{\mathfrak{q}-1}$, $\frac{\iota_{\mathfrak{q}}}{\mathfrak{v}_{\mathfrak{q}}}\leq\frac{1+\Delta}{\Delta}$, $\frac{\iota_{\mathfrak{q}-1}}{\mathfrak{v}_{\mathfrak{q}}}\leq\frac{1}{\Delta}$.
Also the terms

$$\frac{1}{\iota_{\mathfrak{q}}}\sum_{\iota=1}^{\iota_{\mathfrak{q}}}|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|-1\quad\text{and}\quad\frac{1}{\iota_{\mathfrak{q}-1}}\sum_{\iota=1}^{\iota_{\mathfrak{q}-1}}|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|-1,$$

both converges to zero. So, $\frac{1}{\mathfrak{v}_{\mathfrak{q}}}\sum_{\iota\in\mathfrak{I}_{\mathfrak{q}}}|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|\to 1$.

Similarly $\frac{1}{\mathfrak{v}_{\mathfrak{q}}}\sum_{\iota\in\mathfrak{I}_{\mathfrak{q}}}|\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|\to 0$, $\frac{1}{\mathfrak{v}_{\mathfrak{q}}}\sum_{\iota\in\mathfrak{I}_{\mathfrak{q}}}|\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|\to 0$.

So, $\{\mathfrak{V}_\iota\}\in[\mathfrak{W}\mathfrak{N}_\psi^{(\varsigma,\varrho,\nu)}]$. □

Theorem 2.15. *Let* $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ *be an* NMS *and* ψ *be a* Ls. *Then,* $\mathfrak{W}\mathfrak{N}_\psi^{(\varsigma,\varrho,\nu)}\subseteq|\mathfrak{W}\varsigma|^{(\varsigma,\varrho,\nu)}$ *if* $\liminf_{\mathfrak{q}}\rho_{\mathfrak{q}}=1$.

Proof. Let $\liminf_{\mathfrak{q}}\rho_{\mathfrak{q}}=1$ and $\{\mathfrak{V}_\iota\}\in[\mathfrak{W}\mathfrak{N}_\psi^{(\varsigma,\varrho,\nu)}]$. Then, for $\mathfrak{u}>0$ we have

$$\mathfrak{V}_{\mathfrak{q}}=\frac{1}{\mathfrak{v}_{\mathfrak{q}}}\sum_{\iota\in\mathfrak{I}_{\mathfrak{q}}}|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|\to 1,$$
$$\mathfrak{V}'_{\mathfrak{q}}=\frac{1}{\mathfrak{v}_{\mathfrak{q}}}\sum_{\iota\in\mathfrak{I}_{\mathfrak{q}}}|\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|\to 0,\quad\text{and}$$

$$\mathfrak{V}''_{\mathfrak{q}} = \frac{1}{\mathfrak{v}_{\mathfrak{q}}} \sum_{\iota \in \mathfrak{I}_{\mathfrak{q}}} |\nu(\lambda, \mathfrak{V}_\iota, \mathfrak{u}) - \nu(\lambda, \mathfrak{V}, \mathfrak{u})| \to 0,$$

as $\mathfrak{q} \to \infty$. Then for $\varepsilon > 0$, there exists $\mathfrak{q}_0 \in \mathbb{N}$ such that $\mathfrak{V}_{\mathfrak{q}} < 1 + \varepsilon$, for all $\mathfrak{q} > \mathfrak{q}_0$. Also, we can find $\mathfrak{J} > 0$ such that $\mathfrak{V}_{\mathfrak{q}} < \mathfrak{J}$, $\mathfrak{V}'_{\mathfrak{q}} < \mathfrak{J}$ and $\mathfrak{V}''_{\mathfrak{q}} < \mathfrak{J}$, $\mathfrak{q} = 1, 2, \dots$.
Let $\mathfrak{n}$ be an integer with $\iota_{\mathfrak{q}-1} < \mathfrak{n} \le \iota_{\mathfrak{q}}$. Then

$$\begin{aligned}
\frac{1}{\mathfrak{n}} \sum_{\iota=1}^{\mathfrak{n}} |\varsigma(\lambda, \mathfrak{V}_\iota, \mathfrak{u}) - \varsigma(\lambda, \mathfrak{V}, \mathfrak{u})| &\le \frac{1}{\iota_{\mathfrak{q}-1}} \sum_{\iota=1}^{\iota_{\mathfrak{q}}} |\varsigma(\lambda, \mathfrak{V}_\iota, \mathfrak{u}) - \varsigma(\lambda, \mathfrak{V}, \mathfrak{u})| \\
&= \frac{1}{\iota_{\mathfrak{q}-1}} \left[\sum_{\mathfrak{J}_1} |\varsigma(\lambda, \mathfrak{V}_\iota, \mathfrak{u}) - \varsigma(\lambda, \mathfrak{V}, \mathfrak{u})| + \dots + \sum_{\mathfrak{J}_{\mathfrak{q}}} |\varsigma(\lambda, \mathfrak{V}_\iota, \mathfrak{u}) - \varsigma(\lambda, \mathfrak{V}, \mathfrak{u})| \right] \\
&= \sup_{1 \le \mathfrak{q} \le \mathfrak{q}_0} \mathfrak{V}_{\mathfrak{q}} \frac{\iota_{\mathfrak{q}_0}}{\iota_{\mathfrak{q}-1}} + \frac{\mathfrak{v}_{\mathfrak{q}_0+1}}{\iota_{\mathfrak{q}-1}} \mathfrak{V}_{\mathfrak{q}_0+1} + \dots + \frac{\mathfrak{v}_{\mathfrak{q}}}{\iota_{\mathfrak{q}-1}} \mathfrak{V}_{\mathfrak{q}} \\
&< \mathfrak{J} \frac{\iota_{\mathfrak{q}_0}}{\iota_{\mathfrak{q}-1}} + (1+\varepsilon) \frac{\iota_{\mathfrak{q}} - \iota_{\mathfrak{q}_0}}{\iota_{\mathfrak{q}-1}}.
\end{aligned}$$

Since $\iota_{\mathfrak{q}-1} \to \infty$ as $\mathfrak{n} \to \infty$, it follows that

$$\frac{1}{\mathfrak{n}} \sum_{\iota=1}^{\mathfrak{n}} |\varsigma(\lambda, \mathfrak{V}_\iota, \mathfrak{u}) - \varsigma(\lambda, \mathfrak{V}, \mathfrak{u})| \to 1.$$

Similarly, we can prove that

$$\frac{1}{\mathfrak{n}} \sum_{\iota=1}^{\mathfrak{n}} |\varrho(\lambda, \mathfrak{V}_\iota, \mathfrak{u}) - \varrho(\lambda, \mathfrak{V}, \mathfrak{u})| \to 0 \quad \text{and} \quad \frac{1}{\mathfrak{n}} \sum_{\iota=1}^{\mathfrak{n}} |\nu(\lambda, \mathfrak{V}_\iota, \mathfrak{u}) - \nu(\lambda, \mathfrak{V}, \mathfrak{u})| \to 0.$$

Hence $\{\mathfrak{V}_\iota\} \in |\mathfrak{W}\varsigma|^{(\varsigma,\varrho,\nu)}$. □

Theorem 2.16. *If* $\{\mathfrak{V}_\iota\} \in \left[\mathfrak{W}\mathfrak{N}_\psi^{(\varsigma,\varrho,\nu)} \cap |\mathfrak{W}\varsigma|^{(\varsigma,\varrho,\nu)}\right]$, *then*

$$\mathfrak{W}\mathfrak{N}_\psi^{(\varsigma,\varrho,\nu)} - \lim \mathfrak{V}_\iota = |\mathfrak{W}\varsigma|^{(\varsigma,\varrho,\nu)} - \lim \mathfrak{V}_\iota.$$

Proof. Let $[\mathfrak{W}\mathfrak{N}_\psi^{(\varsigma,\varrho,\nu)}] - \lim \mathfrak{V}_\iota = \mathfrak{V}_1$ and $|\mathfrak{W}\varsigma|^{(\varsigma,\varrho,\nu)} - \lim \mathfrak{V}_\iota = \mathfrak{V}_2$.
Given $\varepsilon > 0$, select $\tau > 0$ such that $(1-\tau) \odot (1-\tau) > 1 - \varepsilon$ and $\tau \oplus \tau < \varepsilon$.
Then, for any $\mathfrak{u} > 0$, there exists $\mathfrak{q}_0 \in \mathbb{N}$ such that

$$\frac{1}{\mathfrak{v}_{\mathfrak{q}}} \sum_{\iota \in \mathfrak{I}_{\mathfrak{q}}} \left| \varsigma\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \varsigma\left(\lambda, \mathfrak{V}_1, \frac{\mathfrak{u}}{2}\right) \right| > 1 - \tau,$$

$$\frac{1}{\mathfrak{v}_{\mathfrak{q}}} \sum_{\iota \in \mathfrak{I}_{\mathfrak{q}}} \left| \varrho\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \varrho\left(\lambda, \mathfrak{V}_1, \frac{\mathfrak{u}}{2}\right) \right| < \tau, \quad \text{and}$$

$$\frac{1}{\mathfrak{v}_{\mathfrak{q}}} \sum_{\iota \in \mathfrak{I}_{\mathfrak{q}}} \left| \nu\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \nu\left(\lambda, \mathfrak{V}_1, \frac{\mathfrak{u}}{2}\right) \right| < \tau,$$

for all $\mathfrak{q} > \mathfrak{q}_0$. Also, there exists $\mathfrak{n}_0 \in \mathbb{N}$ such that

$$\frac{1}{\mathfrak{n}} \sum_{\iota=1}^{\mathfrak{n}} \left| \varsigma\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \varsigma\left(\lambda, \mathfrak{V}_2, \frac{\mathfrak{u}}{2}\right) \right| > 1 - \tau$$

$$\frac{1}{\mathfrak{n}} \sum_{\iota=1}^{\mathfrak{n}} \left| \varrho\left(\lambda, \mathfrak{V}_\iota, \frac{\mathfrak{u}}{2}\right) - \varrho\left(\lambda, \mathfrak{V}_2, \frac{\mathfrak{u}}{2}\right) \right| < \tau, \quad \text{and}$$

$$\frac{1}{\mathfrak{n}}\sum_{\iota=1}^{\mathfrak{n}}\left|\nu\left(\lambda,\mathfrak{V}_\iota,\frac{\mathfrak{u}}{2}\right)-\nu\left(\lambda,\mathfrak{V}_2,\frac{\mathfrak{u}}{2}\right)\right|<\tau,$$

for all $\mathfrak{n}>\mathfrak{n}_0$. Consider $\mathfrak{q}_1=\max\{\mathfrak{q}_0,\mathfrak{n}_0\}$. Then, we will get a $l\in\mathbb{N}$ such that

$$\left|\varsigma\left(\lambda,\mathfrak{V}_l,\frac{\mathfrak{u}}{2}\right)-\varsigma\left(\lambda,\mathfrak{V}_1,\frac{\mathfrak{u}}{2}\right)\right|\geq\frac{1}{\mathfrak{v}_\mathfrak{q}}\sum_{\iota\in\mathfrak{I}_\mathfrak{q}}\left|\varsigma\left(\lambda,\mathfrak{V}_\iota,\frac{\mathfrak{u}}{2}\right)-\varsigma\left(\lambda,\mathfrak{V}_1,\frac{\mathfrak{u}}{2}\right)\right|>1-\tau$$

and

$$\left|\varsigma\left(\lambda,\mathfrak{V}_l,\frac{\mathfrak{u}}{2}\right)-\varsigma\left(\lambda,\mathfrak{V}_2,\frac{\mathfrak{u}}{2}\right)\right|\geq\frac{1}{\mathfrak{n}}\sum_{\iota=1}^{\mathfrak{n}}\left|\varsigma\left(\lambda,\mathfrak{V}_\iota,\frac{\mathfrak{u}}{2}\right)-\varsigma\left(\lambda,\mathfrak{V}_2,\frac{\mathfrak{u}}{2}\right)\right|>1-\tau.$$

Therefore $\varsigma(\mathfrak{V}_1-\mathfrak{V}_2,\mathfrak{u})\geq(1-\tau)\odot(1-\tau)>1-\varepsilon$.
Since $\varepsilon>0$ is arbitrary, we get $\varsigma(\mathfrak{V}_1-\mathfrak{V}_2,\mathfrak{u})=1$, for all $\mathfrak{u}>0$, which yields $\mathfrak{V}_1=\mathfrak{V}_2$. □

The following theorem can be proved using the standard techniques, so we state without proof.

Theorem 2.17. *Let* $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ *be an* NMS *and* ψ *be a* Ls. *Then*

(a) $[\mathfrak{W}\mathfrak{N}_\psi^{(\varsigma,\varrho,\nu)}]-\lim\mathfrak{V}_\iota=\mathfrak{V}$ *implies* $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim\mathfrak{V}_\iota=\mathfrak{V}$.

(b) $\{\mathfrak{V}_\iota\}\in l_\infty^{(\varsigma,\varrho,\nu)}$ *and* $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim\mathfrak{V}_\iota=\mathfrak{V}$ *implies* $[\mathfrak{W}\mathfrak{N}_\psi^{(\varsigma,\varrho,\nu)}-\lim\mathfrak{V}_\iota]=\mathfrak{V}$.

(c) $l_\infty^{(\varsigma,\varrho,\nu)}\cap\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}=l_\infty^{(\varsigma,\varrho,\nu)}\cap[\mathfrak{W}\mathfrak{N}_\psi^{(\varsigma,\varrho,\nu)}]$.

Definition 2.18. Let $(\Omega,\varsigma,\varrho,\nu,\odot,\oplus)$ be an *NMS*. A sequence $\{\mathfrak{V}_\iota\}$ is said to be *WLStCa* (or $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}$-Cauchy) with regards to the $NM(\varsigma,\varrho,\nu)$ if there is a subsequence $\{\mathfrak{V}_{\iota'(\mathfrak{q})}\}\in\mathfrak{I}_\mathfrak{q}$ for each $r,(\varsigma,\varrho,\nu)-\lim\mathfrak{V}_{\iota'(\mathfrak{q})}=\mathfrak{V}$ and for each $\varepsilon\in(0,1)$ and $\mathfrak{u}>0$,

$$\lim_{\mathfrak{q}\to\infty}\frac{1}{\mathfrak{v}_\mathfrak{q}}\left|\left\{\begin{array}{l}\iota\in\mathfrak{I}_\mathfrak{q}:\left|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\mathfrak{u})\right|\leq1-\varepsilon\\ \text{or }\ \left|\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\mathfrak{u})\right|\geq\varepsilon\\ \text{and }\ \left|\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\mathfrak{u})\right|\geq\varepsilon\end{array}\right\}\right|=0.$$

Theorem 2.19. *If a sequence* $\{\mathfrak{V}_\iota\}$ *is* WLStC *with regards to the* $NM(\varsigma,\varrho,\nu)$, *then it is WLStCa with regards to the* $NM(\varsigma,\varrho,\nu)$.

Proof. Let $\mathfrak{W}\mathfrak{T}_\psi^{(\varsigma,\varrho,\nu)}-\lim\mathfrak{V}_\iota=\mathfrak{V}$ and for each $\mathfrak{n}$ we write

$$\mathfrak{J}_\mathfrak{n}=\left\{\begin{array}{l}\iota\in\mathbb{N}:|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|>1-\frac{1}{\mathfrak{n}},\\ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|<1\\ \text{and }\ |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|<1\end{array}\right\}.$$

Then $\mathfrak{J}_{n+1}\subseteq\mathfrak{J}_\mathfrak{n}$ for each $\mathfrak{n}$ and $\lim_{\mathfrak{q}\to\infty}\frac{|\mathfrak{J}_\mathfrak{n}\cap\mathfrak{I}_\mathfrak{q}|}{\mathfrak{v}_\mathfrak{q}}=1$. So, there is $\mathfrak{m}_1$ such that $\mathfrak{q}\geq\mathfrak{m}_1$ and $\frac{|\mathfrak{J}_1\cap\mathfrak{I}_\mathfrak{q}|}{\mathfrak{v}_\mathfrak{q}}>0$ i.e. $\mathfrak{J}_1\cap\mathfrak{I}_\mathfrak{q}\neq\emptyset$. We next select $\mathfrak{m}_2>\mathfrak{m}_1$ such that $\mathfrak{q}\geq\mathfrak{m}_2$ implies $\mathfrak{J}_2\cap\mathfrak{I}_\mathfrak{q}\neq\emptyset$.
Then, for each $\mathfrak{q}$ satisfying $\mathfrak{m}_1\leq\mathfrak{q}\leq\mathfrak{m}_2$ we select $\iota'(\mathfrak{q})\in\mathfrak{I}_\mathfrak{q}$ such that $\iota'(\mathfrak{q})\in\mathfrak{J}_1\cap\mathfrak{I}_\mathfrak{q}$.
In general we choose $\mathfrak{m}_{n+1}>\mathfrak{m}_\mathfrak{n}$ such that $\mathfrak{q}\geq\mathfrak{m}_{n+1}$ implies $\iota'(\mathfrak{q})\in\mathfrak{J}_\mathfrak{n}\cap\mathfrak{I}_\mathfrak{q}$.
Thus, $\iota'(\mathfrak{q})\in\mathfrak{I}_\mathfrak{q}$ for each $\mathfrak{q}$ and

$$|\varsigma(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|>1-\frac{1}{\mathfrak{n}},$$

$$|\varrho(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|<\frac{1}{\mathfrak{n}}\ \text{ and}$$

$$|\nu(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|<\frac{1}{\mathfrak{n}}.$$

Hence, $(\varsigma,\varrho,\nu)-\lim \mathfrak{V}_{\iota'(\mathfrak{q})}=\mathfrak{V}$.

Using Theorem 2.5, it is easily seen that

$$\lim_{\mathfrak{q}\to\infty}\frac{1}{\mathfrak{v}_\mathfrak{q}}\left|\left\{\begin{array}{l}\iota\in\mathfrak{J}_\mathfrak{q}:\left|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma\left(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\mathfrak{u}\right)\right|\le 1-\varepsilon\\ \text{or}\ \ \left|\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho\left(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\mathfrak{u}\right)\right|\ge\varepsilon\\ \text{and}\ \ \left|\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho\left(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\mathfrak{u}\right)\right|\ge\varepsilon\end{array}\right\}\right|=0.$$

Conversely, assume that $\{\mathfrak{V}_\iota\}$ is $WLStCa$ sequence with regards to the $NM(\varsigma,\varrho,\nu)$. For $\varepsilon>0$ choose $\tau\in(0,1)$ such that $(1-\tau)\odot(1-\tau)>1-\varepsilon$ and $\tau\oplus\tau<\varepsilon$. Hence, for any $\mathfrak{u}>0$, take

$$\mathfrak{J}_{\varsigma,1}=\left\{\iota\in\mathbb{N}:\left|\varsigma\left(\lambda,\mathfrak{V}_\iota,\frac{\mathfrak{u}}{2}\right)-\varsigma\left(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\frac{\mathfrak{u}}{2}\right)\right|>1-\tau\right\}\ \text{ and}$$

$$\mathfrak{J}_{\varsigma,2}=\left\{\iota\in\mathbb{N}:\left|\varsigma\left(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\frac{\mathfrak{u}}{2}\right)-\varsigma\left(\lambda,\mathfrak{V},\frac{\mathfrak{u}}{2}\right)\right|>1-\tau\right\}.$$

Let $\mathfrak{J}_\varsigma=\mathfrak{J}_{\varsigma,1}\cap\mathfrak{J}_{\varsigma,2}$. Then $\Delta_\psi(\mathfrak{J}_\varsigma)=1$ and for $\iota\in\mathfrak{J}_\varsigma$,

$$\begin{aligned}|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|&\ge\left|\varsigma\left(\lambda,\mathfrak{V}_\iota,\frac{\mathfrak{u}}{2}\right)-\varsigma\left(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\frac{\mathfrak{u}}{2}\right)\right|\odot\left|\varsigma\left(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\frac{\mathfrak{u}}{2}\right)-\varsigma\left(\lambda,\mathfrak{V},\frac{\mathfrak{u}}{2}\right)\right|\\&>1-\varepsilon.\end{aligned}$$

Also, if we take

$$\mathfrak{J}_{\varrho,1}=\left\{\iota\in\mathbb{N}:\left|\varrho\left(\lambda,\mathfrak{V}_\iota,\frac{\mathfrak{u}}{2}\right)-\varrho\left(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\frac{\mathfrak{u}}{2}\right)\right|<\tau\right\}\ \text{ and}$$

$$\mathfrak{J}_{\varrho,2}=\left\{\iota\in\mathbb{N}:\left|\varrho\left(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\frac{\mathfrak{u}}{2}\right)-\varrho\left(\lambda,\mathfrak{V},\frac{\mathfrak{u}}{2}\right)\right|<\tau\right\}.$$

Let $\mathfrak{J}_\varrho=\mathfrak{J}_{\varrho,1}\cap\mathfrak{J}_{\varrho,2}$. Then $\Delta_\psi(\mathfrak{J}_\varrho)=1$ and for $\iota\in\mathfrak{J}_\varrho$,

$$|\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon.$$

Also, if we take

$$\mathfrak{J}_{\nu,1}=\left\{\iota\in\mathbb{N}:\left|\nu\left(\lambda,\mathfrak{V}_\iota,\frac{\mathfrak{u}}{2}\right)-\nu\left(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\frac{\mathfrak{u}}{2}\right)\right|<\tau\right\}\ \text{ and}$$

$$\mathfrak{J}_{\nu,2}=\left\{\iota\in\mathbb{N}:\left|\nu\left(\lambda,\mathfrak{V}_{\iota'(\mathfrak{q})},\frac{\mathfrak{u}}{2}\right)-\nu\left(\lambda,\mathfrak{V},\frac{\mathfrak{u}}{2}\right)\right|<\tau\right\}.$$

Let $\mathfrak{J}_\nu=\mathfrak{J}_{\nu,1}\cap\mathfrak{J}_{\nu,2}$. Then $\Delta_\psi(\mathfrak{J}_\nu)=1$ and for $\iota\in\mathfrak{J}_\nu$,

$$|\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon.$$

Therefore,

$$\lim_{\mathfrak{q}\to\infty}\frac{1}{\mathfrak{v}_\mathfrak{q}}\left|\left\{\begin{array}{l}\iota\in\mathfrak{J}_\mathfrak{q}:|\varsigma(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varsigma(\lambda,\mathfrak{V},\mathfrak{u})|>1-\varepsilon,\\ |\varrho(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\varrho(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon\\ \text{and}\ \ |\nu(\lambda,\mathfrak{V}_\iota,\mathfrak{u})-\nu(\lambda,\mathfrak{V},\mathfrak{u})|<\varepsilon\end{array}\right\}\right|=1.$$

Hence, $\{\mathfrak{V}_\iota\}$ is *WLStC* with regards to the $NM(\varsigma,\varrho,\nu)$. □

3. Conclusion

The *WStC* and *WLStC*-sequences of closed sets in the *NMS* have been examined. Although a few noteworthy findings were obtained. We looked at several fresh ways of combining concepts in *NMS*, namely for Strongly Cesaro summability and *WSC* sequences of closed sets. We analysed the limit of the *LStC* sequences of sets in *NMS* and found a few interesting findings. Although many of the results presented currently are consistent with the studies focus in the proper setting, proofs frequently hire an alternative strategy.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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