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Research Article

# Study of Waves Propagating in Anisotropic Homogeneous Microstretch Elastic Medium

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**Abstract.** The present study deals with the plane waves moving in a solid medium qualifying for anisotropic, homogeneous, microstretch and elastic properties. Primarily, the Christoffel equations have been derived for propagation of waves (coupled longitudinal and coupled transverse) in the medium. A system of homogeneous equations has been established to study polarization of medium particles for wave motion, polarization of medium particles in microrotation and microstretch present in the medium. Condition of solvability for a system of homogeneous linear equations has been applied to derive an equation for determining phase velocities of coupled waves propagating in the medium. Using the software *Mathematica* and hypothetical values for parameters and elastic constants, numerical discussion has been carried out to see the possible number of waves propagating in arbitrarily chosen phase directions in the medium. Finally, a special case of anisotropic homogeneous elastic medium (absence of microstretch) has been discussed to support the results derived in the present study.

**Keywords.** Microstretch, Microrotation, Phase velocity, Polarization, Coupled longitudinal waves, Coupled transverse waves

**Mathematics Subject Classification (2020).** 74J, 74H



## 1. Introduction

A medium can be described completely by considering its both macro and micro studies, structures, movements and properties. One such micro property is micropolar and microstretch. It was due to work of Eringen [4–6] published in 1990, 1999 and 2001 that theory on microstretch elastic solids came into existence. The theory of micropolar elastic solids was extended to develop this theory having three deformable directors associated with each and every material point. Although a wide range of materials are covered in micropolar continua theory, but many solids having microstretch elastic properties like polymers, asphalt, graphite, human bones, composite material with reinforced chopped elastic fibers etc. cannot be modelled by micropolar theory. The development of models for microstretch materials under different conditions could be possible only after the development of microstretch continuum theory. A few studies related to elastic, microstretch and wave motion at macro and micro level have been summarized in this section.

Casolo [3] in 2006 presented a model at macro level. Composite solids having texture of orthotropic nature were taken for modelling. Their in-plane elastic response was modelled microscopically in this study. Kiris and Inan [11] in 2008 used the microstretch theory introduced by Eringen and identified the upper bounds for microstretch elastic moduli of the vibrating plates. Then in 2011, Barsoum and Faleskog [2] analyzed the coalescence and growth of void in the presence of the Lode parameter at micro-mechanical scale. They developed a model showing a band comprised of an array squared in shape and made up of cells having equal size. Also, each cell was also embedded by a void of spherical shape.

Various categories of elastic media and behaviour of waves in these media have been analyzed by many researchers since many decades. Viscoelastic solids exhibit non-linear properties. By taking these materials as initially stressed and having constant temperature, a theory was developed by Schapery [19] in 1966. The theory provided quite simple stress-strain relations. In 2007, Garg [7] investigated harmonic wave motion in viscoelastic anisotropic continua. Biot's theory was used to observe the reaction of harmonic waves in terms of their attenuation and phase velocity under the influence of prestresses. Again in 2009, Garg [8] derived the results for primary waves propagating in anisotropic elastic solid under the influence of initial stress. Specific directions in which primary waves can exist were also suggested in this study. Both primary and secondary waves were analyzed by Sharma [20] in 2010 in media having anisotropic and thermoelastic properties. Wave propagation in microstretch solids has also been a research area of interest of many mathematicians and scientists. In 2006, Tomar and Singh [24] considered the interface comprised of two microstretch elastic half-spaces having different properties. At this interface, they examined Stoneley waves propagation and derived frequency equations for these waves. They found the waves to be dispersive in nature. Based on microstretch theory, Inan and Kiriş [10] in 2007 presented a model for rectangular plates under many boundary conditions. They investigated the propagation of waves by Ritz method and observed new waves as compared to waves described in classical theory of elasticity. Along with this, Chebyshev-Ritz method was also used to find frequency equations for microstretch plate and obtained additional frequencies compared with the values specified by classical theory of elasticity. Tomar and Khurana [23] in 2008 presented a study on electro-microelastic solid free from any stresses. Propagation of elastic waves on the plane boundary of this medium was studied. They investigated the condition along with the frequency range for electric waves existence in electro-microelastic infinite solid. They also noticed the effect of micro-stretch elastic parameter on coefficients of reflection.

Some studies are available in literature for thermo-elastic microstretch half-space which is homogeneous and isotropic in nature. Othman and Lotfy [17] in 2010 derived equations for this medium. Then in 2011, they [14] introduced the effect of rotation on the dynamical equations of this half space but with a thermal shock applied on the surface of medium. Xiong and Tian [25] in 2011 investigated the effect of a time oriented heat shock subjected on the surface of this semi-infinite medium. They concluded that microstretch affects the results related to displacements, stresses etc. to a great extent whereas micropolar slightly affects these results. In 2012, Shaw and Mukhopadhyay [21] studied that how Rayleigh surface waves get affected due to electromagnetic field present in this medium and obtained the governing equations for the waves. Then in 2014, Othman *et al*. [18] contributed for the effects of gravitational force and initial stress present in this medium. Recently, Singh and Goyal [22] in 2017 came up with a study on waves propagating in elastic solid falling in transverse, isotropic and microstretch material class. They derived results on various coupled waves existing in this medium. But these models cannot be applied for the general case of anisotropic medium. A few studies on specific microstretch solids came across while going through the literature. In 2015, on the basis of theory given by Green and Naghdi, a model was generated by Othman and Jahangir [16] for plane wave motion in microstretch elastic medium rotating with fixed angular frequency. One paper of interest published in 2017 and authored by Marin *et al*. [15] in this field with a different outlook was found during literature survey. The authors of this paper used some concepts of Hilbert space and semi-groups theory of linear operators to study the IBVP in microstretch solids. They showed that the solutions of IBVP in these solids continuously depend upon the initial values and supply data. This paper only provided a fact on solutions of the IBVP in microstretch solids but not described any approach to find these solutions.

In the present study, the research gaps namely, most general case of anisotropic elastic medium belonging to homogeneous and microstretch class, has been bridged over. Results on wave motion in simplified class of isotropic, homogeneous, microstretch continuum is available in literature, but not on microstretch anisotropic media. This motivated the authors to work on the objective of present study. The paper will definitely be a precious add-on in microstretch material research world.

## 2. Equations for Wave Motion

### 2.1 Assumptions

To study the dynamical wave equations, the following assumptions are made:

- The medium is homogeneous, anisotropic, microstretched and elastic by nature.
- No body force, initial stress and couple stress are acting on the medium.
- The medium is free from any intrinsic equilibrated body forces.
- A fixed system of rectangular Cartesian axes denoted by $x_i$-axes ($i = 1,2,3$) is used for study.
- Cartesian tensor notation is used in study.

## 2.2 Field Equations, Constitutive Relations, Symmetry Relations and Strain Tensors

In the medium under study, the following equations contribute for the fundamental system of dynamical equations for wave motion.

(a) *Equations of motion*:

$$t_{ji,j} = \rho \ddot{u}_i, \tag{2.1}$$

$$m_{ji,j}\varepsilon_{ijk}t_{jk} = I_{ij}\ddot{\psi}_j. \tag{2.2}$$

(b) *Equation for the balance of equilibrated forces*:

$$\lambda_{i,i} = \rho\kappa\ddot{\phi}. \tag{2.3}$$

The constitutive equations for the medium are given as

$$\left.\begin{aligned} t_{ij} &= A_{ijmn}\varepsilon_{mn} + B_{ijmn}\mu_{mn} + D_{ijk}\gamma_k, \\ m_{ij} &= B_{ijmn}\varepsilon_{mn} + C_{ijmn}\mu_{mn} + E_{ijk}\gamma_k, \\ \lambda_i &= D_{mni}\varepsilon_{mn} + E_{mni}\mu_{mn} + F_{ij}\gamma_j. \end{aligned}\right\} \tag{2.4}$$

Here, $u_i$ and $\psi_i$ represent the components of displacement vector and microrotation vector, respectively. The microstretch is characterized by the scalar function $\phi$. The components of the stress tensor, couple stress tensor and microstress vector have been denoted by $t_{ij}$, $m_{ij}$and $\lambda_i$, respectively. Parameters $\rho$, $I_{ij} = I_{ji}$ and $\kappa$ represent the reference constant mass density, coefficients of microinertia and equilibrated inertia, respectively. Einstein summation convention, subscript $j$ after comma and superposed dot have been used for repeated indices, partial derivative with respect to the spacial coordinate $x_j$ and derivative with respect to time, $t$ respectively. All the subscripts range from 1 to 3. The constitutive coefficients $A_{ijmn}$, $B_{ijmn}$, $C_{ijmn}$, $D_{ijk}$, $E_{ijk}$ and $F_{ij}$, characterizing the properties of the medium, satisfy the following symmetry relations:

$$\left.\begin{aligned} &A_{ijmn} = A_{mnij} = A_{jimn} = A_{ijnm}; \\ &C_{ijmn} = C_{mnij} = C_{jimn} = C_{ijnm}; \\ &D_{ijk} = D_{jik},\ E_{ijk} = E_{jik}, ;F_{ij} = F_{ji}. \end{aligned}\right\} \tag{2.5}$$

The coefficients $B_{ijmn}$ related to coupling of displacement and microrotation fields do not exhibit any kind of symmetry. The strain tensors giving deformation are defined as

$$\left.\begin{aligned} \varepsilon_{ij} &= u_{j,i} + \varepsilon_{ijk}\psi_k, \\ \mu_{ij} &= \psi_j, \\ \gamma_i &= \phi_{,i}, \end{aligned}\right\} \tag{2.6}$$

where $\varepsilon_{ijk}$ is the alternating tensor.

### Resulting Equations of Motion

Making use of equations (2.4) and (2.6) into equations (2.1), (2.2) and (2.3) resulted into a system of three coupled partial differential equations given as

$$\left.\begin{aligned} \rho\ddot{u}_i &= [A_{ijmn}(u_{n,m} + \varepsilon_{nmk}\psi_k) + B_{ijmn}\psi_{n,m} + D_{ijk}\phi_{,k}]_{,j}, \\ I_{ij}\ddot{\psi}_j &= [B_{ijmn}(u_{n,m} + \varepsilon_{nmk}\psi_k) + C_{ijmn}\psi_{n,m} + E_{ijk}\phi_{,k}]_{,j} \\ &\quad + \varepsilon_{ijk}[A_{jkmn}(u_{n,m} + \varepsilon_{nms}\psi_s) + B_{jkmn}\psi_{n,m} + D_{jkr}\phi_{,r}], \\ \rho\kappa\ddot{\phi} &= [D_{mni}(u_{n,m} + \varepsilon_{nms}\psi_s) + E_{mni}\psi_{n,m} + F_{ij}\phi_{,j}]_{,i}. \end{aligned}\right\} \tag{2.7}$$

In a homogeneous medium, these equations reduce to the form

$$\left.\begin{aligned}
&\rho\ddot{u}_i = A_{ijmn}(u_{n,mj}+\varepsilon_{nmk}\psi_{k,j})+B_{ijmn}\psi_{n,mj}+D_{ijk}\phi_{,kj}\,,\\
&I_{ij}\ddot{\psi}_j = B_{ijmn}(u_{n,mj}+\varepsilon_{nmk}\psi_{k,j})+C_{ijmn}\psi_{n,mj}+E_{ijk}\phi_{,kj}\\
&\qquad\quad +\varepsilon_{ijk}[A_{jkmn}(u_{n,m}+\varepsilon_{nms}\psi_s)+B_{jkmn}\psi_{n,m}+D_{jkr}\phi_{,r}],\\
&\rho\kappa\ddot{\phi} = D_{mni}(u_{n,mi}+\varepsilon_{nms}\psi_{s,i})+E_{mni}\psi_{n,mi}+F_{ij}\phi_{,ji}\,.
\end{aligned}\right\}\tag{2.8}$$

### 2.3 Condition for Existence and Uniqueness of Solution

The existence of unique solution of system (2.8) is guaranteed by the following conditions of positivity satisfied by parameters and constitutive coefficients [15]. For any $\xi_{ij}$, $\eta_{ij}$ and $\kappa_i$,

$$\left.\begin{aligned}
&\rho>0,\ I_{ij}>0,\ \kappa>0,\\
&A_{ijmn}\xi_{ij}\xi_{mn}+2B_{ijmn}\xi_{ij}\eta_{mn}+C_{ijmn}\eta_{ij}\eta_{mn}+2D_{ijs}\xi_{ij}\kappa_s+2E_{ijs}\eta_{ij}\kappa_s+F_{ij}\kappa_i\kappa_j\\
&\qquad \ge \alpha(\xi_{ij}\xi_{ij}+\eta_{ij}\eta_{ij}+\kappa_i\kappa_i),\quad \alpha>0.
\end{aligned}\right\}\tag{2.9}$$

## 3. Propagation of Plane Waves

The harmonic solution for the system of equations (2.8) is assumed as,

$$(u_j,\psi_j,\phi)=(U_j,\Psi_j,\Phi)\exp\left[\iota\omega\left(\frac{1}{v}n_kx_k-t\right)\right],\quad (j=1,2,3),\tag{3.1}$$

where wave propagates along $(n_1,n_2,n_3)$, a unit vector orthogonal to wave surface, with phase velocity $v$ and angular frequency $\omega$.

Represent the direction of phase propagation by a row vector $N=(n_1,n_2,n_3)$. Then use of harmonic solution (3.1) and symmetry relations (2.5) in equations of motion (2.8) results in the following system of equations,

$$\left.\begin{aligned}
&[E-\rho v^2\delta_{in}]U_n+H\Psi_n+G\Phi=0,\\
&HU_n+(L-v^2I)\Psi_n+M\Phi=0,\\
&\zeta U_n+\chi\Psi_n+(NFN^T-\rho\kappa v^2)\Phi=0,
\end{aligned}\right\}\tag{3.2}$$

where $E$, $H$, $L$, $I$, $F$ are square matrices of order 3, $G$ and $M$ are column vectors having three elements, and $\zeta$, $\chi$ are row vectors having 3 elements. These matrices represent the tensors given as,

$$\left.\begin{aligned}
&E=A_{ijmn}n_mn_j,\ H=B_{ijmn}n_mn_j,\ L=C_{ijmn}n_mn_j,\ I=I_{ij},\ NFN^T=F_{ij}n_jn_i,\\
&G=D_{ijk}n_kn_j,\ M=E_{ijmk}n_kn_j,\ \zeta=D_{mni}n_mn_i,\ \chi=E_{mni}n_mn_i.
\end{aligned}\right\}\tag{3.3}$$

Here, $n$ takes the values 1, 2 and 3. Representing the column vectors $(U_1,U_2,U_3)^T$ and $(\Psi_1,\Psi_2,\Psi_3)^T$ by $U$ and $\Psi$ respectively, the system of 7 equations (3.2) in 7 variables looks like a system of 3 homogeneous equations in 3 variables $U$, $\Psi$, and $\Phi$, given as

$$\left.\begin{aligned}
&[E-\rho v^2I]U+H\Psi+G\Phi=0,\\
&HU+(L-v^2I)\Psi+M\Phi=0,\\
&\zeta U+\chi\Psi+(NFN^T-\rho\kappa v^2)\Phi=0.
\end{aligned}\right\}\tag{3.4}$$

Here, $I$ is the identity matrix of order 3. For the condition $NFN^T-\rho v^2\kappa\neq 0$, this system is transformed into the following simpler system of three homogeneous equations in three variables $U_1$, $U_2$ and $U_3$ known as Christoffel equations for the propagation of waves (coupled longitudinal and coupled transverse waves) in the medium.

$$\left[\left(E-\rho v^2 I-\frac{G\zeta}{NFN^T-\rho v^2\kappa}\right)-\left(\frac{G\chi}{NFN^T-\rho v^2\kappa}-H\right)\left(L-v^2 I-\frac{M\chi}{NFN^T-\rho v^2\kappa}\right)^{-1}\right.$$
$$\left.\cdot\left(\frac{M\zeta}{NFN^T-\rho v^2\kappa}-H\right)\right]U=0. \tag{3.5}$$

The expressions for $\Psi$ and $\Phi$ in terms of $U$ obtained from system (3.4) are given as

$$\Psi=\left(L-v^2 I-\frac{M\chi}{NFN^T-\rho v^2\kappa}\right)^{-1}\left(\frac{M\zeta}{NFN^T-\rho v^2\kappa}-H\right)U \tag{3.6}$$

and

$$\Phi=\frac{-1}{NFN^T-\rho v^2\kappa}\left[\zeta+\chi\left(L-v^2 I-\frac{M\chi}{NFN^T-\rho v^2\kappa}\right)^{-1}\left(\frac{M\zeta}{NFN^T-\rho v^2\kappa}-H\right)\right]U. \tag{3.7}$$

Non-trivial solution of system (3.5) is guaranteed by the condition

$$\det\left[\left(E-\rho v^2 I-\frac{G\zeta}{NFN^T-\rho v^2\kappa}\right)-\left(\frac{G\chi}{NFN^T-\rho v^2\kappa}-H\right)\left(L-v^2 I-\frac{M\chi}{NFN^T-\rho v^2\kappa}\right)^{-1}\right.$$
$$\left.\cdot\left(\frac{M\zeta}{NFN^T-\rho v^2\kappa}-H\right)\right]=0, \tag{3.8}$$

where det stands for the determinant of a square matrix. Equation (3.8) is a polynomial equation in $v^2$. Positive square roots of real solutions of this polynomial equation give the phase velocities of the existing waves in the medium. The phase velocities when used in the system (3.5) make it possible to evaluate $(U_1,U_2,U_3)$, the polarization of particles of the medium for propagation of corresponding wave. The corresponding polarization of the particles in microrotation, $(\Psi_1,\Psi_2,\Psi_3)$ and the microstretch present in the wave are then calculated by the relations (3.6) and (3.7), respectively.

## 4. Special Case: Absence of Microstretch

In the absence of microstretch, the coefficients characterizing the microstretch properties of the medium vanish given as

$$B_{ijmn}=C_{ijmn}=D_{ijk}=E_{ijk}=F_{ij}=0 \tag{4.1}$$

which results in the following:

$$G=H=L=M=K=\zeta=\chi=0. \tag{4.2}$$

As a result, the Christoffel equations for the propagation of plane waves in anisotropic homogeneous elastic medium will be

$$(E-\rho v^2 I)U=0 \tag{4.3}$$

which witness three plane waves propagating in the medium. This system being an eigen system of a symmetric matrix $E$ confirms that the polarizations of these three waves are mutually orthogonal, which is in agreement with the results for anisotropic medium available in literature. On the contrary, due to absence of this type of symmetry in matrices present in system (3.5), the displacement vectors corresponding to each wave in anisotropic homogeneous microstretch elastic medium may not be mutually orthogonal in an arbitrary phase direction.

## 5. Numerical Discussion

Due to non-availability of micromechanics experimental data for anisotropic homogeneous microstretch elastic medium in literature, hypothetical values for parameters and constitutive coefficients following some similar literature works [1, 9, 12, 13, 22] satisfying the positivity assumptions (2.9), are taken to evaluate the phase velocities of waves. The elastic constants are taken in GPa. Elastic moduli of the medium in two-suffixed notations are denoted as, $A_{ijmn}$ by a symmetric square matrix $A = A_{IJ}$ of order 6, $B_{ijmn}$ by a square matrix $B = B_{MN}$ of order 9, and $C_{ijmn}$ by a symmetric square matrix $C = C_{IJ}$ of order 6.

Also, represent the coefficients of microinertia, $I_{ij}$ by a symmetric square matrix $MI$ of order 3.

*Parameter values*:

$$\rho = 2.74\times 10^3\,\text{kg/m}^3,\quad \kappa = 0.279\,\text{m}^2;$$

*Matrix MI* (*in* m$^2$):

$$MI = \begin{bmatrix} 0.793 & 0.072 & 0.112 \\ 0.072 & 0.572 & 0.012 \\ 0.112 & 0.012 & 0.543 \end{bmatrix};$$

*Elastic constants matrices/values* (*in* GPa):

$$A = \begin{bmatrix} 17.8 & 5.23 & 7.59 & 6.94 & 8.23 & 7.99 \\ 5.23 & 9.7 & 9.76 & 7.69 & 8.97 & 4.42 \\ 7.59 & 9.76 & 18.43 & 5.97 & 7.97 & 7.79 \\ 6.94 & 7.69 & 5.97 & 7.62 & 9.56 & 4.79 \\ 8.23 & 8.97 & 7.97 & 9.56 & 4.357 & 1.89 \\ 7.99 & 8.94 & 7.79 & 4.79 & 1.89 & 4.42 \end{bmatrix};$$

$$B = \begin{bmatrix} 2.14 & 2.11 & 2.23 & 1.97 & 1.74 & 0.192 & 0.211 & 0.199 & 0.212 \\ 2.16 & 1.98 & 1.75 & 2.13 & 2.31 & 0.179 & 0.178 & 0.187 & 0.198 \\ 2.25 & 1.78 & 2.15 & 2.21 & 2.23 & 0.201 & 0.203 & 0.15 & 0.13 \\ 1.99 & 2.16 & 2.25 & 2.75 & 2.95 & 0.16 & 0.11 & 0.121 & 0.131 \\ 1.77 & 2.36 & 2.25 & 2.92 & 2.13 & 0.09 & 0.08 & 0.19 & 0.20 \\ 0.19 & 0.17 & 0.207 & 0.19 & 0.07 & 2.15 & 0.097 & 0.08 & 0.07 \\ 0.23 & 0.18 & 0.20 & 0.13 & 0.07 & 0.09 & 2.22 & 0.051 & 0.17 \\ 0.20 & 0.19 & 0.16 & 0.125 & 0.197 & 0.088 & 0.054 & 2.17 & 0.25 \\ 0.215 & 0.195 & 0.139 & 0.133 & 0.199 & 0.077 & 0.175 & 0.255 & 2.21 \end{bmatrix};$$

$$C = \begin{bmatrix} 0.023 & 0.012 & 0.013 & 0.0019 & 0.00191 & 0.0018 \\ 0.012 & 0.021 & 0.017 & 0.0018 & 0.0009 & 0.0011 \\ 0.013 & 0.017 & 0.0201 & 0.00182 & 0.00191 & 0.00193 \\ 0.0019 & 0.0018 & 0.00182 & 0.0201 & 0.00187 & 0.0009 \\ 0.0019 & 0.0009 & 0.00191 & 0.00187 & 0.0241 & 0.0008 \\ 0.0018 & 0.0011 & 0.00193 & 0.0009 & 0.0008 & 0.0221 \end{bmatrix};$$

$D_{111} = 0.0173$, $D_{112} = 0.0165$, $D_{113} = 0.0101$, $D_{121} = 0.0091$, $D_{122} = 0.0087$, $D_{123} = 0.0075$,

$D_{131} = 0.0054$, $D_{132} = 0.0097$, $D_{133} = 0.0099$, $D_{211} = D_{121}$, $D_{212} = D_{122}$, $D_{213} = D_{123}$,

$D_{221} = 0.0078$, $D_{222} = 0.0077$, $D_{223} = 0.0073$, $D_{231} = 0.0087$, $D_{232} = 0.0089$, $D_{232} = 0.0088$,

$D_{311} = D_{131}$, $D_{312} = D_{132}$, $D_{313} = D_{133}$, $D_{321} = D_{231}$, $D_{322} = D_{232}$, $D_{323} = D_{233}$,

$D_{331} = 0.0165$, $D_{332} = 0.0176$, $D_{333} = 0.0191$;

$E_{111} = 0.00021$, $E_{112} = 0.00013$, $E_{113} = 0.00017$, $E_{121} = 0.00016$, $E_{122} = 0.0001$,

$E_{123} = 0.00023$, $E_{131} = 0.00019$, $E_{132} = 0.00018$, $E_{133} = 0.0002$;

$E_{211} = E_{121}$, $E_{212} = E_{122}$, $E_{213} = E_{123}$,

$E_{221} = 0.00024$, $E_{222} = 0.00015$, $E_{223} = 0.00011$, $E_{231} = 0.00009$, $E_{232} = 0.00007$,

$E_{233} = 0.00024$, $E_{311} = E_{131}$, $E_{312} = E_{132}$, $E_{313} = E_{133}$, $E_{321} = E_{231}$, $E_{322} = E_{232}$,

$E_{323} = E_{233}$, $E_{331} = 0.00007$, $E_{332} = 0.0005$, $E_{333} = 0.00022$;

$F_{11} = 0.003$, $F_{12} = 0.0009$, $F_{13} = 0.0007$,

$F_{21} = F_{12}$, $F_{22} = 0.002$, $F_{23} = 0.0008$,

$F_{31} = F_{13}$, $F_{32} = F_{23}$, $F_{33} = 0.004$.

Using the above hypothetical values and the software *Mathematica*, the obtained positive squared phase velocity values calculated in arbitrarily chosen phase directions and hence the possible number of waves propagating in those directions are given as,

- If $\theta = 0$ and $\phi = 0$ to $\frac{\pi}{2}$, then $v^2$:

| | | | | | |
|---|---|---|---|---|---|
| 0.0864533 | 0.0404048 | 0.0404048 | 0.00101941 | $5.27856\times10^{-6}$ | $5.23245\times10^{-6}$ |
| $5.23245\times10^{-6}$ | $5.23245\times10^{-6}$ | $5.23245\times10^{-6}$ | $5.23245\times10^{-6}$ | $5.23245\times10^{-6}$ | |

  Number of possible waves: 11 distinct waves.
  Along the directions of zero polar angle and arbitrary azimuth angle, eleven waves came into existence. In these phase directions, five types of waves have been observed, categorized according to their squared phase velocity.

- If $\theta = 0.2$ to $\frac{\pi}{2}$ and $\phi = 0$ to $\frac{\pi}{2}$, then $v^2$: no positive real value.

  Number of possible waves: no wave.
  For arbitrary azimuth, if we go on increasing the polar angle beyond a point (in the present case, this limit point exists between 0.1 and 0.2 radians), then no wave is found in these phase directions.

- If $\theta = 0.1$ and $\phi = 0$, then $v^2$:

| | | | | |
|---|---|---|---|---|
| 0.102444 | 0.0399805 | 0.0399805 | 0.0342426 | 0.0341956 |
| 0.0341956 | 0.0294655 | 0.0290779 | 0.0290779 | $5.40133\times10^{-6}$ |
| $5.40133\times10^{-6}$ | $5.40133\times10^{-6}$ | $5.39227\times10^{-6}$ | $5.39227\times10^{-6}$ | $4.64544\times10^{-6}$ |

  Number of possible waves: 15 distinct waves.
  For 0.1 radians polar angle and zero azimuth, nine types of waves come into existence.

- If $\theta = 0.1$ and $\phi = 0.5$, then $v^2$:

| | | | | |
|---|---|---|---|---|
| 0.110076 | 0.0431058 | 0.0431058 | 0.0328957 | 0.0321738 |
| 0.0321738 | 0.0291699 | 0.0286164 | 0.0286164 | $5.48561\times10^{-6}$ |
| $5.48561\times10^{-6}$ | $5.48561\times10^{-6}$ | $5.4758\times10^{-6}$ | $5.4758\times10^{-6}$ | $5.38626\times10^{-6}$ |

Number of possible waves: 15 distinct waves.
In this phase direction also, nine types of waves appear.

- If $\theta = 0.1$ and $\phi = 1$, then $v^2$:

| | | | | |
|---|---|---|---|---|
| 0.112018 | 0.0454911 | 0.0454911 | 0.03177 | 0.0309273 |
| 0.0309273 | 0.02926 | 0.0280515 | 0.0280515 | $5.49409\times10^{-6}$ |
| $5.49409\times10^{-6}$ | $5.49409\times10^{-6}$ | $5.48401\times10^{-6}$ | $5.48401\times10^{-6}$ | $5.41458\times10^{-6}$ |

Number of possible waves: 15 distinct waves.
This phase direction also witnesses nine types of waves.

- If $\theta = 0.1$ and $\phi = 1.5$, then $v^2$:

| | | | | | |
|---|---|---|---|---|---|
| 0.107855 | 0.0468969 | 0.468969 | 0.0307044 | 0.0307044 | 0.0272652 |
| 0.0272652 | $5.42835\times10^{-6}$ | $5.42835\times10^{-6}$ | $5.42835\times10^{-6}$ | $5.28099\times10^{-6}$ | |

Number of possible waves: 11 distinct waves.
In this phase direction, six types of waves make their propagation path.

- If $\theta = 0.01$ and $\phi = 0$, then $v^2$:

| | | | | |
|---|---|---|---|---|
| 0.0880305 | 0.0401376 | 0.0401376 | 0.0300389 | 0.0300389 |
| 0.00096106 | $5.30093\times10^{-6}$ | $5.25063\times10^{-6}$ | $5.25063\times10^{-6}$ | $5.25063\times10^{-6}$ |

Number of possible waves: 10 distinct waves.
Six types of waves follow this phase direction.

- If $\theta = 0.01$ and $\phi = 0.5$, then $v^2$:

| | | | | | |
|---|---|---|---|---|---|
| 0.088821 | 0.0404644 | 0.0404644 | 0.0300135 | 0.0300135 | 0.000937967 |
| $5.31015\times10^{-6}$ | $5.25849\times10^{-6}$ | $5.25849\times10^{-6}$ | $5.25849\times10^{-6}$ | $5.25054\times10^{-6}$ | $5.25054\times10^{-6}$ |

Number of possible waves: 12 distinct waves.
In this phase direction, seven types of waves can be seen.

- If $\theta = 0.01$ and $\phi = 1$, then $v^2$:

| | | | | |
|---|---|---|---|---|
| 0.0890333 | 0.0407724 | 0.0407724 | 0.0299115 | 0.0299115 |
| 0.000936141 | $5.31138\times10^{-6}$ | $5.25984\times10^{-6}$ | $5.25984\times10^{-6}$ | $5.25984\times10^{-6}$ |
| $5.25984\times10^{-6}$ | $5.25984\times10^{-6}$ | $5.25187\times10^{-6}$ | $5.25187\times10^{-6}$ | |

Number of possible waves: 14 distinct waves.
Seven types of waves prefer to propagate in this phase direction.

- If $\theta = 0.01$ and $\phi = 1.5$, then $v^2$:

| | | | |
|---|---|---|---|
| 0.0886175 | 0.0297527 | 0.0297527 | 0.00095655 |
| $5.30426\times10^{-6}$ | $5.24644\times10^{-6}$ | $5.24644\times10^{-6}$ | |

Number of possible waves: 7 distinct waves.
This phase direction observes five types of waves.

## 6. Conclusion

In an anisotropic homogeneous microstretch elastic solid medium, the following conclusions may be stated based upon the theoretical and numerical discussion:

- Different phase directions may witness different number of waves propagating in the medium.
- There may exist some phase directions where not a single wave propagates.
- Number of distinct waves propagating in different phase directions may vary from 0 to 15.
- There is possibility of nine types of waves propagating in the medium.
- Coupling of microstretch and microrotation fields with displacement fields may be responsible for affecting the number of distinct waves and number of types of waves propagating in different phase directions.
- The results derived in the present study will definitely be of great utility for real experimental data based problems in an anisotropic homogeneous microstretch elastic solid medium.

## Competing Interests

The authors declare that they have no competing interests.

## Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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