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Research Article

Finite Population and Finite Capacity Single Server Batch Service Queue With Single Vacation and Impatient

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Abstract. Two single server queues are considered in this paper. The models are (i) Finite source and (ii) Finite waiting line models. For both the models, if there are no customers at a service completion epoch, the server takes single vacation. Also, the waiting customers may become impatient and leaves the queue without getting service called reneging behaviour of the customer. In addition the services are given in batches of fixed service. It is assumed that the inter arrival times, service times, vacation times and reneging times all follows different exponential distributions. Steady state analysis is carried out for both models. Cost and profit analyses are also provided. The two models are compared numerically.

Keywords. Finite population queue, Batch service, Steady state probability, Performance measures, Single vacation

Mathematics Subject Classification (2020). 90B22, 60K25, 60K30

1. Introduction

The population of potential customers, called calling population. The calling population can be finite or infinite. In an infinite population model, the arrival rate is not affected by the number of customers in the system. Whereas, the arrival rate of finite population model, the arrival rate is affected by the population size. In an infinite population size model, the number of arrival can be controlled by restricting the queue size. If the queue size is finite, then the model is finite capacity model. In this paper, we consider both the models. Finite source queueing system has attention in terms of machine repairman problem (Stecke [25], Ke and Wang [17]). Definition of finite source models with analytical results and literature review can be found in Sztrik [27]. For solving machine interference problem, homogeneous $M/M/r//N$ queueing system has been used by many authors. It should be noted that the analysis of the system is presented in Kleinrock [18], Gross *et al*. [11], Carmichael [3], and Allen [1].

Satty [23] analyzed finite population model, called machine interference model. Queue with infinite population have been developed and analyzed by many authors. But relatively few researches worked on finite population queue. In 1975, Kleinrock considered the finite population model $M/M/c//m$. An application of the model $M/M/N//N$ in electronics is considered by Konigsberg [19]. The $M/G/1//N$ system, some times called machine interference problem (Satty [23]) is analyzed in detail by Takacs [28], Jaiswal [14], Cooper [7], Stecke [25], Stecke and Aronson [24], and Bunday [2], in their papers mainly concentrate on cyclic queues and manufacturing system modelling. Some other notable works in this area are Papadopoulos and Heavey [22], Syski [26], Ching [6], and Chakravarthy and Agarwal [4]. A detail survey of the machine interference problem has been given by Haque and Armstrong [12]. Jain [13] obtained the product form solution for the stationary state distribution for the finite population queueing model with a queue dependent servers. Queueing models with batch service have been extensively studied by many researches (see Chaudhry and Templetan [5], Medhi [20, 21] and Dshalalow [10]). In transportation problems, one can note that the service is in bulk. Relatively few works are cited in this area. Notable among them are, Kalyanaraman and Saritha [16] have analyzed a finite population model with bulk service rule and with accessible and non-accessible batches. Kalyanaraman and Saritha [15] have studied a multiserver batch service queue with delay and with finite population.

In a queueing system, if there are no customers waiting for service, if the server has the option to leaves the service station for a random period of time called vacation period. Vacation models are very reasonable models by seeing real life situations. Many modifications vacation period are defined and analyzed by researchers. The importance of vacation models has been established in survey of Doshi [8, 9], Teghem [31] and also in the monograph of Takagi [29]. At a service completion epoch, if there are no customers in the system, the server takes vacation of random period. After completion of this random period, the server returns to the system without considering the number of customers in the system. This vacation policy is called single vacation policy. Zhang and Tian [32] considered an $M/M/C$ queue with a single vacation. Zhang and Tian [33] analyzed an $M/M/C$ queue as a quasi-birth-death process.

The content of this article is a finite source Markovian queue with single vacation. Based on service rule two models are defined. The models are analyzed in sections 2 and 3. Some numerical results are provided in section 4. Cost and profit model are defined in section 5. A conclusion is given in section 6.

2. The Finite Population Queue and Analysis

A single server finite population model with the following assumption have been considered:

(i) The arrival follows Poisson process.

(ii) The service time distribution is negative exponential.

(iii) The services are given in batches of size K. At a service completion point, if there are less than K customer in the queue the server waits, that is server becomes idle until, the size reaches K.

(iv) At the completion of a service, if there are no customer in the queue the server takes vacation of duration V, which follows exponential distribution with rate θ.

(v) A customer waiting for service may get impatient due to delay and decide to renege from the queue. Reneging times are exponential with rate α. The average reneging rate is $(n-1)\alpha$ for $n \geq 1$.

(vi) The population size is N.

(vii) The arrival rate is $\lambda_j = \begin{cases} (N-j)\lambda, & j = 0,1,2,\dots,N-1, \\ 0, & j = N, N+1, \dots. \end{cases}$

(viii) The service rate is $\mu_j = \begin{cases} \mu, & j = 1,2,\dots,LK, \\ \mu_1, & j = LK+1,\dots,N. \end{cases}$

That is, up to L batches the service rate is μ and after L batches the service rate is μ_1.

(ix) The queue discipline is First in First out.

2.1 The Queue Length Distribution

To analyze the queueing model we define the following notations: Let $X(t)$ be the number of customers in the system at time t, $Y(t)$ be the number of customers in the source at time t, $N(t) = X(t) + Y(t)$and $Z(t)$ be the server state at time t where

$$Z(t) = \begin{cases} 1, & \text{the server is in busy state,} \\ 2, & \text{the server is in single vacation state.} \end{cases}$$

The process $\{(X(t), Z(t)) : t \geq 0\}$ is a finite Markov process with state space $S = \{(i,1) : i = 0,1, 2,\dots,N\} \cup \{(i,2) : i = 1,2,\dots,N\}$. Let $p_{ij}(t) = \Pr\{X(t) = i, Z(t) = j\}$ be the corresponding probability distribution. The differential difference equations satisfied by $p_{ij}(t)$ are obtained, using birth death arguments as

$$\frac{d}{dt} p_{01}(t) = -N\lambda p_{01}(t) + \mu p_{K1}(t), \tag{2.1}$$

$$\frac{d}{dt} p_{j1}(t) = -(N-j)\lambda p_{j1}(t) + \mu p_{K+j1}(t) + (N-j+1)\lambda p_{j-11} + \theta p_{j2}(t); \quad j = 1,2,\dots,K-1, \tag{2.2}$$

$$\frac{d}{dt} p_{j1}(t) = -(\mu + (N-j)\lambda) p_{j1}(t) + \mu p_{K+j1}(t) + (N-j+1)\lambda p_{j-11}(t) + \theta p_{j2}(t);$$
$$j = K, K+1, \dots, LK-1, \tag{2.3}$$

$$\frac{d}{dt} p_{LK1}(t) = -(\mu + (N-LK)\lambda) p_{LK1}(t) + \mu_1 p_{K+LK1}(t) + (N-LK+1)\lambda p_{LK-11}(t) + \theta p_{LK2}(t), \tag{2.4}$$

$$\frac{d}{dt}p_{j1}(t) = -(\mu_1 + (N-j)\lambda)p_{j1}(t) + \mu_1 p_{K+j1}(t) + (N-j+1)\lambda p_{j-11}(t) + \theta p_{j2}(t);$$
$$j = LK+1, LK+2, \dots, N-K, \tag{2.5}$$

$$\frac{d}{dt}p_{j1}(t) = -(\mu_1 + (N-j)\lambda)p_{j1}(t) + (N-j+1)\lambda p_{j-11}(t) + \theta p_{j2}(t);$$
$$j = N-K+1, N-K+2, \dots, N-1 \tag{2.6}$$

$$\frac{d}{dt}p_{N1}(t) = -\mu_1 p_{N1}(t) + \lambda p_{N-11}(t) + \theta p_{N2}(t), \tag{2.7}$$

$$\frac{d}{dt}p_{j2}(t) = -(\theta + (N-j)\lambda + (j-1)\alpha)p_{j2}(t) + (N-j+1)\lambda p_{j-12}(t) + j\alpha p_{j+12}(t);$$
$$j = 1, 2, \dots, N-1, \tag{2.8}$$

$$\frac{d}{dt}p_{N2}(t) = -(\theta + (N-1)\alpha)p_{N2}(t) + \lambda p_{N-12}(t). \tag{2.9}$$

In steady state, $\lim_{n\to\infty} p_n(t) = p_n$, the equations (2.1) to (2.9) becomes

$$N\lambda p_{01} = \mu p_{K1}, \tag{2.10}$$

$$(N-j)\lambda p_{j1} = \mu p_{K+j1} + (N-j+1)\lambda p_{j-11} + \theta p_{j2}; \quad j = 1, 2, \dots, K-1 \tag{2.11}$$

$$(\mu + (N-j)\lambda)p_{j1} = \mu p_{K+j1} + (N-j+1)\lambda p_{j-11} + \theta p_{j2}; \quad j = K, K+1, \dots, LK-1, \tag{2.12}$$

$$(\mu + (N-LK)\lambda)p_{LK1} = \mu_1 p_{K+LK1} + (N-LK+1)\lambda p_{LK-11} + \theta p_{LK2}, \tag{2.13}$$

$$(\mu_1 + (N-j)\lambda)p_{j1} = \mu_1 p_{K+j1} + (N-j+1)\lambda p_{j-11} + \theta p_{j2}; \quad j = LK+1, \dots, N-K, \tag{2.14}$$

$$(\mu_1 + (N-j)\lambda)p_{j1} = (N-j+1)\lambda p_{j-11} + \theta p_{j2}; \quad j = N-K+1, \dots, N-1, \tag{2.15}$$

$$\mu_1 p_{N1} = \lambda p_{N-11} + \theta p_{N2}, \tag{2.16}$$

$$(\theta + (N-j)\lambda + (j-1)\alpha)p_{j2} = (N-j+1)\lambda p_{j-12} + j\alpha p_{j+12}; \quad j = 1, 2, \dots, N-1, \tag{2.17}$$

$$(\theta + (N-1)\alpha)p_{N2} = \lambda p_{N-12}. \tag{2.18}$$

The equations (2.10) to (2.18)are recursively solved and the probabilities are obtained as

$$p_{N-i2} = B''_i p_{N2}; \quad i = 1, 2, 3, \dots, N, \tag{2.19}$$

$$p_{N-i1} = (B_i - C'_i)p_{N2}; \quad i = 1, 2, 3, \dots, K, \tag{2.20}$$

$$p_{N-i1} = (C_i - D_i)p_{N2}; \quad i = K+1, K+2, \dots, 2K+1, \tag{2.21}$$

$$p_{N-i1} = (E_i - E'_i)p_{N2}; \quad i = 2K+2, 2K+3, \dots, N-LK, \tag{2.22}$$

$$\begin{aligned} P_{LK-11} = \Bigg\{ & \frac{\mu + (N-LK)\lambda}{(N-LK+1)\lambda}(C_{N-LK} - A_{N-(LK+1)}C_{N-(K+LK+1)} - A_{N-LK}C_{N-(K+LK)} \\ & - D_{N-LK} + A_{N-(LK+1)}D_{N-(K+LK+1)} - A''_{N-LK}B''_{LK+1}) \\ & - A_{N-LK}D_{N-(K+LK)} + A''_{N-LK}B''_{LK}\Bigg\} p_{N2}, \end{aligned} \tag{2.23}$$

$$p_{N-i1} = (H_i - H'_i)p_{N2}; \quad i = N-LK+2, N-LK+3, \dots, N-K+1, \tag{2.24}$$

$$\begin{aligned} p_{i1} = & \frac{N-(i+1)}{N-i}[(H_{N-(i+1)} - A'_{N-(i+1)}H_{N-(K+i+1)}) - H'_{N-(i+1)} + A'_{N-(i+1)}H'_{N-(K+i+1)} \\ & + A''_{N-(i+1)}B'_{i+1}]p_{N2}; \quad i = K-1, K-2, \dots, 0, \end{aligned} \tag{2.25}$$

where

$$B_i = \prod_{j=1}^{i} \frac{(j-1)\lambda + \mu_1}{j\lambda}, \tag{2.26}$$

$$B'_i = \prod_{j=1}^{i} \frac{(j-1)\lambda + \theta}{j\lambda}, \tag{2.27}$$

$$B''_i = \prod_{j=1}^{i} \frac{(N-j)\alpha + (j-1)\lambda + \theta}{j\lambda} - \frac{(N-(i-1))\alpha}{i\lambda}, \tag{2.28}$$

$$C_i = B_i - \sum_{l=K+1}^{i-1} \prod_{j=1}^{i-l} \frac{\mu_1 + (i-j)\lambda}{(i-(j-1))\lambda} A_{l-1} B_{l-(K+1)} - A_{i-1} B_{i-(K+1)}, \tag{2.29}$$

$$C'_i = \prod_{j=1}^{i-1} \frac{\mu_1 + (i-j)\lambda}{(i-j+1)\lambda} B'_1 + \sum_{l=2}^{i-1} \prod_{j=1}^{i-l} \frac{\mu_1 + (i-j)\lambda}{(i-j+1)\lambda} A''_{l-1} B''_{l-1} + A''_{i-1} B''_{i-1}, \tag{2.30}$$

$$\begin{aligned} D_i = C'_i - \prod_{j=1}^{i-(K+2)} \frac{\mu_1 + (i-j)\lambda}{(i-j+1)\lambda} A_{K+1} B''_1 \\ - \sum_{l=K+3}^{i-1} \prod_{j=1}^{i-l} \frac{\mu_1 + (i-j)\lambda}{(i-(j-1))\lambda} A''_{l-1} C'_{l-(K+1)} - A''_{i-1} C'_{i-(K+1)}, \end{aligned} \tag{2.31}$$

$$E_i = C_i - A_{i-1} C_{i-(K+1)}, \tag{2.32}$$

$$E'_i = D_i - A_{i-1} D_{i-(K+1)} + A''_i B''_i, \tag{2.33}$$

$$\begin{aligned} H_i = \prod_{j=1}^{i-(N-LK)} \frac{\mu + (i-j)\lambda}{(i-(j-1))\lambda} C_{N-LK} - A_{N-(LK+1)} C_{N-(K+LK+1)} \\ - \prod_{j=1}^{i-(N-LK+1)} \frac{\mu + (i-j)\lambda}{(i-(j-1))\lambda} A_{N-LK} C_{N-(K+LK)} \\ - \sum_{m=N-LK+2}^{i-1} \prod_{j=1}^{i-m} \frac{\mu + (i-j)\lambda}{(i-(j-1))\lambda} A'_{m-1} C_{(m-1)-K} - A'_{i-1} C_{(i-1)-K}, \end{aligned} \tag{2.34}$$

$$\begin{aligned} H'_i = \prod_{j=1}^{i-(N-LK)} \frac{\mu + (i-j)\lambda}{(i-(j-1))\lambda} D_{N-LK} - A_{N-(LK+1)} D_{N-(K+LK+1)} \\ - \prod_{j=1}^{i-(N-LK+1)} \frac{\mu + (i-j)\lambda}{(i-(j-1))\lambda} (A_{N-LK} D_{N-(K+LK)} + A''_{N-LK} B''_{LK+1}) \\ - \sum_{m=N-LK+2}^{i-1} \prod_{j=1}^{i-n} \frac{\mu + (i-j)\lambda}{(i-(j-1))\lambda} (A'_{n-1} D_{(n-1)-K} + A''_{n-1} B''_{n-N}) + A''_{i-1} B''_{N-(i-1)}, \end{aligned} \tag{2.35}$$

$$A_i = \frac{\mu_1}{(i+1)\lambda}, \tag{2.36}$$

$$A'_i = \frac{\mu}{(i+1)\lambda}, \tag{2.37}$$

$$A''_i = \frac{\theta}{i\lambda}. \tag{2.38}$$

Using the normalization condition $\sum_{i=0}^{N} p_{i1} + \sum_{i=1}^{N} p_{i2} = 1$, we can obtain p_{N2} as,

$$p_{N2} = \Big[1 + \sum_{i=0}^{L-2} \frac{N-(i+1)}{N-i}(H_{N-(i+1)} - A'_{N-(i+1)}H_{N-(K+i+1)} - H'_{N-(i+1)} + A'_{N-(i+1)}H'_{N-(K+i+1)} + A''_{N-(i+1)}B'_{i+1}) + \sum_{i=L-1}^{L+K-2}(H_i - H'_i) + \sum_{i=L+K-1}^{N-2K-2}(E_i - E'_i) + \sum_{i=N-2K-1}^{N-K-1}(C_i - D_i) + \sum_{i=N-K}^{N-1}(B_i - C'_i) + \sum_{i=0}^{N-1} B''_i\Big]^{-1} p_{N2}. \tag{2.39}$$

Equations (2.19) to (2.39) represents the steady state probabilities of the model discussed in this paper.

Theorem 2.1. *The steady state probabilities for the model discussed in this section are*

$$p_{N-i2} = B''_i p_{N2}; \quad i = 1,2,3,\dots,N,$$
$$p_{N-i1} = (B_i - C'_i)p_{N2}; \quad i = 1,2,3,\dots,K,$$
$$p_{N-i1} = (C_i - D_i)p_{N2}; \quad i = K+1, K+2,\dots,2K+1,$$
$$p_{N-i1} = (E_i - E'_i)p_{N2}; \quad i = 2K+2, 2K+3,\dots,N-LK,$$
$$P_{LK-11} = \Big[\frac{\mu + (N-LK)\lambda}{(N-LK+1)\lambda}(C_{N-LK} - A_{N-(LK+1)}C_{N-(K+LK+1)} - A_{N-LK}C_{N-(K+LK)} - D_{N-LK} + A_{N-(LK+1)}D_{N-(K+LK+1)} - A''_{N-LK}B''_{LK+1}) - A_{N-LK}D_{N-(K+LK)} + A''_{N-LK}B''_{LK}\Big] p_{N2},$$
$$p_{N-i1} = (H_i - H'_i)p_{N2}; \quad i = N-LK+2, N-LK+3,\dots,N-K+1,$$
$$p_{i1} = \frac{N-(i+1)}{N-i}[(H_{N-(i+1)} - A'_{N-(i+1)}H_{N-(K+i+1)}) - H'_{N-(i+1)} + A'_{N-(i+1)}H'_{N-(K+i+1)} + A''_{N-(i+1)}B'_{i+1}]p_{N2}; \quad i = K-2, K-1,\dots,0,$$

where B_i, C_i, D_i, E_i, F_i, H_i, B''_i, C'_i, E'_i, F'_i *and* H'_i *are given in equations* (2.26) *to* (2.35)*.*

2.2 Some Performance Measures

Using straight forward calculations the following performance measures are calculated:

2.2.1 Mean Number of Customers in the System

$$L = \sum_{i=0}^{N} i p_{i1} + \sum_{i=1}^{N} i p_{i2}$$

$$L = \Big\{1 + \sum_{i=0}^{L-2} \frac{N-(i+1)}{N-i} i(H_{N-(i+1)} - A'_{N-(i+1)}H_{N-(K+i+1)} - H'_{N-(i+1)} + A'_{N-(i+1)}H'_{N-(K+i+1)} + A''_{N-(i+1)}B'_{i+1}) + \sum_{i=L-1}^{L+K-2} i(H_i - H'_i) + \sum_{i=L+K-1}^{N-2K-2} i(E_i - E'_i) + \sum_{i=N-2K-1}^{N-K-1} i(C_i - D_i) + \sum_{i=N-K}^{N-1} i(B_i - C'_i) + \sum_{i=0}^{N-1} iB''_i + N\Big\} p_{N2}$$

2.2.2 Idle Probability

$$p_{01} = \frac{N-1}{N}[(H_{N-1} - A'_{N-1}H_{N-(K+1)}) - H'_{N-1} + A'_{N-1}H'_{N-(K+1)} + +A''_{N-1}B'_1]p_{N2}$$

2.2.3 Probability that the Server is Busy

$$p_b = \sum_{i=1}^{N} p_{i1}$$

2.2.4 Probability That the Server Is an Vacation

$$p_v = \sum_{i=1}^{N} p_{i2}$$

2.2.5 Expected Number of Customers Served per Unit Time

$$L_1 = \mu \sum_{i=1}^{L} i p_{i1} + \mu_1 \sum_{i=L+1}^{N} i p_{i1}$$

2.2.6 Effective Input Rate

$$\begin{aligned}\lambda' &= \lambda(1 - p_{N2})\\ &= \lambda\Bigg[1 - \Bigg(\Bigg(1 + \sum_{i=0}^{L-2} \frac{N-(i+1)}{N-i} i(H_{N-(i+1)} - A'_{N-(i+1)}H_{N-(K+i+1)} - H'_{N-(i+1)}\\ &\qquad + A'_{N-(i+1)}H'_{N-(K+i+1)} + A''_{N-(i+1)}B'_{i+1}) + \sum_{i=L-1}^{L+K-2} i(H_i - H'_i) + \sum_{i=L+K-1}^{N-2K-2} i(E_i - E'_i)\end{aligned}$$

The expression for the notations B_i, C_i, D_i, E_i, B'_i, C'_i, E'_i are given in equations (2.26) to (2.35).

2.2.7 Utilization Factor of the Service Station

$$\rho' = \frac{\lambda'}{\mu + \mu_1 + \theta}$$

2.2.8 Mean Waiting Time in the System (using Little's Law)

$$W = \frac{L}{\lambda'}$$

2.3 Particular Model

If $K = 1$, $L = 1$, $\mu_1 = \mu$, $\theta = 0$, $\alpha = 0$ the model $M/M/1//N$ coincide with the model in [18, 106 – 107].

3. The Finite Waiting Line Model and Analysis

As a modification of the model discussed in section 2, we assume that the source population is infinite but the waiting line capacity is $N-1$.

The arrival rate is

$$\lambda_j = \begin{cases} \lambda, & j = 0,1,2,\ldots,N-1, \\ 0, & j = N, N+1,\ldots. \end{cases}$$

All other assumptions are as in the case of the model in section 2 except the population size.

3.1 The Queue Length Distribution

To analyze the queueing model we define the following notations:

Let $X(t)$ be the number of customers in the system at time t, $Z(t)$ be the server state at time t where

$$Z(t) = \begin{cases} 1, & \text{the server is in busy state,} \\ 2, & \text{the server is in single vacation state} \end{cases}$$

The process $\{(X(t), Z(t)) : t \geq 0\}$ is a finite Markov process with state space

$$S = \{(i,1) : i = 0,1,2,\ldots,N\} \cup \{(i,2) : i = 1,2,\ldots,N\}.$$

Let $p_{ij}(t) = \Pr\{X(t) = i, Z(t) = j\}$ be the corresponding probability distribution.

$$\lambda p_{01} = \mu p_{K1}, \tag{3.1}$$

$$\lambda p_{j1} = \mu p_{K+j1} + \lambda p_{j-11} + \theta p_{j2}, \quad j = 1,2,\ldots,K-1, \tag{3.2}$$

$$(\mu+\lambda) p_{j1} = \mu p_{K+j1} + \lambda p_{j-11} + \theta p_{j2}, \quad j = K, K+1,\ldots,LK-1, \tag{3.3}$$

$$(\mu+\lambda) p_{LK1} = \mu_1 p_{K+LK1} + \lambda p_{LK-11} + \theta p_{LK2}, \tag{3.4}$$

$$(\mu_1+\lambda) p_{j1} = \mu_1 p_{K+j1} + \lambda p_{j-11} + \theta p_{j2}, \quad j = LK+1,\ldots,N-K, \tag{3.5}$$

$$(\mu_1+\lambda) p_{j1} = \lambda p_{j-11} + \theta p_{j2}, \quad j = N-K+1, N-K+2,\ldots,N-1, \tag{3.6}$$

$$\mu_1 p_{N1} = \lambda p_{N-11} + \theta p_{N2}, \tag{3.7}$$

$$(\theta + \lambda + (j-1)\alpha) p_{j2} = \lambda p_{j-12} + j\alpha p_{j+12}, \quad j = 1,2,\ldots,N-1, \tag{3.8}$$

$$(\theta + (N-1)\alpha) p_{N2} = \lambda p_{N-12}. \tag{3.9}$$

The equations (3.1) to (3.9) are recursively solved and the probabilities are obtained as

$$p_{N-i2} = b_i p_{N2}, \quad i = 1,2,3,\ldots,N, \tag{3.10}$$

$$p_{N-i1} = (c_i' - c_i'') p_{N2}, \quad i = 1,2,\ldots,K, \tag{3.11}$$

$$p_{N-i1} = (d_i' - d_i'') p_{N2}, \quad i = K+1, K+2,\ldots,2K+1, \tag{3.12}$$

$$p_{N-i1} = (f_i' - f_i'') p_{N2}, \quad i = 2K+2, 2K+3,\ldots,N-LK, \tag{3.13}$$

$$\begin{aligned} p_{LK-11} = \bigg(& \frac{(1+\gamma)}{\gamma} f'_{N-LK} - \frac{1}{\rho}(f'_{N-(K+LK+1)} + f'_{N-(K+LK)}) \\ & - \frac{(1+\gamma)}{\gamma} f''_{N-LK} + \frac{1}{\rho}(f''_{N-(K+LK+1)} + f''_{N-(K+LK)}) - \frac{1}{\rho_1} b_{LK} \bigg) p_{N2}, \end{aligned} \tag{3.14}$$

$$p_{N-i1} = (e_i' - e_i'') p_{N2}, \quad i = N-LK+2, N-LK+3,\ldots,N-K+1, \tag{3.15}$$

$$p_{i1} = \left(e'_{N-(i+1)} - \frac{1}{\gamma} e'_{N-(i+K+1)} - e''_{N-(i+1)} + \frac{1}{\gamma} e''_{N-(i+K+1)} - \frac{1}{\rho_1} b_i \right) p_{N2}, \quad i = K-2, K-3,\ldots,0, \tag{3.16}$$

where

$$b_i = \frac{(1+\rho_2)^{i-1}}{\rho_2{}^i} - \frac{N-(i-1)}{\alpha}, \tag{3.17}$$

$$c'_i = \frac{(1+\rho)^{i-1}}{\rho^i}, \tag{3.18}$$

$$c''_i = \frac{1}{\rho_1}\sum_{K=1}^{i}\sum_{j=1}^{K}\left(\frac{1+\rho}{\rho}\right)^{i-j} b_{j-1}, \tag{3.19}$$

$$d'_i = c'_i - \frac{1}{\rho^{i-K}}((1+\rho)^{i-(K+1)} + (i-(K+1))(1+\rho)^{i-(K+2)}), \tag{3.20}$$

$$d''_i = \left(\frac{1+\rho}{\rho}\right)^{i-(K+1)} c''_{K+1} - \sum_{j=K+2}^{i}\left[\frac{(1+\rho)^{i-j}}{\rho^{i-j+1}}c''_{j-(K+1)} + \frac{1}{\rho_1}\left(\frac{1+\rho}{\rho}\right)^{i-j} b_{j-(K+1)}\right], \tag{3.21}$$

$$\begin{aligned} e'_i = {} & \left(\frac{1+\gamma}{\gamma}\right)^{i-(N-LK)} f'_{N-LK} - \frac{1}{\rho}\left(\frac{1+\gamma}{\gamma}\right)^{i-(N-LK+1)} \\ & - \left(\frac{1+\alpha}{\alpha} f'_{N-(K+LK+1)} + f'_{N-(K+LK)}\right) - \frac{1}{\gamma}\sum_{j=N-LK+2}^{i}\left(\frac{1+\alpha}{\alpha}\right)^{i-j} f'_{j-(K+1)}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} e''_i = {} & \left(\frac{1+\gamma}{\gamma}\right)^{i-(N-LK)} f''_{N-LK} - \frac{1}{\rho}\left(\frac{1+\gamma}{\gamma}\right)^{i-(N-LK+1)}\left(\frac{1+\gamma}{\gamma} f''_{N-(K+LK+1)} + f''_{N-(K+LK)} + b_{LK-1}\right) \\ & - \frac{1}{\gamma}\sum_{j=N-LK+2}^{i}\left(\frac{1+\gamma}{\gamma}\right)^{i-j} f''_{j-(K+1)} + \frac{1}{\rho_1}\sum_{j=N-LK+2}^{i}\left(\frac{1+\gamma}{\gamma}\right)^{i-j} b_{j-(N-LK)}, \end{aligned} \tag{3.23}$$

$$\rho = \frac{\lambda}{\mu_1}, \tag{3.24}$$

$$\rho_1 = \frac{\lambda}{\theta}, \tag{3.25}$$

$$\rho_2 = \frac{\lambda}{\theta + (N-1)\alpha}, \tag{3.26}$$

$$\gamma = \frac{\lambda}{\mu}. \tag{3.27}$$

Theorem 3.1. *The steady state probabilities for the model discussed in this section are*

$$p_{N-i2} = b_i p_{N2}, \quad i = 1,2,3,\dots,N,$$

$$p_{N-i1} = (c'_i - c''_i)p_{N2}, \quad i = 1,2,\dots,K,$$

$$p_{N-i1} = (d'_i - d''_i)p_{N2}, \quad i = K+1, K+2,\dots,2K+1,$$

$$p_{N-i1} = (f'_i - f''_i)p_{N2}, \quad i = 2K+2, 2K+3,\dots,N-LK,$$

$$\begin{aligned} p_{LK-11} = \Bigg(& \frac{(1+\gamma)}{\gamma} f'_{N-LK} - \frac{1}{\rho}(f'_{N-(K+LK+1)} + f'_{N-(K+LK)}) \\ & - \frac{(1+\gamma)}{\gamma} f''_{N-LK} + \frac{1}{\rho}(f''_{N-(K+LK+1)} + f''_{N-(K+LK)}) - \frac{1}{\rho_1} b_{LK}\Bigg) p_{N2}, \end{aligned}$$

$$p_{N-i1} = (e'_i - e''_i)p_{N2}, \quad i = N-LK+2, N-LK+3,\dots,N-K+1,$$

$$p_{i1}=\left(e'_{N-(i+1)}-\frac{1}{\gamma}e'_{N-(i+K+1)}-e''_{N-(i+1)}+\frac{1}{\gamma}e''_{N-(i+K+1)}-\frac{1}{\rho_1}b_i\right)p_{N2},\quad i=K-2,K-3,\dots,0,$$

where b_i, c'_i, d'_i, d''_i, e'_i, *and* e''_i *are given in equations* (3.17) *to* (3.23).

3.2 Some Performance Measures

Using straight forward calculation the following performance measures can be calculated:

3.2.1 Mean Number of Customers in the System

$$\begin{aligned}L_2&=\sum_{i=1}^{N}ip_i\\&=\left(\sum_{i=1}^{K-2}i\left(e'_{N-(i+1)}-\frac{1}{\gamma}e'_{N-(i+K+1)}-e''_{N-(i+1)}+\frac{1}{\gamma}e''_{N-(i+K+1)}-\frac{1}{\rho_1}b_i\right)+\sum_{i=K-1}^{LK-1}i(e'_i-e''_i)\right.\\&\quad\left.+\sum_{i=LK}^{N-2K-2}i(f'_i-f''_i)+\sum_{i=N-2K-1}^{N-K-1}i(d'_i-d''_i)+\sum_{i=N-K}^{N-1}i(c'_i-c''_i)+\sum_{i=1}^{N-1}ib_i+N\right)p_{N2}.\end{aligned}$$

3.2.2 Idle Probability

$$p_{01}=(e'_{N-(i+1)}-\frac{1}{\gamma}e'_{N-(i+K+1)}-e''_{N-(i+1)}+\frac{1}{\gamma}e''_{N-(i+K+1)}.-\frac{1}{\rho_1}b_i)p_{N2}.$$

3.2.3 Blocking Probability

$$\begin{aligned}p_{N2}&=\left(1+\sum_{i=1}^{K-2}\left(e'_{N-(i+1)}-\frac{1}{\gamma}e'_{N-(i+K+1)}-e''_{N-(i+1)}+\frac{1}{\gamma}e''_{N-(i+K+1)}-\frac{1}{\rho_1}b_i\right)+\sum_{i=K-1}^{LK-1}(e'_i-e''_i)\right.\\&\quad\left.+\sum_{i=LK}^{N-2K-2}(f'_i-f''_i)+\sum_{i=N-2K-1}^{N-K-1}(d'_i-d''_i)+\sum_{i=N-K}^{N-1}(c'_i-c''_i)+\sum_{i=1}^{N-1}b_i\right)^{-1}.\end{aligned}$$

3.2.4 Probability that the server is busy

$$p_b=\sum_{i=1}^{N}p_{i1}$$

3.2.5 Probability that the server is an vacation

$$p_v=\sum_{i=1}^{N}p_{i2}$$

3.2.6 Expected number of customers served per unit time

$$L_1=\mu\sum_{i=1}^{L}ip_{i1}+\mu_1\sum_{i=L+1}^{N}ip_{i1}$$

3.2.7 Effective input rate

$$\begin{aligned}\lambda'_1&=\lambda(1-p_{N2})\\&=\lambda\left[1-\left(\left(1+\sum_{i=1}^{K-2}\left(e'_{N-(i+1)}-\frac{1}{\gamma}e'_{N-(i+K+1)}-e''_{N-(i+1)}+\frac{1}{\gamma}e''_{N-(i+K+1)}\right)-\frac{1}{\rho_1}b_i\right)\right.\right.\\&\quad\left.\left.+\sum_{i=K-1}^{LK-1}(e'_i-e''_i)+\sum_{i=LK}^{N-2K-2}(f'_i-f''_i)+\sum_{i=N-2K-1}^{N-K-1}(d'_i-d''_i)+\sum_{i=N-K}^{N-1}i(c'_i-c''_i)+\sum_{i=1}^{N-1}b_i\right)^{-1}\right].\end{aligned}$$

The expression for the notations are b_i, c'_i, d'_i, d''_i, e'_i, and e''_i are given in equations (3.17) to (3.23).

3.2.8 Utilization factor of service station is

$$\rho'_2 = \frac{\lambda'_1}{\mu + \mu_1 + \theta}$$

3.2.9 Mean waiting time in the system (using Little's law)

$$W_1 = \frac{L_2}{\lambda'_1}$$

3.3 Particular Model

If $K = 1$, $L = 1$, $\mu_1 = \mu$, $\theta = 0$, $\alpha = 0$ the model coincide with the model $M/M/1/N$ in [, Kleinrock, 103 – 104] (1975).

4. Some Numerical Results and Comparison of the Models

This section shows the numerical tractability of the performance measures provided in sections 2.2 (for model I(MI)) and 3.2 (for model II(MII)).

Numerical results for probabilities are presented in Tables 1 to 8 and system performance measures are displayed in Figures 1 to 4. Tables 1 to 8 depicts the probabilities by fixing $N = 10$, $L = 4$ and changing $K = 8, 9, 10$. Also, we fix $\mu = 2.5$, $\mu_1 = 3.9$, $\theta = 3.7$, $\alpha = 2.8$ and we vary the arrival rate λ from 1.1 to 2.0. Figures 1 and 2 displays the mean number of customers in the system for MI and MII against arrival rate λ. In general, the mean system size increases if λ increases for both the models. The first row of each table shows idle probability of MI and MII. It is observed that the idle probabilities p_{01} decreases with increasing arrival rate λ. Comparing with MI, the idle probability of MII is too small. p_{10} for MII are called blocking probabilities, the value decreases as λ increases, we experiences the same in all the tables.

Figures 1 and 2, shows the mean number of customers in the system for both the models and for various values of K. The values of mean number of customers increases arrival rate increases. The functions are increasing functions with respect to arrival rate λ. Figures 3 and 4, shows the mean waiting time for various values of λ, for $K = 8, 9$, and 10. The mean function of MII is an increasing function with respect to λ. For MI, the mean waiting time function is a combination of increasing, convex and concave with respect to the arrival rate λ.

Table 1. $N = 10$, $K = 8$, $L = 9$, $\mu = 2.5$, $\mu_1 = 3.9$, $\theta = 3.7$, $\alpha = 2.8$

p_i	$\lambda = 1.1$		$\lambda = 1.2$		$\lambda = 1.3$	
	MI	MII	MI	MII	MI	MII
p_{01}	0.2866	0.1532	0.2810	0.1513	0.2758	0.1509
p_{11}	0.1268	0.1431	0.1245	0.1425	0.1224	0.1456
p_{21}	0.0828	0.1261	0.0819	0.1378	0.0811	0.1436
p_{31}	0.0698	0.1247	0.0606	0.1337	0.0604	0.1418
p_{41}	0.0495	0.1018	0.0478	0.1235	0.0480	0.1246

Table Contd.

p_i	$\lambda = 1.1$		$\lambda = 1.2$		$\lambda = 1.3$	
	MI	MII	MI	MII	MI	MII
p_{51}	0.0480	0.0986	0.0468	0.1141	0.0457	0.0839
p_{61}	0.0476	0.0691	0.0392	0.0730	0.0397	0.0777
p_{71}	0.0388	0.0380	0.0334	0.0417	0.0338	0.4370
p_{81}	0.0325	0.0209	0.0332	0.0238	0.0334	0.0258
p_{91}	0.0285	0.0092	0.0289	0.0189	0.0296	0.0183
p_{12}	0.0261	0.0087	0.0268	0.0136	0.0274	0.0152
p_{22}	0.0256	0.0063	0.0267	0.0077	0.0273	0.0094
p_{32}	0.0246	0.0045	0.0255	0.0060	0.0266	0.0090
p_{42}	0.0244	0.0040	0.0254	0.0055	0.0261	0.0053
p_{52}	0.0237	0.0034	0.0245	0.0044	0.0253	0.0036
p_{62}	0.0230	0.0031	0.0238	0.0020	0.0247	0.0019
p_{72}	0.0225	0.0006	0.0234	0.0004	0.0242	0.0001
p_{82}	0.0217	0.0001	0.0230	0.0001	0.0240	0.0000
p_{92}	0.0216	0.0000	0.0227	0.0000	0.0238	0.0000
p_{102}	0.0015	0.0000	0.0009	0.0000	0.0007	0.0000

Table 2. $N = 10$, $K = 8$, $L = 9$, $\mu = 2.5$, $\mu_1 = 3.9$, $\theta = 3.7$, $\alpha = 2.8$

p_i	$\lambda = 1.4$		$\lambda = 1.5$	
	MI	MII	MI	MII
p_{01}	0.2750	0.1496	0.2658	0.1484
p_{11}	0.1228	0.1485	0.1186	0.1376
p_{21}	0.0818	0.1388	0.0796	0.1296
p_{31}	0.0614	0.1229	0.0607	0.1290
p_{41}	0.0499	0.1179	0.0483	0.1185
p_{51}	0.0489	0.0946	0.0437	0.0815
p_{61}	0.0409	0.0836	0.0405	0.0745
p_{71}	0.0350	0.0483	0.0350	0.0569
p_{81}	0.0341	0.0350	0.0335	0.0493
p_{91}	0.0317	0.0195	0.0308	0.0296
p_{12}	0.0293	0.0105	0.0293	0.0215
p_{22}	0.0290	0.0099	0.0284	0.0097
p_{32}	0.0289	0.0077	0.0282	0.0087
p_{42}	0.0280	0.0076	0.0274	0.0032
p_{52}	0.0265	0.0034	0.0267	0.0019
p_{62}	0.0264	0.0021	0.0263	0.0001
p_{72}	0.0256	0.0001	0.0259	0.0000
p_{82}	0.0253	0.0000	0.0257	0.0000
p_{92}	0.0249	0.0000	0.0252	0.0000
p_{102}	0.0006	0.0000	0.0004	0.0000

Table 3. $N = 10$, $K = 9$, $L = 9$, $\mu = 2.5$, $\mu_1 = 3.9$, $\theta = 3.7$, $\alpha = 2.8$

p_i	$\lambda = 1.1$		$\lambda = 1.2$		$\lambda = 1.3$	
	MI	MII	MI	MII	MI	MII
p_{01}	0.2869	0.1416	0.2816	0.1393	0.2767	0.1345
p_{11}	0.1269	0.1392	0.1248	0.1327	0.1228	0.1319
p_{21}	0.0829	0.1370	0.0821	0.1319	0.0813	0.1293
p_{31}	0.0609	0.1256	0.0607	0.1252	0.0606	0.1271
p_{41}	0.0480	0.1124	0.0479	0.1189	0.0481	0.1026
p_{51}	0.0477	0.0935	0.0469	0.0967	0.0458	0.0987
p_{61}	0.0388	0.0615	0.0393	0.0581	0.0398	0.0949
p_{71}	0.0334	0.0514	0.0325	0.0552	0.0339	0.0683
p_{81}	0.0326	0.0333	0.0332	0.0380	0.0335	0.0344
p_{91}	0.0286	0.0282	0.0290	0.0315	0.0294	0.0262

Table 4. $N = 10$, $K = 9$, $L = 9$, $\mu = 2.5$, $\mu_1 = 3.9$, $\theta = 3.7$, $\alpha = 2.8$

p_i	$\lambda = 1.1$		$\lambda = 1.2$		$\lambda = 1.3$	
	MI	MII	MI	MII	MI	MII
p_{12}	0.0261	0.0205	0.0268	0.0228	0.0274	0.0203
p_{22}	0.0247	0.0189	0.0254	0.0143	0.0262	0.0120
p_{32}	0.0242	0.0155	0.0251	0.0145	0.0260	0.0065
p_{42}	0.0239	0.0085	0.0250	0.0058	0.0256	0.0062
p_{52}	0.0237	0.0047	0.0245	0.0053	0.0253	0.0042
p_{62}	0.0230	0.0045	0.0239	0.0033	0.0248	0.0016
p_{72}	0.0225	0.0025	0.0234	0.0012	0.0243	0.0003
p_{82}	0.0221	0.0009	0.0230	0.0002	0.0240	0.0001
p_{92}	0.0217	0.0002	0.0228	0.0001	0.0235	0.0000
p_{102}	0.0014	0.0001	0.0011	0.0000	0.0010	0.0000

Table 5. $N = 10$, $K = 9$, $L = 9$, $\mu = 2.5$, $\mu_1 = 3.9$, $\theta = 3.7$, $\alpha = 2.8$

p_i	$\lambda = 1.4$		$\lambda = 1.5$	
	MI	MII	MI	MII
p_{01}	0.2757	0.1310	0.2667	0.1296
p_{11}	0.1226	0.1273	0.1190	0.1284
p_{21}	0.0777	0.1262	0.0798	0.1259
p_{31}	0.0613	0.1253	0.0603	0.1257
p_{41}	0.0482	0.1182	0.0485	0.1205
p_{51}	0.0456	0.1052	0.0438	0.1043
p_{61}	0.0350	0.0867	0.0407	0.0836
p_{71}	0.0342	0.0615	0.0354	0.0653
p_{81}	0.0308	0.0374	0.0336	0.0356

Table Contd.

p_i	$\lambda = 1.4$		$\lambda = 1.5$	
	MI	MII	MI	MII
p_{91}	0.0305	0.0229	0.0309	0.0319
p_{12}	0.0285	0.0228	0.0285	0.0219
p_{22}	0.0274	0.0138	0.0276	0.0180
p_{32}	0.0272	0.0084	0.0275	0.0095
p_{42}	0.0268	0.0060	0.0271	0.0063
p_{52}	0.0266	0.0051	0.0268	0.0056
p_{62}	0.0261	0.0015	0.0264	0.0010
p_{72}	0.0256	0.0002	0.0260	0.0001
p_{82}	0.0250	0.0001	0.0257	0.0000
p_{92}	0.0245	0.0000	0.0250	0.0000
p_{102}	0.0009	0.0000	0.0007	0.0000

Table 6. $N = 10$, $K = 10$, $L = 9$, $\mu = 2.5$, $\mu_1 = 3.9$, $\theta = 3.7$, $\alpha = 2.8$

p_i	$\lambda = 1.1$		$\lambda = 1.2$		$\lambda = 1.3$	
	MI	MII	MI	MII	MI	MII
p_{01}	0.2877	0.1395	0.2824	0.1386	0.2769	0.1375
p_{11}	0.1273	0.1289	0.1252	0.1250	0.1231	0.1236
p_{21}	0.0831	0.1196	0.0820	0.1158	0.0815	0.1125
p_{31}	0.0610	0.1157	0.0609	0.1145	0.0607	0.1029
p_{41}	0.0481	0.1089	0.0480	0.1085	0.0482	0.0986
p_{51}	0.0478	0.0958	0.0470	0.0863	0.0459	0.0846
p_{61}	0.0390	0.0875	0.0394	0.0689	0.0399	0.0680
p_{71}	0.0335	0.0619	0.0336	0.0678	0.0340	0.0678
p_{81}	0.0326	0.0319	0.0333	0.0450	0.0336	0.0549
p_{91}	0.0286	0.0286	0.0291	0.0359	0.0295	0.0418
p_{12}	0.0262	0.0190	0.0268	0.0256	0.0274	0.0379
p_{22}	0.0247	0.0154	0.0255	0.0216	0.0262	0.0276
p_{32}	0.0242	0.0115	0.0252	0.0195	0.0260	0.0199
p_{42}	0.0238	0.0093	0.0246	0.0099	0.0254	0.0109
p_{52}	0.0231	0.0090	0.0240	0.0079	0.0249	0.0065
p_{62}	0.0225	0.0073	0.0235	0.0063	0.0248	0.0045
p_{72}	0.0221	0.0059	0.0231	0.0023	0.0240	0.00004
p_{82}	0.0218	0.0036	0.0228	0.0005	0.0238	0.0001
p_{92}	0.0213	0.0006	0.0223	0.0001	0.0233	0.0000
p_{102}	0.0016	0.0001	0.0013	0.0000	0.0009	0.0000

Table 7. $N = 10$, $K = 10$, $L = 9$, $\mu = 2.5$, $\mu_1 = 3.9$, $\theta = 3.7$, $\alpha = 2.8$

	$\lambda = 1.4$		$\lambda = 1.5$	
p_i	MI	MII	MI	MII
p_{01}	0.2760	0.1355	0.2671	0.1339
p_{11}	0.1234	0.1214	0.1192	0.1206
p_{21}	0.0802	0.1115	0.0800	0.1109
p_{31}	0.0617	0.1015	0.0603	0.1005
p_{41}	0.0496	0.0945	0.0486	0.0913
p_{51}	0.0430	0.0819	0.0439	0.0856
p_{61}	0.0353	0.0715	0.0418	0.0801
p_{71}	0.0343	0.0653	0.0330	0.0739
p_{81}	0.0311	0.0596	0.0321	0.0685
p_{91}	0.0308	0.0575	0.0309	0.0639

Table 8. $N = 10$, $K = 10$, $L = 9$, $\mu = 2.5$, $\mu_1 = 3.9$, $\theta = 3.7$, $\alpha = 2.8$

	$\lambda = 1.4$		$\lambda = 1.5$	
p_i	MI	MII	MI	MII
p_{12}	0.0286	0.0419	0.0307	0.0331
p_{22}	0.0274	0.0296	0.0286	0.0173
p_{32}	0.0267	0.0215	0.0276	0.0416
p_{42}	0.0261	0.0035	0.0269	0.0049
p_{52}	0.0257	0.0029	0.0264	0.0008
p_{62}	0.0254	0.0003	0.0260	0.0001
p_{72}	0.0251	0.0001	0.0257	0.0000
p_{82}	0.0246	0.0000	0.0255	0.0000
p_{92}	0.0244	0.0000	0.0251	0.0000
p_{102}	0.0007	0.0000	0.0006	0.0000

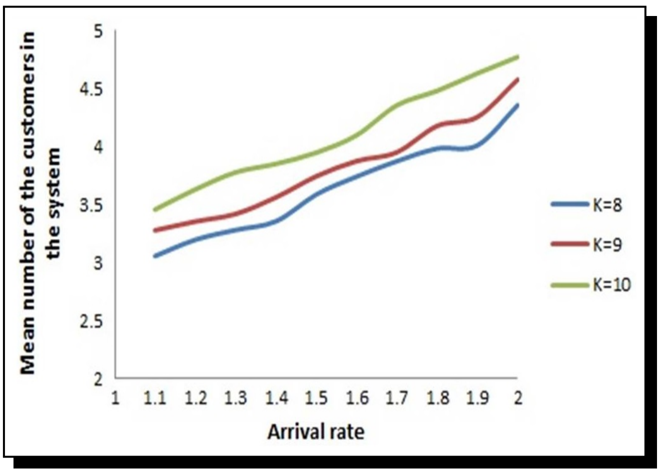

Figure 1. Mean number of the customers in the system for MI

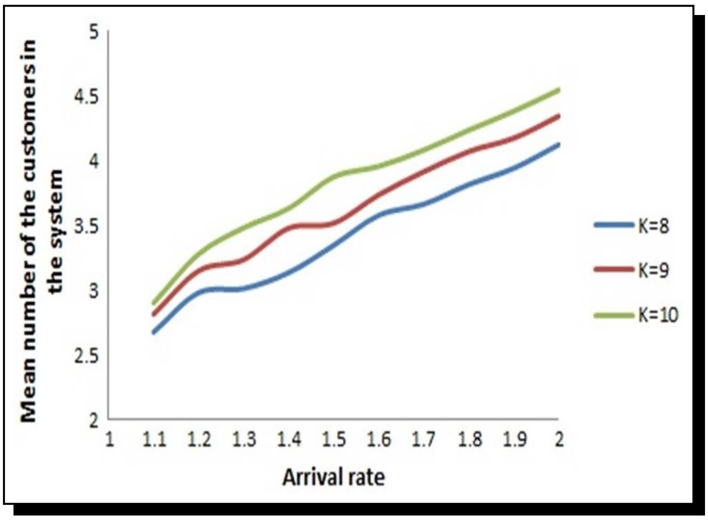

Figure 2. Mean number of the customers in the system for MII

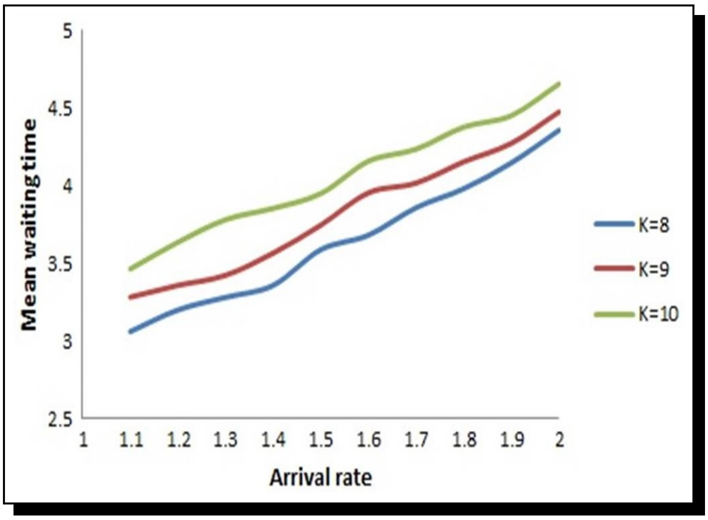

Figure 3. Mean waiting time for MI

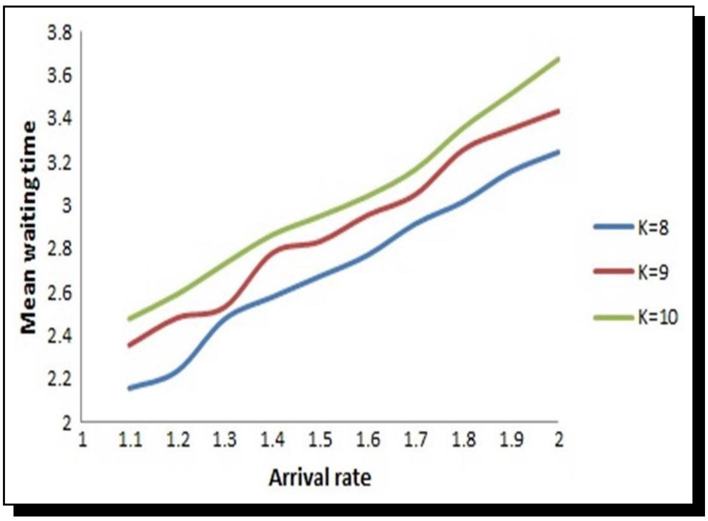

Figure 4. Mean waiting time for MII

5. Cost and Profit Analysis

In this section, we define a cost structure to the model discussed in this paper. To carry out the cost and profit analysis we introduce the following cost and profit related elements.

C_h: Unit's holding cost per unit time

C_{bb}: Cost per unit time when the server busy with batch service

C_l: Cost associate each lost cost per unit time

C_v: Cost per unit time when the server is an vacation

C_1: Cost per service by server1 per unit time

C_2: Cost per service by server2 per unit time

C: Cost

P: Profit

R: Revenue

$T(C)$: Total expected cost per unit time

$T(R)$: Total expected revenue of the system

$T(P)$: Total expected profit of the system

For model (MI), the total expected cost per unit time of the systems is

$$T = C_{bb}p_{bb} + C_v p_v + C_h L + C_1\mu + C_2\mu_1 .$$

For model (MII), the total expected cost per unit time of the systems is

$$T = C_{bb}p_{bb} + C_v p_v + C_h L + C_l \lambda p_N .$$

Total expected revenue of the system is $T(R) = R(\mu+\mu_1)L_1$

Total expected profit of the systems is $T(P) = T(R) - T(C)$

By fixing the cost parameters $C_{bb} = 8$, $C_v = 6$, $C_h = 13$, $C_1 = 7$, $C_2 = 4$, $R = 8$, $L = 5$, the queueing $\mu = 3.9$, $\mu_1 = 4.1$, $N = 10$ and varying $\lambda = 1.1$ to 2.0, $K = 8,9,10$, the functional values $T(C)$, $T(R)$ and $T(P)$ are obtained and are depicted in the Tables 9, 10 and 11, respectively. Also, the corresponding graphs are drawn in the Figures 5 to 10.

From the numerical calculations, for MI it is clear that the total expected cost per unit time is minimum at $\lambda = 1.7$ for $K = 8$, at $\lambda = 1.6$ for $K = 9$ and at $\lambda = 1.9$ for $K = 10$. The total expected profit is maximum at $\lambda = 1.6$ for $K = 8$, at $\lambda = 1.6$ for $K = 9$ and $\lambda = 1.7$ for $K = 10$. The minimum of total expected revenue at $\lambda = 1.7$ for $K = 8$, at $\lambda = 1.5$ for $K = 9$ and $\lambda = 1.8$ for $K = 10$.

For MII it is clear that the total expected cost per unit time is minimum at $\lambda = 1.5$ for $K = 8$, at $\lambda = 1.8$ for $K = 9$ and at $\lambda = 1.4$ for $K = 10$. The total expected profit is maximum at $\lambda = 1.6$ for $K = 8$, at $\lambda = 1.4$ for $K = 9$ and at $\lambda = 1.3$ for $K = 10$. The minimum of total expected revenue at $\lambda = 1.6$ for $K = 8$, at $\lambda = 1.4$ for $K = 9$ and $\lambda = 1.8$ for $K = 10$.

Table 9. Total expected cost per unit time $T(C)$ ($C_{bb} = 8$, $C_v = 6$, $C_h = 13$, $C_1 = 7$, $C_2 = 4$, $R = 8$, $L = 5$, $\mu = 3.9$, $\mu_1 = 4.1$)

	$K = 8$		$K = 9$		$K = 10$	
λ	MI	MII	MI	MII	MI	MII
1.1	89.7546	84.8198	93.2193	87.2004	96.1127	88.8846
1.2	92.0373	89.2215	94.6288	91.9508	98.7894	94.2017
1.3	93.6158	90.1123	95.8042	93.4084	100.9513	97.0954
1.4	96.2211	92.3507	97.7484	96.6512	102.1074	**73.2435**
1.5	98.5028	**69.3738**	100.6211	97.6779	103.5851	102.6270
1.6	100.7166	98.5873	**76.3292**	100.7471	105.5951	103.7529
1.7	**65.2375**	99.7290	104.1279	102.8282	112.4542	105.4355
1.8	103.9009	101.7164	106.2373	**65.8386**	113.6530	107.4563
1.9	104.3142	103.4016	107.2356	106.1631	**73.6259**	109.3759
2.0	108.8637	105.8333	111.4001	108.3437	114.4898	111.5271

Table 10. Total expected profit per unit time $T(P)$ ($C_{bb} = 8$, $C_v = 6$, $C_h = 13$, $C_1 = 7$, $C_2 = 4$, $R = 8$, $L = 5$, $\mu = 3.9$, $\mu_1 = 4.1$)

	$K = 8$		$K = 9$		$K = 10$	
λ	MI	MII	MI	MII	MI	MII
1.1	58.5670	69.5183	59.5669	71.3581	60.7915	74.8576
1.2	59.3675	70.9715	60.3786	72.7635	61.3619	75.1935
1.3	61.7834	71.6346	61.1739	73.1419	63.4054	**76.8319**
1.4	62.8999	72.3599	63.3599	**75.9783**	65.3689	45.9736
1.5	63.6350	73.1783	65.4819	43.8769	67.2515	44.5419
1.6	**65.5636**	**74.4756**	**67.7345**	40.8856	69.4856	43.4377
1.7	43.4787	49.3577	40.5986	39.1736	**73.8990**	40.6569
1.8	40.1579	47.4981	39.3579	38.9907	36.9786	39.1974
1.9	39.7430	45.7386	37.4531	37.1989	35.1993	38.8926
2.0	38.9136	44.5993	36.7816	36.3964	33.1786	36.9810

Table 11. Total expected revenue per unit time $T(R)$ ($C_{bb} = 8$, $C_v = 6$, $C_h = 13$, $C_1 = 7$, $C_2 = 4$, $R = 8$, $L = 5$, $\mu = 3.9$, $\mu_1 = 4.1$)

	$K = 8$		$K = 9$		$K = 10$	
λ	MI	MII	MI	MII	MI	MII
1.1	148.3216	173.3659	152.7862	175.8559	154.6741	179.3615
1.2	149.9986	174.1735	153.3635	176.3517	155.8529	180.7840
1.3	150.7816	175.8993	154.5416	177.8931	156.1457	181.8019
1.4	151.3589	176.9875	155.6953	**139.4563**	157.5316	182.7388
1.5	152.7677	177.7319	**120.4382**	179.7919	158.3103	183.9781
1.6	154.4639	**135.8426**	156.9186	180.8515	159.4935	185.3659
1.7	**112.3519**	178.5519	158.1078	181.4366	161.8859	187.4566
1.8	157.5760	179.3679	160.0763	182.567	**123.7326**	**143.7310**
1.9	158.4837	180.4815	161.3845	183.3514	162.6583	188.8531
2.0	160.2919	182.5989	162.2647	184.1819	163.1938	190.6354

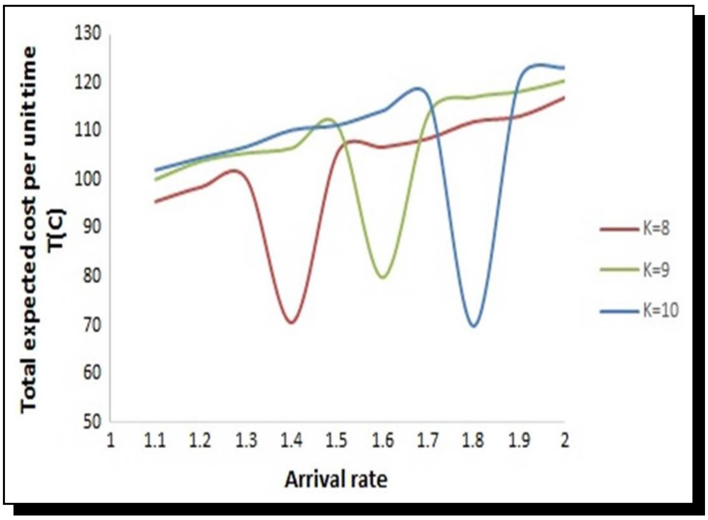

Figure 5. Total expected cost per unit time for MI

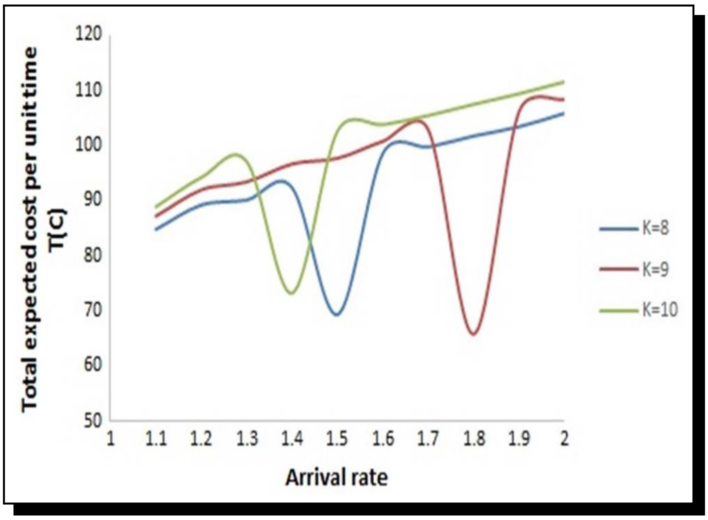

Figure 6. Total expected cost per unit time for MII

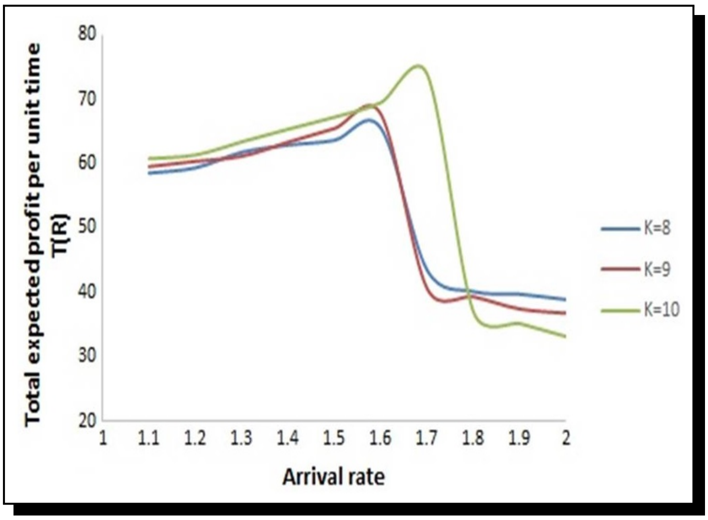

Figure 7. Total expected profit per unit time for MI

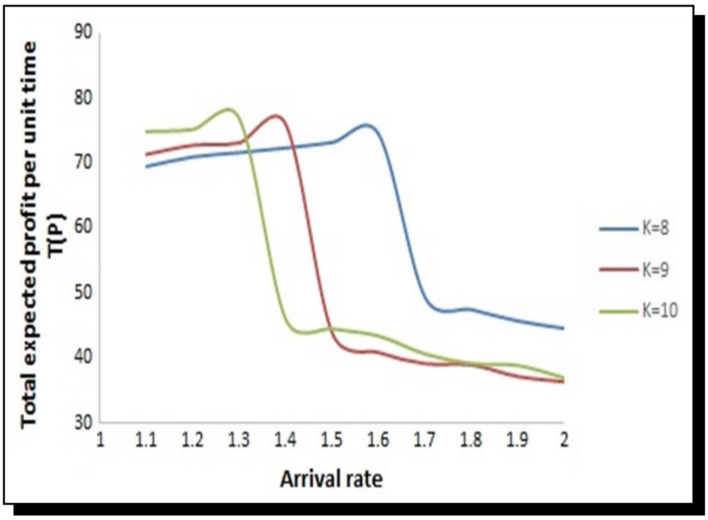

Figure 8. Total expected profit per unit time for MII

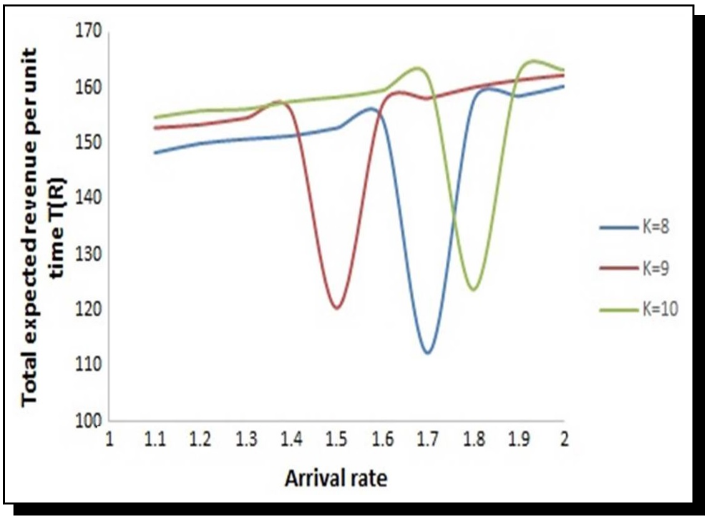

Figure 9. Total expected revenue per unit time for MI

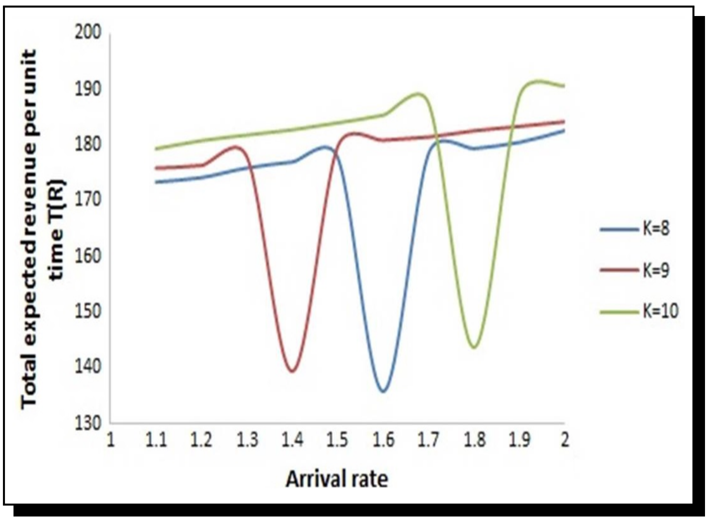

Figure 10. Total expected revenue per unit time for MII

6. Conclusion

Single vacation, batch service, impatient customers are the characters of the queueing systems analyzed in this paper with the above characters, two models are defined and analyzed by assuming finite population customers, infinite population customers with finite waiting line separately. The two models are completely analyzed in steady state. The efficiency of the models are identified by defining cost and profit structure. The models are compared numerically.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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