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On the Negatively Subscripted Padovan and Perrin Matrix Sequences

Research Article

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Abstract. The first main idea of this paper is to develop the *matrix sequences* that represent negatively subscripted Padovan and Perrin numbers. Then, by taking into account matrix properties for these new matrix sequences, some behaviours of negatively subscripted Padovan and Perrin numbers have been investigated. Moreover, we present the important relationships between negatively subscripted Padovan and Perrin matrix sequences.

Keywords. Padovan matrix sequence; Perrin matrix sequence; generating function

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1. Introduction and Preliminaries

There are so many studies in the literature that concern about the special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan and Perrin (see, for example [2, 4, 7, 8, 11], and the references cited therein). On the other hand, the matrix sequences have taken so much interest for different type of numbers (cf. [1, 3, 9, 10]). Therefore, it is worth to study a new matrix sequence related to less known numbers. In the light of this thought, the goal of this paper is to define the related *matrix sequences* for negatively subscripted Padovan and Perrin numbers as the first time in the literature. Actually the most important difference with some other similar studies is, in here the study contains three-dimensional matrices instead of two as in Fibonacci, Lucas and Pell.

In Fibonacci numbers, there clearly exists the term Golden ratio which is defined as the ratio of two consecutive of Fibonacci numbers that converges to $\alpha = \frac{1+\sqrt{5}}{2}$. It is also clear that the ratio has so many applications in, specially, Physics, Engineering, Architecture, etc. [5, 6].

In a similar manner, the ratio of two consecutive Padovan numbers converges to

$$\alpha_P = \sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}}+\sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}}$$

that is named as *Plastic ratio* and was firstly defined in 1924 by Gérard Cordonnier.

Although the study of Perrin numbers started in the beginning of 19th century under different names, the master study was published in 2006 by Shannon et al. in [7]. In this reference, the authors defined the Perrin $\{R_n\}_{n\in\mathbb{N}}$ and Padovan $\{P_n\}_{n\in\mathbb{N}}$ sequences as in the forms

$$R_{n+3} = R_{n+1} + R_n, \quad \text{where} \quad R_0 = 3,\ R_1 = 0,\ R_2 = 2$$

and

$$P_{n+3} = P_{n+1} + P_n, \quad \text{where} \quad P_0 = P_1 = P_2 = 1,$$

respectively. Also, as with any sequence defined by the recurrence relations, Padovan P_n and Perrin numbers R_n for $n < 0$ can be defined by rewriting the recurrence relation as

$$P_n = P_{n+3} - P_{n+1}, \text{ where } P_0 = P_1 = P_2 = 1, \tag{1.1}$$

$$R_n = R_{n+3} - R_{n+1}, \text{ where } R_0 = 3,\ R_1 = 0,\ R_2 = 2. \tag{1.2}$$

Starting with $n = -1$ and working backwards, we extend P_n and R_n to negative indices:

Table 1

n	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10
P_n	1	0	1	0	0	1	−1	1	0	−1	2
R_n	3	−1	1	2	−3	4	−2	−1	5	−7	6

It is well-known that the relationship between $\{R_n\}$ and $\{P_n\}$ is presented by

$$R_n = 3P_{n-5} + 2P_{n-4}, \quad \text{where } n \in \mathbb{Z}. \tag{1.3}$$

In [9], Yilmaz and Taskara considered the following definition of Padovan and Perrin matrix sequences and investigated properties of these sequences.

Definition 1 ([9]). The Padovan $(\mathscr{P}_n)_{n\in\mathbb{N}}$ and Perrin matrix sequences $(\mathscr{R}_n)_{n\in\mathbb{N}}$ are defined by

$$\mathscr{P}_{n+3} = \mathscr{P}_{n+1} + \mathscr{P}_n, \tag{1.4}$$

$$\mathscr{R}_{n+3} = \mathscr{R}_{n+1} + \mathscr{R}_n, \tag{1.5}$$

with initial conditions $\mathscr{P}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathscr{P}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $\mathscr{P}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathscr{R}_0 = \begin{pmatrix} 4 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$, $\mathscr{R}_1 = \begin{pmatrix} -3 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$, $\mathscr{R}_2 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 3 \end{pmatrix}$.

Proposition 2 ([9]). *For $m,n \in \mathbb{N}$, the following properties are hold:*

$$\begin{cases} \mathscr{P}_{n+m} = \mathscr{P}_m \mathscr{P}_n = \mathscr{P}_n \mathscr{P}_m, \\ \mathscr{P}_m \mathscr{R}_n = \mathscr{R}_n \mathscr{P}_m = \mathscr{R}_{n+m}, \\ \mathscr{P}_n = A_1 x^n + B_1 y^n + C_1 z^n, \\ \mathscr{R}_n = A_2 x^n + B_2 y^n + C_2 z^n, \end{cases}$$

where

$$A_1 = \frac{x\mathscr{P}_2 + x^2\mathscr{P}_1 + \mathscr{P}_0}{x(x-y)(x-z)}, \quad B_1 = \frac{y\mathscr{P}_2 + y^2\mathscr{P}_1 + \mathscr{P}_0}{y(y-x)(y-z)}, \quad C_1 = \frac{z\mathscr{P}_2 + z^2\mathscr{P}_1 + \mathscr{P}_0}{z(z-x)(z-y)}$$

and

$$A_2 = \frac{x\mathscr{R}_2 + x^2\mathscr{R}_1 + \mathscr{R}_0}{x(x-y)(x-z)}, \quad B_2 = \frac{y\mathscr{R}_2 + y^2\mathscr{R}_1 + \mathscr{R}_0}{y(y-x)(y-z)}, \quad C_2 = \frac{z\mathscr{R}_2 + z^2\mathscr{R}_1 + \mathscr{R}_0}{z(z-x)(z-y)}.$$

This paper is divided in two sections except the first one. In Section 2, the matrix sequences of negative indices Padovan and Perrin numbers will be defined as the first time in the literature. Then, by giving the generating functions, Binet formulas and summation formulas over these new matrix sequences, we will obtain some fundamental properties on negative indices Padovan and Perrin numbers. In Section 3, we will present the relationship between these matrix sequences. Since we are studying on three-dimensional matrices and so sequences for negative indices Padovan and Perrin numbers, there exist some difficulties in the meaning of the investigation of properties of negative indices Padovan and Perrin numbers. However, by the results in Sections 2 and 3 of this paper, we have a great opportunity to compare and obtain some new properties over these numbers. This is the main aim of this paper.

2. The Matrix Sequences of Negative Indices Padovan and Perrin Numbers

In this section, we will mainly focus on the matrix sequences of negative indices Padovan and Perrin numbers to get some important results. In fact, as a middle step, we also present the related Binet formulas, summations and generating functions. Besides, the new Binet formulas will be used in Section 3.

Hence, in the following, we firstly define the negative indices Padovan and Perrin matrix sequences.

Definition 3. For $n > 2$, the negative indices Padovan ($\mathscr{P}_{-n}$) and Perrin matrix sequences ($\mathscr{R}_{-n}$) are defined by

$$\mathscr{P}_{-n} = \mathscr{P}_{-n+3} - \mathscr{P}_{-n+1}, \tag{2.1}$$

$$\mathscr{R}_{-n} = \mathscr{R}_{-n+3} - \mathscr{R}_{-n+1}, \tag{2.2}$$

respectively, with initial conditions

$$\mathscr{P}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathscr{P}_{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathscr{P}_{-2} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathscr{R}_0 = \begin{pmatrix} 4 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}, \mathscr{R}_{-1} = \begin{pmatrix} -2 & -3 & 4 \\ 4 & 2 & -3 \\ -3 & 1 & 2 \end{pmatrix}, \mathscr{R}_{-2} = \begin{pmatrix} -1 & 4 & -2 \\ -2 & -3 & 4 \\ 4 & 2 & -3 \end{pmatrix}.$$

The first main result gives the nth general terms of the sequences in (2.1) and (2.2) via negative indices Padovan and Perrin numbers as in the following.

Theorem 4. *For any integer $n \geq 0$, we have the matrix sequences*

$$\mathscr{P}_{-n} = \begin{pmatrix} P_{-n-5} & P_{-n-3} & P_{-n-4} \\ P_{-n-4} & P_{-n-2} & P_{-n-3} \\ P_{-n-3} & P_{-n-1} & P_{-n-2} \end{pmatrix} \tag{2.3}$$

and

$$\mathscr{R}_{-n} = \begin{pmatrix} R_{-n-5} & R_{-n-3} & R_{-n-4} \\ R_{-n-4} & R_{-n-2} & R_{-n-3} \\ R_{-n-3} & R_{-n-1} & R_{-n-2} \end{pmatrix}, \tag{2.4}$$

respectively.

Proof. The proof will be done by induction steps.

First of all, let us consider Table 1. Thus we obtain

$$\mathscr{P}_0 = \begin{pmatrix} P_{-5} & P_{-3} & P_{-4} \\ P_{-4} & P_{-2} & P_{-3} \\ P_{-3} & P_{-1} & P_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which gives the following first step of the induction. Secondly, again considering Table 1 and initial conditions in Definition 3, we also get

$$\mathscr{P}_{-1} = \begin{pmatrix} P_{-6} & P_{-4} & P_{-5} \\ P_{-5} & P_{-3} & P_{-4} \\ P_{-4} & P_{-2} & P_{-3} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Actually, by iterating this procedure and assuming the equation in (2.3) holds for all $n = k \in \mathbb{Z}^+$, we can end up the proof if we manage to show that the case also holds for $n = k+1$:

$$\begin{aligned} \mathscr{P}_{-k-1} &= \mathscr{P}_{-k+2} - \mathscr{P}_{-k} \\ &= \begin{pmatrix} P_{-k-3} & P_{-k-1} & P_{-k-2} \\ P_{-k-2} & P_{-k} & P_{-k-1} \\ P_{-k-1} & P_{-k+1} & P_{-k} \end{pmatrix} - \begin{pmatrix} P_{-k-5} & P_{-k-3} & P_{-k-4} \\ P_{-k-4} & P_{-k-2} & P_{-k-3} \\ P_{-k-3} & P_{-k-1} & P_{-k-2} \end{pmatrix} \\ &= \begin{pmatrix} P_{-k-6} & P_{-k-4} & P_{-k-5} \\ P_{-k-5} & P_{-k-3} & P_{-k-4} \\ P_{-k-4} & P_{-k-2} & P_{-k-3} \end{pmatrix}. \end{aligned}$$

Hence that is the result.

For the truthness of the negative indices Perrin matrix sequence, we need the follow almost same approximation by considering (1.2). Similarly as in the above case, the final step of the induction can be obtained by $\mathscr{R}_{-k-1} = \mathscr{R}_{-k+2} - \mathscr{R}_{-k}$ as in the following

$$\mathscr{R}_{-k-1} = \begin{pmatrix} R_{-k-6} & R_{-k-4} & R_{-k-5} \\ R_{-k-5} & R_{-k-3} & R_{-k-4} \\ R_{-k-4} & R_{-k-2} & R_{-k-3} \end{pmatrix}.$$

This completes the proof. □

Theorem 5. *For every $n \in \mathbb{N}$, we can write the Binet formulas for the negative indices Padovan and Perrin matrix sequences as the form*

$$\mathscr{P}_{-n} = A_1 x^{-n} + B_1 y^{-n} + C_1 z^{-n} \quad \text{and} \quad \mathscr{R}_{-n} = A_2 x^{-n} + B_2 y^{-n} + C_2 z^{-n},$$

where

$$A_1 = \frac{x\mathscr{P}_{-2} + x^3\mathscr{P}_{-1} + x^2\mathscr{P}_0}{(x-y)(x-z)}, \quad B_1 = \frac{y\mathscr{P}_{-2} + y^3\mathscr{P}_{-1} + y^2\mathscr{P}_0}{(y-x)(y-z)}, \quad C_1 = \frac{z\mathscr{P}_{-2} + z^3\mathscr{P}_{-1} + z^2\mathscr{P}_0}{(z-x)(z-y)}$$

and

$$A_2 = \frac{x\mathscr{R}_{-2} + x^3\mathscr{R}_{-1} + x^2\mathscr{R}_0}{(x-y)(x-z)}, \quad B_2 = \frac{y\mathscr{R}_{-2} + y^3\mathscr{R}_{-1} + y^2\mathscr{R}_0}{(y-x)(y-z)}, \quad C_2 = \frac{z\mathscr{R}_{-2} + z^3\mathscr{R}_{1} + z^2\mathscr{R}_0}{(z-x)(z-y)},$$

such that x, y, z are roots of characteristic equations of (2.1) *and* (2.2).

Proof. We note that the proof will be based on the recurrence relations (2.1) and (2.2) in Definition 3. As in the previous result, we will only show the thruthness of the Binet formula for negative indices Padovan matrix sequence and will omit the proof of same formula for negative indices Perrin matrix sequence since they have same characteristic equations.

So let us consider (2.1). By the assumption, the roots of the characteristic equation of (2.1) are x, y and z. Hence the general solution of it is given by

$$\mathscr{P}_{-n} = A_1 x^{-n} + B_1 y^{-n} + C_1 z^{-n}.$$

Using initial conditions in Definition 3 and also applying fundamental linear algebra operations, we clearly get the matrices A_1, B_1 and C_1, as desired. This implies the formula for $\mathscr{P}_{-n}$. □

In [7], the authors obtained the Binet formulas for Padovan and Perrin numbers. Now as a different approximation and so as a consequence of Theorems 4 and 5, in the following corollary, we will present the formulas for negative indices of these numbers via related matrix sequences. In fact, in the proof of this corollary, we will just compare the linear combination of the 3rd row and 2nd column entries of the matrices

- A_1, B_1 and C_1 with the matrix $\mathscr{P}_{-n}$ in (2.3) and, similarly,
- A_2, B_2 and C_2 with the matrix $\mathscr{R}_{-n}$ in (2.4).

Corollary 6. *The Binet formulas for negative indices Padovan and Perrin numbers in terms of their matrix sequences are given by*

$$P_{-n-1} = \frac{x^{-n+3}}{(x-y)(x-z)} + \frac{y^{-n+3}}{(y-x)(y-z)} + \frac{z^{-n+3}}{(z-x)(z-y)},$$

and

$$R_{-n-1} = x^{-n-1} + y^{-n-1} + z^{-n-1},$$

where $n \geq 0$.

Now, for negative indices Padovan and Perrin matrix sequences, we give the summations according to specified rules as we depicted at the beginning of this section.

Theorem 7. *For* $m > j \geq 0$ *and* $n > 0$, *there exist*

$$\sum_{i=0}^{n-1} \mathscr{P}_{-mi-j} = \frac{\mathscr{P}_{-mn+m-j} + \mathscr{P}_{-mn-m-j} + (1-R_{-m})\mathscr{P}_{-mn-j} - \mathscr{P}_{m-j}}{R_{-m} - R_m} - \frac{\mathscr{P}_{-m-j} - (R_{-m}-1)\mathscr{P}_{-j}}{R_{-m} - R_m} \tag{2.5}$$

and

$$\sum_{i=0}^{n-1} \mathscr{R}_{-mi-j} = \frac{\mathscr{R}_{-mn+m-j} + \mathscr{R}_{-mn-m-j} + (1-R_{-m})\mathscr{R}_{-mn-j} - \mathscr{R}_{m-j}}{R_{-m} - R_m} - \frac{\mathscr{R}_{-m-j} - (R_{-m}-1)\mathscr{R}_{-j}}{R_{-m} - R_m}. \tag{2.6}$$

Proof. The main point of the proof will be touched just the result Theorem 5, in other words the Binet formulas of related matrix sequences. Differently as previous results, we will consider the proof over negative indices Perrin matrix sequence and will omit the case of Padovan. Thus

$$\begin{aligned}\sum_{i=0}^{n-1} \mathscr{R}_{-mi-j} &= \sum_{i=0}^{n-1} \left(A_2 x^{-mi-j} + B_2 y^{-mi-j} + C_2 z^{-mi-j}\right), \\ &= A_2 x^{-j}\left(\frac{x^{-mn}-1}{x^{-m}-1}\right) + B_2 y^{-j}\left(\frac{y^{-mn}-1}{y^{-m}-1}\right) + C_2 z^{-j}\left(\frac{z^{-mn}-1}{z^{-m}-1}\right).\end{aligned}$$

In here, simplifying the last equality in above will be implied (2.6) as required. □

If we state almost same explanation as before Corollary 6, then the following result will be clear for the summations of negative indices Padovan and Perrin numbers as a consequence of Theorem 7.

Corollary 8. *For* $m > j \geq 0$ *and* $n > 0$, *we have*

$$\sum_{i=0}^{n-1} P_{-mi-j-1} = \frac{P_{-mn+m-j-1} + P_{-mn-m-j-1} + (1-R_{-m})P_{-mn-j-1}}{R_{-m} - R_m} - \frac{P_{m-j-1} + P_{-m-j-1} - (R_{-m}-1)P_{-j-1}}{R_{-m} - R_m}$$

and

$$\sum_{i=0}^{n-1} R_{-mi-j-1} = \frac{R_{-mn+m-j-1} + R_{-mn-m-j-1} + (1-R_{-m})R_{-mn-j-1}}{R_{-m} - R_m} - \frac{R_{m-j-1} + R_{-m-j-1} - (R_{-m}-1)R_{-j-1}}{R_{-m} - R_m}.$$

As we noted in the beginning of this section, our aim of this paper was to present generating functions of our new matrix sequences. The next result deals with it.

Theorem 9. *For negative indices Padovan and Perrin matrix sequences, we have the generating functions*

$$\sum_{i=0}^{\infty} \mathscr{P}_{-i} x^i = \frac{1}{1+x-x^3} \begin{pmatrix} 1 & x^2 & x \\ x & 1+x & x^2 \\ x^2 & x+x^2 & 1+x \end{pmatrix}$$

and

$$\sum_{i=0}^{\infty} \mathscr{R}_{-i} x^i = \frac{1}{1+x-x^3} \begin{pmatrix} 4+2x-3x^2 & 2-x+x^2 & -3+x+2x^2 \\ -3+x+2x^2 & 1+3x-x^2 & 2-x+x^2 \\ 2-x+x^2 & -1+3x^2 & 1+3x-x^2 \end{pmatrix},$$

respectively.

Proof. We will again omit Padovan case since the proof will be quite similar.

Assume that $G(x)$ is the generating function for the sequence $\{\mathscr{R}_{-n}\}_{n\in\mathbb{N}}$. Then we have

$$\begin{aligned} G(x) &= \sum_{i=0}^{\infty} \mathscr{R}_{-i} x^i \\ &= \mathscr{R}_0 + \mathscr{R}_{-1}x + \mathscr{R}_{-2}x^2 + \sum_{i=3}^{\infty} (\mathscr{R}_{-i+3} - \mathscr{R}_{-i+1}) x^i. \end{aligned}$$

From Definition 3, we obtain

$$\begin{aligned} G(x) &= \mathscr{R}_0 + \mathscr{R}_{-1}x + \mathscr{R}_{-2}x^2 + \mathscr{R}_0 x + \mathscr{R}_{-1}x^2 + x^3 \sum_{i=0}^{\infty} \mathscr{R}_{-i} x^i - x \sum_{i=0}^{\infty} \mathscr{R}_{-i} x^i \\ &= \mathscr{R}_0 + (\mathscr{R}_{-1} + \mathscr{R}_0)x + (\mathscr{R}_{-2} + \mathscr{R}_{-1})x^2 + x^3 G(x) - xG(x). \end{aligned}$$

Now rearrangement the equation implies that

$$G(x) = \frac{\mathscr{R}_0 + (\mathscr{R}_{-1} + \mathscr{R}_0)x + (\mathscr{R}_{-2} + \mathscr{R}_{-1})x^2}{1+x-x^3}$$

which equal to the $\sum_{i=0}^{\infty} \mathscr{R}_{-i} x^i$ in theorem.

Hence that is the result. □

In [8], the authors obtained the generating functions for Padovan and Perrin numbers. However, in here, we can obtain these functions in terms of negative indices Padovan and Perrin matrix sequences as a consequence of Theorem 9. To do that we will again compare the 3rd row and 2nd column entries with the matrices in Theorem 9. Hence we have the following corollary.

Corollary 10. *There exist*

$$\sum_{i=0}^{\infty} P_{-i-1}x^i = \frac{x+x^2}{1+x-x^3} \quad \text{and} \quad \sum_{i=0}^{\infty} R_{-i-1}x^i = \frac{-1+3x^2}{1+x-x^3}.$$

3. The Relationships Between New Matrix Sequences

The following proposition (which will be needed for some of our results in this section) express us that there always exist some interpasses between the Padovan and Perrin matrix sequences. In fact the proof of it can be seen directly by considering Theorem 4 and the equality in (1.3).

Proposition 11. *For the matrix sequences $(\mathscr{P}_{-n})_{n\in\mathbb{N}}$ and $(\mathscr{R}_{-n})_{n\in\mathbb{N}}$, we have the following equalities:*

(i) $\mathscr{R}_{-n} = 3\mathscr{P}_{-n-5} + 2\mathscr{P}_{-n-4}$,

(ii) $\mathscr{R}_0\mathscr{P}_{-n} = \mathscr{P}_{-n}\mathscr{R}_0 = \mathscr{R}_{-n}$,

(iii) $\mathscr{R}_0\mathscr{P}_{-n} = 2\mathscr{P}_{-n-2} + \mathscr{P}_{-n-5} = \mathscr{R}_{-n}$.

Remark 12. We remind that the interpass between Padovan and Perrin numbers was stated as in (1.3) as the expression of a Perrin number in terms of Padovan numbers. In addition to this, by taking into account Proposition 11, one can also obtain

$$P_{-n-1} = \frac{1}{23}(R_{-n-3} + 8R_{-n-2} + 10R_{-n-1}) \tag{3.1}$$

as a new interpass for the same numbers. Notice that the relation in (3.1) based on the expression of the negative indices Padovan number in terms of negative indices Perrin numbers.

The following theorem express us that there exist the relation between the positive indices and negative indices Padovan matrix sequences.

Theorem 13. *For $n \geq 0$, we have*

$$\mathscr{P}_{-n} = (\mathscr{P}_n)^{-1}. \tag{3.2}$$

Proof. The proof will be done by induction steps.

First of all, let us consider Definition 3. Thus we obtain

$$\mathscr{P}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\mathscr{P}_0)^{-1},$$

which gives the following first step of the induction. Secondly, again considering initial conditions in Definition 1 and 3, we also get

$$\mathscr{P}_{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (\mathscr{P}_1)^{-1}.$$

Actually, by iterating this procedure and assuming the equation in (3.2) holds for all $n = k \in \mathbb{Z}^+$, we can end up the proof if we manage to show that the case also holds for $n = k+1$. By taking account Proposition 2, we have

$$\begin{aligned}
(\mathscr{P}_{k+1})^{-1} &= (\mathscr{P}_k \mathscr{P}_1)^{-1} \\
&= (\mathscr{P}_1)^{-1} (\mathscr{P}_k)^{-1} \\
&= \mathscr{P}_{-1} \mathscr{P}_{-k} \\
&= \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{-k-5} & P_{-k-3} & P_{-k-4} \\ P_{-k-4} & P_{-k-2} & P_{-k-3} \\ P_{-k-3} & P_{-k-1} & P_{-k-2} \end{pmatrix} \\
&= \begin{pmatrix} P_{-k-6} & P_{-k-4} & P_{-k-5} \\ P_{-k-5} & P_{-k-3} & P_{-k-4} \\ P_{-k-4} & P_{-k-2} & P_{-k-3} \end{pmatrix} = \mathscr{P}_{-k-1}.
\end{aligned}$$

Hence the result. □

Theorem 14. *For $m, n \in \mathbb{N}$, the following equalities are hold:*

(i) $\mathscr{P}_{-m} \mathscr{P}_{-n} = \mathscr{P}_{-m-n}$

(ii) $\mathscr{P}_m \mathscr{P}_{-n} = \mathscr{P}_{m-n}$,

(iii) $\mathscr{P}_{-m} \mathscr{R}_{-n} = \mathscr{R}_{-n} \mathscr{P}_{-m} = \mathscr{R}_{-m-n}$,

(iv) $\mathscr{P}_m \mathscr{R}_{-n} = \mathscr{R}_m \mathscr{P}_{-n} = \mathscr{R}_{m-n}$,

(v) $\mathscr{R}_{-m} \mathscr{R}_{-n} = 4\mathscr{P}_{-m-n-4} + 4\mathscr{P}_{-m-n-7} + \mathscr{P}_{-m-n-10}$,

(vi) $\mathscr{R}_{-m} \mathscr{R}_{-n} = \mathscr{R}_0 \mathscr{R}_{-m-n}$,

(vii) $\mathscr{R}_m \mathscr{R}_{-n} = \mathscr{R}_0 \mathscr{R}_{m-n}$

Proof. (i) We will use Theorem 13 and Proposition 2. Then we obtain

$$\begin{aligned}
\mathscr{P}_{-m} \mathscr{P}_{-n} &= (\mathscr{P}_m)^{-1} (\mathscr{P}_n)^{-1} \\
&= (\mathscr{P}_n \mathscr{P}_m)^{-1} \\
&= (\mathscr{P}_{m+n})^{-1} = \mathscr{P}_{-m-n}.
\end{aligned}$$

(ii) From Theorem 5 with its assumptions and Proposition 2, we can have

$$\begin{aligned}\mathscr{P}_m\mathscr{P}_{-n} &= \left(A_1x^m+B_1y^m+C_1z^m\right)\left(A_1x^{-n}+B_1y^{-n}+C_1z^{-n}\right)\\ &= A_1A_1x^{m-n}+A_1B_1x^my^{-n}+A_1C_1x^mz^{-n}\\ &\quad +B_1A_1y^mx^{-n}+B_1B_1y^{m-n}+B_1C_1y^mz^{-n}\\ &\quad +C_1A_1z^mx^{-n}+C_1B_1z^my^{-n}+C_1C_1z^{m-n}.\end{aligned}$$

In here, since $x+y+z=0$ and $xyz=1$, a simple matrix calculations imply that $A_1^2=A_1$, $B_1^2=B_1$, $C_1^2=C_1$ and

$$A_1B_1=A_1C_1=B_1A_1=B_1C_1=C_1A_1=C_1B_1=[0]\,.$$

Then we obtain

$$\mathscr{P}_m\mathscr{P}_{-n}=A_1x^{m-n}+B_1y^{m-n}+C_1z^{m-n}=\mathscr{P}_{m-n}.$$

(iii) Here, we will just show that the truthness of the equality $\mathscr{P}_{-m}\mathscr{R}_{-n}=\mathscr{R}_{-m-n}$ since the other can be done similarly. Now, by Proposition 11(ii), we write

$$\mathscr{P}_{-m}\mathscr{R}_{-n}=\mathscr{P}_{-m}\mathscr{P}_{-n}\mathscr{R}_0.$$

On the other hand, by the above (i) and again Proposition 11(ii), we finally have $\mathscr{P}_{-m}\mathscr{R}_{-n}=\mathscr{P}_{-m-n}\mathscr{R}_0=\mathscr{R}_{-m-n}$.

(iv) As in (iii), we will just show the first equality of this condition. So, by Proposition 11(ii), we write

$$\mathscr{P}_m\mathscr{R}_{-n}=\mathscr{P}_m\mathscr{P}_{-n}\mathscr{R}_0.$$

On the other hand, by the above (ii) and again Proposition 11(ii), we finally have $\mathscr{P}_m\mathscr{R}_{-n}=\mathscr{P}_{m-n}\mathscr{R}_0=\mathscr{R}_{m-n}$.

(v) By Proposition 11(iii), we have

$$\mathscr{R}_{-m}\mathscr{R}_{-n}=(2\mathscr{P}_{-m-2}+\mathscr{P}_{-m-5})(2\mathscr{P}_{-n-2}+\mathscr{P}_{-n-5}).$$

It is easy to see that one can use the above (i) in this latest equality. Thus, applying sufficient operations, we then obtain

$$\mathscr{R}_{-m}\mathscr{R}_{-n}=4\mathscr{P}_{-m-n-4}+4\mathscr{P}_{-m-n-7}+\mathscr{P}_{-m-n-10},$$

as desired.

(vi) Here, we will just show that the truthness of the equality $\mathscr{R}_{-m}\mathscr{R}_{-n}=\mathscr{R}_0\mathscr{R}_{-m-n}$. Now, by Proposition 11(ii), we write

$$\mathscr{R}_{-m}\mathscr{R}_{-n}=\mathscr{R}_0\mathscr{P}_{-m}\mathscr{R}_{-n}.$$

On the other hand, by the above (iii), we finally have $\mathscr{R}_{-m}\mathscr{R}_{-n}=\mathscr{R}_0\mathscr{R}_{-m-n}$.

(vii) By Proposition 2, we write

$$\mathscr{R}_m\mathscr{R}_{-n}=\mathscr{R}_0\mathscr{P}_m\mathscr{R}_{-n}.$$

On the other hand, by the above iv), we finally have $\mathscr{R}_m\mathscr{R}_{-n}=\mathscr{R}_0\mathscr{R}_{m-n}$. □

Now, we consider again the proof of Theorem 7 by using Theorem 14(ii). Let us take $\sum_{i=0}^{n-1} \mathscr{P}_{-mi-j} = S$. Then we obtain

$$\begin{aligned} S\mathscr{P}_m &= \sum_{i=0}^{n-1} \mathscr{P}_{-mi-j+m} \\ &= \mathscr{P}_{m-j} + S - \mathscr{P}_{-m(n-1)-j}. \end{aligned}$$

By elementary operations, we have

$$\begin{aligned} \mathscr{P}_{-m(n-1)-j} - \mathscr{P}_{m-j} &= S(\mathscr{P}_0 - \mathscr{P}_m) \\ \sum_{i=0}^{n-1} \mathscr{P}_{-mi-j} &= \left(\mathscr{P}_{-mn+m-j} - \mathscr{P}_{m-j}\right)(\mathscr{P}_0 - \mathscr{P}_m)^{-1}. \end{aligned}$$

Comparing matrix entries and then using Theorem 14, we obtain the following result.

Corollary 15. *We have the following identities for Padovan and Perrin numbers:*

- $P_{-m-3}P_{-n-3} + P_{-m-1}P_{-n-2} + P_{-m-2}P_{-n-1} = P_{-m-n-1}$,
- $P_{m-3}P_{-n-3} + P_{m-1}P_{-n-2} + P_{m-2}P_{-n-1} = P_{m-n-1}$,
- $P_{-m-3}R_{-n-3} + P_{-m-1}R_{-n-2} + P_{-m-2}R_{-n-1} = R_{-m-n-1}$,
- $P_{m-3}R_{-n-3} + P_{m-1}R_{-n-2} + P_{m-2}R_{-n-1} = R_{m-n-1}$,
- $R_{-m-3}R_{-n-3} + R_{-m-1}R_{-n-2} + R_{-m-2}R_{-n-1} = 4P_{-m-n-5} + 4P_{-m-n-8} + P_{-m-n-11}$,
- $R_{-m-3}R_{-n-3} + R_{-m-1}R_{-n-2} + R_{-m-2}R_{-n-1} = 2R_{-m-n-3} + R_{-m-n-6}$,
- $R_{m-3}R_{-n-3} + R_{m-1}R_{-n-2} + R_{m-2}R_{-n-1} = 2R_{m-n-3} + R_{m-n-6}$.

The following theorem express us that there exist the relation between the positive indices and negative indices Perrin matrix sequences.

Theorem 16. *For $n \geq 0$, we have*

$$\mathscr{R}_{-n} = (\mathscr{R}_0)^{1-n}(\mathscr{R}_{-1})^n.$$

Proof. The proof will Theorem 14(vi). Then we have

$$\mathscr{R}_0\mathscr{R}_{-n} = \mathscr{R}_{-n+1}\mathscr{R}_{-1}. \tag{3.3}$$

By multiplying equation (3.3) with $\mathscr{R}_0$, then we write

$$\begin{aligned} \mathscr{R}_0\mathscr{R}_0\mathscr{R}_{-n} &= \mathscr{R}_0\mathscr{R}_{-n+1}\mathscr{R}_{-1} \\ &= \mathscr{R}_{-n+2}\mathscr{R}_{-1}\mathscr{R}_{-1}. \end{aligned}$$

By iterative process, we hence obtain

$$\mathscr{R}_0^{n-1}\mathscr{R}_{-n} = \mathscr{R}_{-1}^n.$$

So, by applying some elementary operations, we get

$$\mathscr{R}_{-n} = \mathscr{R}_0^{1-n}\mathscr{R}_{-1}^n.$$

Hence the result. □

Also, in Theorem 16, by taking account Theorem 14(iv), we can write

$$\begin{aligned}\mathscr{R}_{-n} &= (\mathscr{R}_n\mathscr{P}_{-n})^{1-n}\mathscr{R}_{-1}^n \\ &= \mathscr{P}_{-n}^{1-n}\mathscr{R}_n^{1-n}\mathscr{R}_{-1}^n.\end{aligned}$$

Then we consider Theorem 13, we obtain

$$\mathscr{R}_{-n} = \left(\mathscr{P}_n\mathscr{R}_n^{-1}\mathscr{R}_{-1}\right)^{n-1}\mathscr{R}_{-1}.$$

In the light of the above results, the following theorems provide the convenience us to obtain the powers of Padovan and Perrin matrix sequences.

Theorem 17. *For $m,n,r \in \mathbb{N}$ and $n \geq r$, the following equalities are hold:*

(i) $(\mathscr{P}_{-n})^m = \mathscr{P}_{-mn}$,

(ii) $(\mathscr{P}_{-n-1})^m = (\mathscr{P}_{-1})^m\mathscr{P}_{-mn}$,

(iii) $\mathscr{P}_{-n-r}\mathscr{P}_{-n+r} = (\mathscr{P}_{-n})^2 = (\mathscr{P}_{-2})^n$.

Proof. (i) By Theorem 13, we write

$$(\mathscr{P}_{-n})^m = \left((\mathscr{P}_n)^{-1}\right)^m = \left((\mathscr{P}_n)^m\right)^{-1}.$$

From Proposition 2 and again Theorem 13, we clearly obtain

$$(\mathscr{P}_{-n})^m = (\mathscr{P}_{mn})^{-1} = \mathscr{P}_{-mn}.$$

(ii) Let us consider the left-hand side of the equality. By considering similar approximation in (i) and Theorem 14(i), we write

$$\begin{aligned}(\mathscr{P}_{-n-1})^m &= \mathscr{P}_{m(-n-1)} \\ &= \mathscr{P}_{-m}\mathscr{P}_{-mn} \\ &= \mathscr{P}_{-1}\mathscr{P}_{-m+1}\mathscr{P}_{-mn}.\end{aligned}$$

Similarly, we can write $\mathscr{P}_{-m+1} = \mathscr{P}_{-1}\mathscr{P}_{-m+2}$. By iterative process, we hence obtain

$$(\mathscr{P}_{-n-1})^m = \underbrace{\mathscr{P}_{-1}\mathscr{P}_{-1}\cdots\mathscr{P}_{-1}}_{m \text{ times}}\mathscr{P}_{-mn} = (\mathscr{P}_{-1})^m\mathscr{P}_{-mn}.$$

The proof of (iii) can be seen quite similarly as the proof of (ii). □

Theorem 18. *For $m,n,r \in \mathbb{N}$ and $n \geq r$, the equalities*

$$\mathscr{R}_{-n-r}\mathscr{R}_{-n+r} = (\mathscr{R}_{-n})^2 \quad \text{and} \quad (\mathscr{R}_{-n})^m = \mathscr{R}_0^m \mathscr{P}_{-mn} = \mathscr{R}_0^{m-1}\mathscr{R}_{-mn}$$

always hold.

Proof. In the first part of the proof, we mainly consider Theorem 14(vi). Hence we can write

$$\mathscr{R}_{-n-r}\mathscr{R}_{-n+r} = \mathscr{R}_0\mathscr{R}_{-2n} = \mathscr{R}_0\mathscr{R}_{-n-n} = \mathscr{R}_{-n}\mathscr{R}_{-n},$$

By applying some elementary operations, we obtain

$$\mathscr{R}_{-n-r}\mathscr{R}_{-n+r} = (\mathscr{R}_{-n})^2,$$

as required.

Secondly, let us consider the left-hand side of the equality $(\mathscr{R}_{-n})^m = \mathscr{R}_0^m \mathscr{P}_{-mn}$. By Proposition 11(ii), we have

$$(\mathscr{R}_{-n})^m = (\mathscr{R}_0\mathscr{P}_{-n})^m = \mathscr{R}_0^m (\mathscr{P}_{-n})^m.$$

By considering Theorem 14(i), we obtain

$$(\mathscr{R}_{-n})^m = \mathscr{R}_0^m \mathscr{P}_{-mn}.$$

So, we write $(\mathscr{R}_{-n})^m = \mathscr{R}_0^{m-1}\mathscr{R}_0\mathscr{P}_{-mn}$ and from Proposition 11(ii), we finally obtain $(\mathscr{R}_{-n})^m = \mathscr{R}_0^{m-1}\mathscr{R}_{-mn}$.

Hence that is the result. □

References

[1] H. Civciv and R. Türkmen, On the (s,t)-Fibonacci and Fibonacci matrix sequences, *Ars Combinatoria* **87** (2008), 161–173.

[2] S. Falcon and A. Plaza, On the Fibonacci k-numbers, *Chaos, Solitons & Fractal* **32** (2007), 1615–1624.

[3] H.H. Gulec and N. Taskara, On the (s,t)-Pell and (s,t)-Pell-Lucas sequences and their matrix representations, *Applied Mathematics Letter* **25** (2012), 1554–1559.

[4] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons Inc., NY, 2001.

[5] L. Marek-Crnjac, On the mass spectrum of the elementary particles of the standard model using El Naschie's golden field theory, *Chaos, Solutions & Fractals* **15**(4) (2003), 611–618.

[6] L. Marek-Crnjac, The mass spectrum of high energy elementary particles via El Naschie's golden mean nested oscillators, the Dunkerly-Southwell eigenvalue theorems and KAM, *Chaos, Solutions & Fractals* **18**(1) (2003), 125–133.

[7] A. G. Shannon, P. G. Anderson and A. F. Horadam, Properties of Cordonnier, Perrin and Van der Laan numbers, *International Journal of Mathematical Education in Science and Technology* **37**(7) (2006), 825–831.

[8] A. G. Shannon and A. F. Horadam, Generating functions for powers of third order recurrence sequences, *Duke Mathematical Journal* **38** (1971), 791–794.

[9] N. Yilmaz and N. Taskara, Matrix sequences in terms of Padovan and Perrin numbers, *Journal of Applied Mathematics* (2013), Article ID 941673.

[10] Y. Yazlik, N. Taskara, K. Uslu and N. Yilmaz, The Generalized (s,t)-Sequence and its Matrix Sequence, *American Institute of Physics* (*AIP*) *Conf. Proc.* **1389** (2012), 381–384.

[11] Y. Yazlik and N. Taskara, A note on generalized k-Horadam sequence, *Computers and Mathematics with Applications* **63** (2012), 36–41.