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Research Article

A Comparative Study on the Solutions of 4th Order Differential Equations With Boundary Conditions

C. Balarama Krishna*[1], S. Vishwa Prasad Rao[2] and Purushotham Endla[3]

[1] *Department of Mathematics, SR University, Warangal, India*
[2] *Department of Mathematics and Humanities, Kakatiya Institute of Technology and Science, Warangal, India*
[3] *Department of Physics, SR University, Warangal, India*
***Corresponding author:** cbrk2004@gmail.com

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Abstract. In the present study, we have investigated the differential equations of order four to evolve the methods to achieve the solution for differential equations. *Absolute Stability Region* (ASR) of the differential equations has been examined. *Numerical Differentiation* (ND) and *Differential Transform Method* (DTM) which are suggested and derived in this article are much suitable to understand the solutions of differential equations of fourth order. Both the methods are applied to some differential equations, numerical examples and results are presented to outline the capability and robustness of our strategies and compared them with that of exact solution.

Keywords. Numerical differentiation, Differential transform method, Absolute stability region, Boundary conditions

Mathematics Subject Classification (2020). 65L05, 65L06

1. Introduction

For finding the solution of IVPs, we have different single and multistep techniques. While single step techniques utilize the information about the preceding values of x_n, multistep methods use several previous computed steps to calculate x_{n+1}. There are a many number of multistep method formulas for approximating solutions of differential equations.

Solution of differential equations of 2nd order by linear multistep methods are discussed in Eskandari and Dahaghin [1]. Gear [2] has shown the numerical differentiation techniques for differential equations of 1st order are stiffly stable. Gragg and Statter [3], and Henrici [4] performed an in-depth analysis of single and multistep approaches. Stiffly stable methods of higher order have been discussed by Jain [5]. Special multistep techniques using numerical differentiation to solve the IVPs have been discussed by Rao [9]. In particular, the approaches mentioned in this article are focused on the assumption that the solution is better estimated by polynomials. The inspiration behind the present article is that of Henrici [4], and Rao [9]. Special multistep techniques were derived by substituting $y(x)$ by interpolating polynomial and four times differentiating it. These implicit methods are observed to be of $(k-3)$rd order. The *absolute stability region* (ASR) for these methods is achieved. The computational efficiency measures of the techniques are calculated by solving differential equations and by contrasting them with that of exact/real solution. The outcomes recorded demonstrate the validity of the methods.

The differential transformation is a technique for solving ODE (ordinary differential equations). Zhou [12] has introduced this method and it is useful for solving the linear/non-linear IVPs in understanding the analysis of electric circuits. The DT method is employed for determining the solution for the differential equations by Rao *et al*. [8]. It is witnessed that *Differential Transform Technique* is an efficient and dependable instrument for the result of differential equations. The method provides the solution in series form. Precision of the results can be improvised by considering more number of terms. In most of the cases, results obtained by using differential transform method can be articulated in closed form. These methods are also considered in solving some mathematical problems (Krishna *et al*. [7]). Multistep methods were employed to solve 6th order boundary value problems by Krishna *et al*. [6]. For solving Bratu, Lane-Emden–type and other singular boundary value problems, B-spline collocation method was discussed in ([10, 11]).

2. Linear Multistep Method for BVPs of 4th Order

The BVPs of 4th order are in the following form

$$y^{iv} = f(x,y), \quad y(0) = y_0, \quad y'(0) = y_0', \quad y''(0) = y_0'', \quad y'''(0) = y_0''' \tag{1}$$

and they occur in many branches of engineering and technology.
The multistep method for solution of (1) is

$$y_{n+1} = \sum_{j-1}^{k} a_j y_{n+1-j} + h^4 \sum_{j=0}^{k} b_j y_{n+1-j}. \tag{2}$$

Here 'h' is the step size and a_j and b_j are constants.
Taking the polynomials

$$\rho(\xi) = \xi^k - \sum_{j=1}^{k} a_j \xi^{k-1} \text{ and } \sigma(\xi) = \sum_{j=1}^{k} b_j \xi^{k-1}. \tag{3}$$

Equation (2) becomes as

$$\rho(E)y_{n-k+1} - h^4\sigma(E)y^{iv}_{n-k+1} = 0. \tag{4}$$

Applying (4) to $y^{iv} = \lambda y$, we get

$$\sigma(\xi) - \bar{h}\sigma(\xi) = 0, \tag{5}$$

where $\bar{h} = \lambda h^4$.

The solutions ξ_i of above equation $\bar{h}$ are complex. *Absolute Stability Region* (ASR) is described as complex region $\bar{h}$-plane where the roots/solutions of equation (5) lies inside the circle of unit radius, every time $\bar{h}$ lies inside the region. Denoting the *absolute stability region* (ASR) by R, its boundary with ∂R, locus of the boundary is

$$\bar{h}(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}, \quad 0 \le \theta \le 2\pi. \tag{6}$$

3. Derivation of Proposed Methods

Let $p(x)$ be the interpolating polynomial of $y(x)$. This polynomial

$$p(x) = \sum_{m=0}^{k} (-1)^m \binom{-s}{m} \nabla^m y_{n+1}, \quad s = \frac{(x - x_{n+!})}{h} \tag{7}$$

on differentiating equation (7) four times with respect to x, we get

$$p^{iv}(x) = \left(\frac{1}{h^4}\right) \sum_{m=0}^{k} \frac{d^4}{ds^4}\left[(-1^m)\binom{-s}{m}\right] \nabla^m y_{n+1}.$$

Substituting $s = -r$ and taking $y^{iv}(x)$ as $p^{iv}(x)$ in (1), we obtain

$$\sum_{m=0}^{k} \delta_{r,m} \nabla^m y_{n+1} = h^4 f_{n+1-r}, \tag{8}$$

where

$$\delta_{r,m} = \frac{d^4}{ds^4}\left[(-1^m)\binom{-s}{m}\right]. \tag{9}$$

Taking $r = 1/2$ in (8), we can obtain the method as

$$\sum_{m=0}^{k} \delta_{\frac{1}{2},m} \nabla^m y_{n+1} = h^4 f_{n+\frac{1}{2}}. \tag{10}$$

These coefficient values $\delta_{\frac{1}{2},m}$ are given in Table 1.
After simplification, the equation (10) reduces to

$$\sum_{j=0}^{k} a_j y_{n+1-j} = h^4 f_{n+\frac{1}{2}}. \tag{11}$$

The coefficient values a_j are given in Table 2.

Table 1. Coefficient values of $\delta_{\frac{1}{2},m}$; $m = 0$ to 9

M	0	1	2	3	4	5	6	7	8	9
$\delta_{\frac{1}{2},m}$	0	0	0	0	1	$\frac{3}{2}$	$\frac{41}{24}$	$\frac{85}{48}$	$\frac{3861}{1920}$	$\frac{6587}{3840}$

Table 2. Coefficient values of $\delta_{\frac{1}{2},m}$; $m = 0$ to 9

k	j									
	0	1	2	3	4	5	6	7	8	9
4	1	-4	6	-4	1					
5	$\frac{5}{2}$	$\frac{-23}{2}$	$\frac{42}{2}$	$\frac{-38}{2}$	$\frac{17}{2}$	$\frac{-3}{2}$				
6	$\frac{101}{24}$	$\frac{-522}{24}$	$\frac{1119}{24}$	$\frac{-1276}{24}$	$\frac{819}{24}$	$\frac{-282}{24}$	$\frac{41}{24}$			
7	$\frac{287}{48}$	$\frac{-1639}{48}$	$\frac{4023}{48}$	$\frac{-5527}{48}$	$\frac{4613}{48}$	$\frac{-2349}{48}$	$\frac{677}{48}$	$\frac{-85}{48}$		
8	$\frac{15341}{1920}$	$\frac{-96448}{1920}$	$\frac{269028}{1920}$	$\frac{-434790}{1920}$	$\frac{457290}{1920}$	$\frac{-310176}{1920}$	$\frac{135188}{1920}$	$\frac{-34288}{1920}$	$\frac{3861}{1920}$	
9	$\frac{37269}{3840}$	$\frac{-252179}{3840}$	$\frac{775188}{3840}$	$\frac{-1427900}{3840}$	$\frac{1739542}{3840}$	$\frac{-1450314}{3840}$	$\frac{823684}{3840}$	$\frac{-305708}{3840}$	$\frac{67005}{3840}$	$\frac{-6587}{3840}$

The error term of (11) is

$$LTE = \delta_{0,k+1} h^{k+1} y^{k+1}(\eta). \tag{12}$$

It is clear that method (14) has $(k-3)$rd order and it is absolutely stable in the region $-4 \le h \le 0$. For method (11),

$$\rho(\xi) = \sum_{j=0}^{k} a_j \xi^{k-j} \quad \text{and} \quad \sigma(\xi) = \xi^k. \tag{13}$$

The ASRs of the discussed method for $k = 4$ to 9 are presented in Figures 1 and 2. The area beyond the boundary denotes ASR.

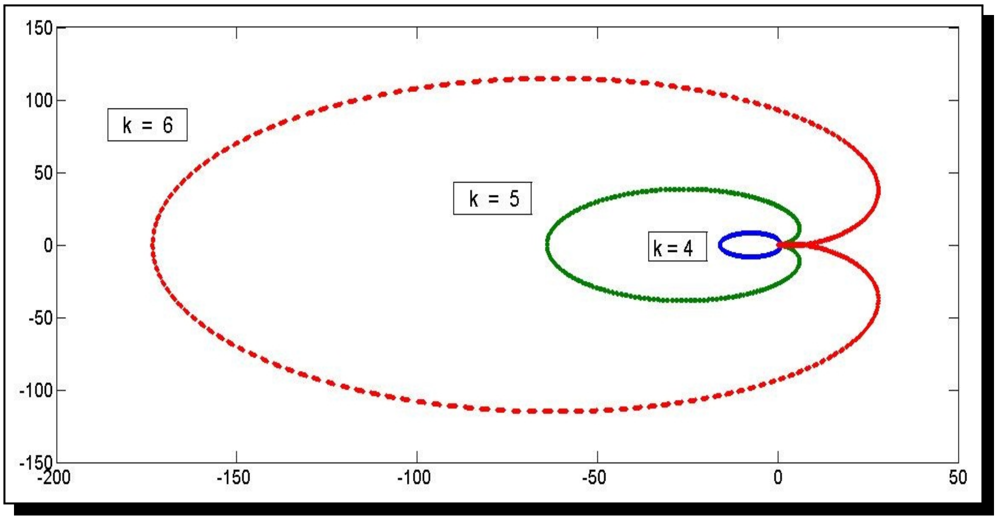

Figure 1. ASR for the method taking $k = 4, 5$ and 6

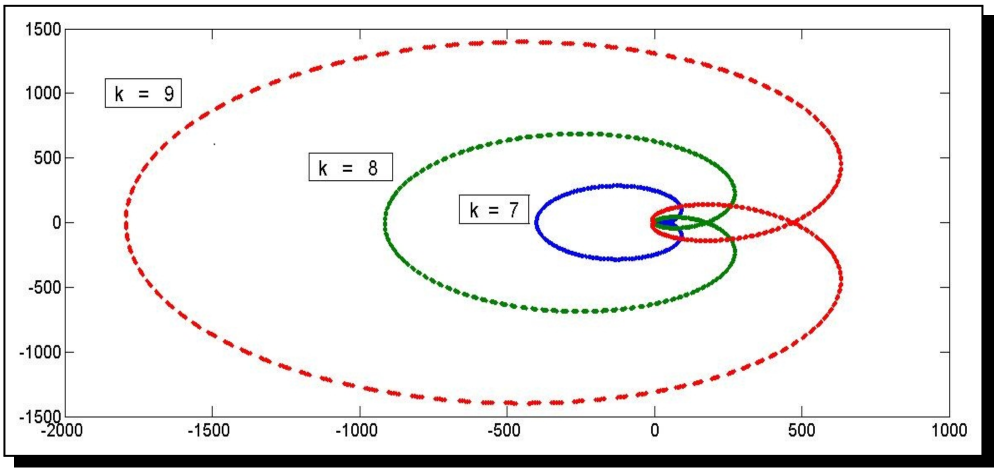

Figure 2. ASR for the method taking $k = 7, 8$ and 9

4. Differential Transform Method

Differential transform technique is employed to get the solution of some ODEs. Some ODE were solved to show the capability and robustness of this method with numerical examples.

An analytic function $y(x)$ be defined in some domain R and $x = x_0$ represents a point of R. Then, $y(x)$ is expressed as power series for which center is at x_0. For $\frac{d^k}{dx^k}y(x)$, its differential transform in one variable is defined in the following manner:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=0} \tag{14}$$

In (14), $Y(k)$ is transformation of $y(x)$.

Inverse differential transformation of $Y(k)$ is given as

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k) \tag{15}$$

From (14) and (15), we obtain

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right] \tag{16}$$

Equation (16) indicates that differential transformation concept is originated from expansion of Taylor series. It is quite easy to understand from equations (14) and (15) and acquire the below given operations:

Function	Transformed function
$y(x) = f(x) \pm g(x)$	$Y(k) = F(k) \pm G(k)$
$y(x) = cf(x)$	$Y(k) = cF(k)$
$y(x) = e^{ax}$	$Y(k) = a^k/k!$
$y(x) = \frac{d^n f(x)}{dx^n}$	$Y(k) = \frac{(k+n)!}{k!}F(k+n)$
$y(x) = \sin(ax+b)$	$Y(k) = \frac{a^k}{k!}\sin\left(\frac{k\pi}{2}+b\right)$

5. Numerical Example

The ND method is taken into consideration to work out the solution of the BVP

$$y^{iv} = y + \sin 2x, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -1, \quad y'''(0) = 0 \tag{17}$$

in $[0,2]$, step size $h = 0.01$ and 0.02. Results are given in Tables 3 and 4.

Table 3. The solution by ND method of 5th order for $k = 6$ and $h = 0.01$

x	Solution by 5th order ND	Solution by DT method	Exact solution	Absolute error (ND method)	Absolute error (DT method)
0	–3.3008700374E-14	0.000000000000000E+00	0.0000000000E+00	3.3008700374E-14	0.000000000000000E+00
0.1	9.5000248452E-02	9.500024845247710E-02	9.5000248452E-02	3.2335245592E-14	4.770905892570450E-13
0.2	1.8000789084E-01	1.800078908426860E-01	1.8000789084E-01	3.2029934260E-14	2.686073585778100E-12
0.3	2.5505939225E-01	2.550593922335340E-01	2.5505939225E-01	2.7755575616E-14	1.646632830087920E-11
0.4	3.2024773559E-01	3.202477353329160E-01	3.2024773559E-01	2.2926105459E-14	2.570836321602600E-10
0.5	3.7574733428E-01	3.757473318860100E-01	3.7574733428E-01	1.7708057243E-14	2.393990394811140E-09
0.6	4.2183572380E-01	4.218357089580910E-01	4.2183572380E-01	1.0380585280E-14	1.484190859102340E-08
0.7	4.5891144532E-01	4.589113756679960E-01	4.5891144532E-01	4.6074255522E-15	6.965200438635580E-08
0.8	4.8750765774E-01	4.875073914127170E-01	4.8750765774E-01	4.3853809473E-15	2.663272828384860E-07
0.9	5.0830115974E-01	5.083002883963640E-01	5.0830115974E-01	1.1768364061E-14	8.713436356888590E-07
1	5.2211666367E-01	5.221141415576070E-01	5.2211666367E-01	1.8429702209E-14	2.522112393155140E-06
1.1	5.2992633315E-01	5.299197186350600E-01	5.2992633315E-01	2.2759572005E-14	6.614514939884410E-06
1.2	5.3284476873E-01	5.328287716157310E-01	5.3284476873E-01	2.8532731733E-14	1.599711426925050E-05
1.3	5.3211979428E-01	5.320836363137710E-01	5.3211979428E-01	3.4861002973E-14	3.615796622868930E-05
1.4	5.2911955478E-01	5.290423761014190E-01	5.2911955478E-01	3.7636560535E-14	7.717867858103580E-05
1.5	5.2531657675E-01	5.251597242771510E-01	5.2531657675E-01	3.8191672047E-14	1.568524728494670E-04
1.6	5.2226956144E-01	5.219640312638640E-01	5.2226956144E-01	4.1189274214E-14	3.055301761363260E-04
1.7	5.2160377253E-01	5.210302904782960E-01	5.2160377253E-01	4.2410519541E-14	5.734820517043770E-04
1.8	5.2499094176E-01	5.239490816380270E-01	5.2499094176E-01	4.0523140399E-14	1.041860121973050E-03
1.9	5.3412964448E-01	5.322909124502780E-01	5.3412964448E-01	3.7414515930E-14	1.838732029721510E-03
2	5.5072709313E-01	5.475649376733980E-01	5.5072709313E-01	3.2196467714E-14	3.162155456602480E-03

Table 4. The solution by ND method of 5th order for $k = 6$ and $h = 0.02$

x	Solution by 5th order ND	Solution by DT method	Exact solution	Absolute error (ND method)	Absolute error (DT method)
0	1.0667583069E-09	0.000000000000000E+00	0.0000000000E+00	1.0667583069E-09	0.000000000000000E+00
0.1	9.5000239003E-02	9.500024845247710E-02	9.5000248452E-02	9.4491035968E-09	4.770905892570450E-13
0.2	1.8000787149E-01	1.800078908426860E-01	1.8000789084E-01	1.9354154918E-08	2.686073585778100E-12
0.3	2.5505936387E-01	2.550593922335340E-01	2.5505939225E-01	2.8376869077E-08	1.646632830087920E-11
0.4	3.2024769932E-01	3.202477353329160E-01	3.2024773559E-01	3.6268672343E-08	2.570836321602600E-10
0.5	3.7574729146E-01	3.757473318860100E-01	3.7574733428E-01	4.2814860823E-08	2.393990394811140E-09
0.6	4.2183567596E-01	4.218357089580910E-01	4.2183572380E-01	4.7844171902E-08	1.484190859102340E-08
0.7	4.5891139408E-01	4.589113756679960E-01	4.5891144532E-01	5.1236667353E-08	6.965200438635580E-08
0.8	4.8750760481E-01	4.875073914127170E-01	4.8750765774E-01	5.2929533867E-08	2.663272828384860E-07
0.9	5.0830110682E-01	5.083002883963640E-01	5.0830115974E-01	5.2920682281E-08	8.713436356888590E-07
1	5.2211661240E-01	5.221141415576070E-01	5.2211666367E-01	5.1269945178E-08	2.522112393155140E-06
1.1	5.2992628506E-01	5.299197186350600E-01	5.2992633315E-01	4.8097817951E-08	6.614514939884410E-06
1.2	5.3284472515E-01	5.328287716157310E-01	5.3284476873E-01	4.3581887432E-08	1.599711426925050E-05
1.3	5.3211975633E-01	5.320836363137710E-01	5.3211979428E-01	3.7950970588E-08	3.615796622868930E-05
1.4	5.2911952330E-01	5.290423761014190E-01	5.2911955478E-01	3.1477311202E-08	7.717867858103580E-05
1.5	5.2531655228E-01	5.251597242771510E-01	5.2531657675E-01	2.4467117221E-08	1.568524728494670E-04
1.6	5.2226954419E-01	5.219640312638640E-01	5.2226956144E-01	1.7249739859E-08	3.055301761363260E-04
1.7	5.2160376237E-01	5.210302904782960E-01	5.2160377253E-01	1.0166110065E-08	5.734820517043770E-04
1.8	5.2499093820E-01	5.239490816380270E-01	5.2499094176E-01	3.5566819490E-09	1.041860121973050E-03
1.9	5.3412964673E-01	5.322909124502780E-01	5.3412964448E-01	2.2504385022E-09	1.838732029721510E-03
2	5.5072710008E-01	5.475649376733980E-01	5.5072709313E-01	6.9508315770E-09	3.162155456602480E-03

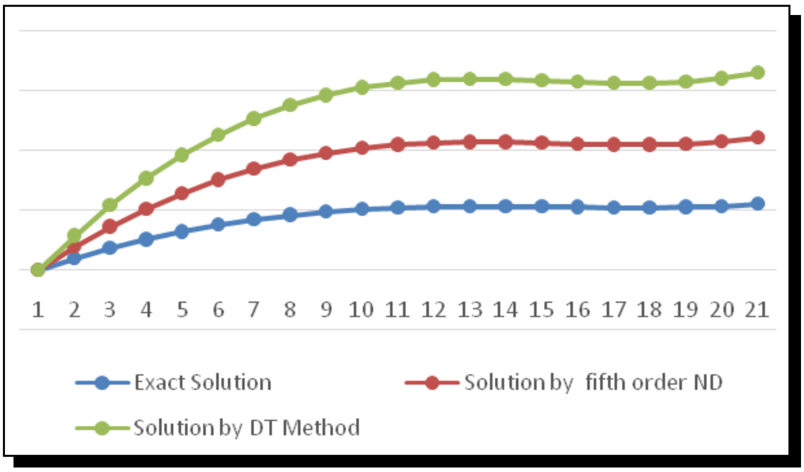

Figure 3. ND method ($h = 0.01$) v/s DT method

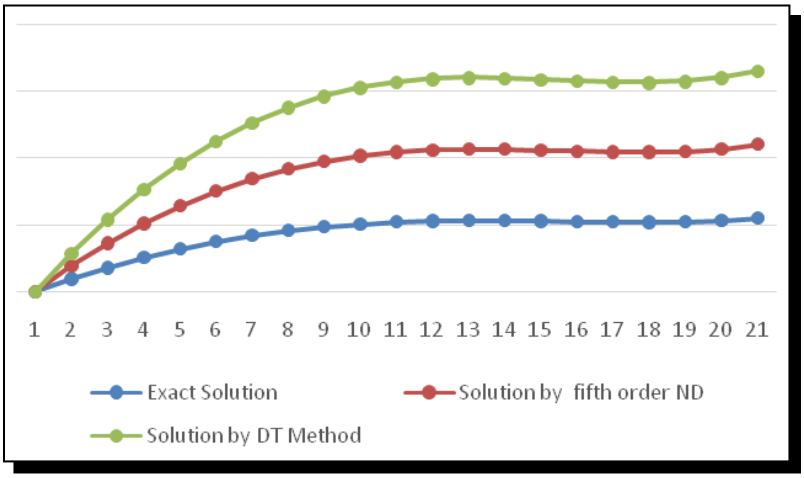

Figure 4. ND method ($h = 0.02$) v/s DT method

The 5th order ND method obtained for $k = 6$ is

$$y_{n+1} = \frac{522}{101}y_n - \frac{1119}{101}y_{n-1} + \frac{1276}{101}y_{n-2} - \frac{819}{101}y_{n-3} + \frac{282}{101}y_{n-4} - \frac{41}{101}y_{n-5} + \frac{24}{101}h^4 f_{n+\frac{1}{2}} \tag{18}$$

The 6th order ND method obtained for $k = 7$ is

$$y_{n+1} = \frac{1639}{287}y_n - \frac{4023}{287}y_{n-1} + \frac{5527}{287}y_{n-2} - \frac{4613}{287}y_{n-3} + \frac{2349}{287}y_{n-4} - \frac{677}{287}y_{n-5} + \frac{85}{287}y_{n-6} + \frac{48}{287}h^4 f_{n+\frac{1}{2}}. \tag{19}$$

Now applying the DT method to the same differential equation (17), applying the differential transformation on either side of (17), we get

$$Y(k+4) = \frac{k!}{(k+4)!}\left[Y(K) + \frac{2^k}{k!}\sin\frac{k\pi}{2}\right] \tag{20}$$

and

$$Y(0) = 0,\quad Y(1) = 1,\quad Y(2) = -\frac{1}{2}, Y(3) = 0. \tag{21}$$

Putting equation (21) in (20) and by the recursive method, we get

$$
\begin{array}{llll}
Y(4)=0 & Y(5)=\dfrac{1}{40} & Y(6)=\dfrac{-1}{720} & Y(7)=\dfrac{-1}{630} \\
Y(8)=0 & Y(9)=\dfrac{1}{10368} & Y(10)=\dfrac{-1}{3628800} & Y(11)=\dfrac{-1}{39916800} \\
Y(12)=0 & Y(13)=\dfrac{547}{6227020800} & \cdots &
\end{array}
$$

The solution can be expressed as

$$y(x)=\sum_{k=0}^{\infty} x^k Y(k)=x-\frac{x^2}{2}+\frac{x^5}{40}-\frac{x^6}{720}-\frac{x^7}{630}+\frac{x^9}{10368}+\frac{x^{10}}{3628800}-\frac{136x^{11}}{39916800}+\dots$$

6. Conclusion

Outside of some closed boundaries, the ND methods are found to be stable. The solutions obtained by ND methods in this article are more precise. The absolute errors were very small. The results obtained by DT method are also very near to the exact solution. However, by analyzing the results, values recorded in Tables 3 and 4 and the Figure 3 and 4, it is apparent that the results by ND method are superior to the results obtained by using DT method.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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