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Research Article

Stability of a Quadratic-Reciprocal Functional Equation: Direct Method

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Abstract. In this paper, we establish the Hyers-Ulam, Hyers-Ulam-Rassias, generalized Hyers-Ulam-Rassias, and Rassias stability results of the quadratic-reciprocal functional equation:

$$f(x+y)=\frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}}$$

connected with fuzzy homomorphisms and fuzzy derivations between fuzzy Banach algebras using direct method.

Keywords. Quadratic reciprocal functional equation, Ulam-Hyers stability, Fuzzy Banach algebra

Mathematics Subject Classification (2020). 39B82, 97I70, 46J99

1. Introduction

Functional equations are equations that involve functions rather than just numbers. They are used in mathematics to solve for unknown functions. In a functional equation, the unknown functions are usually denoted by variables, and the functional equation provides a relationship between the variables that must be satisfied for the equation to hold true. Functional equations can be used to study a wide range of problems, such as describing the behavior of physical systems, solving mathematical puzzles, and understanding data.

Functional equation stability is the study of how solutions to a functional equation change when the terms of the equation are modified. Specifically, it studies how the solutions of a functional equation vary when the parameters of the equation are changed. The study of functional equation stability examines how small changes to the equation can cause large changes in the solutions. It is an important concept in mathematics, economics, and other fields.

In the fall of 1940, S.M. Ulam [23] gave a wide-ranging talk before a Mathematical Colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms:

Let $(G_1,*)$ be a group and let $(G_2,\diamond,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exists $\delta(\epsilon)>0$ such that if $h:G_1\to G_2$ satisfies the inequality

$$d(h(x*y),h(x)\diamond h(y))<\delta, \quad \forall\, x,y\in G_1,$$

then there is a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$.

If the answer is affirmative, we say that the functional equation for homomorphisms is stable.

In the next year, Hyers [8] gave a affirmative answer to this question for additive groups under the assumption that groups are Banach spaces. He brilliantly answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces. The result of Hyers is stated as follows:

Theorem 1.1. *Let $f:E_1\to E_2$ be a function between Banach spaces such that*

$$\|f(x+y)-f(x)-f(y)\|\le\epsilon \tag{1.1}$$

for all $x,y\in E_1$ and $\epsilon>0$ is a constant. Then the limit

$$A(x)=\lim_{n\to\infty}2^{-n}f(2^nx) \tag{1.2}$$

exists for each $x\in E_1$ and $A:E_1\to E_2$ is unique additive mapping satisfying

$$\|f(x)-A(x)\|\le\epsilon \tag{1.3}$$

for all $x\in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x\in E_1$, then the function A is linear.

Taking this famous result into consideration, the additive Cauchy functional equation $f(x+y)=f(x)+f(y)$ is said to have the *Hyers-Ulam stability* on (E_1,E_2) if for every function $f:E_1\to E_2$ satisfying the inequality (1.1) for some $\epsilon\ge 0$ and for all $x,y\in E_1$, there exists an additive function $A:E_1\to E_2$ such that $f-A$ is bounded on E_1.

The method in (1.2) provided by Hyers which produces the additive function A will be called a *direct method*. This method is the most important and powerful tool to study the stability of various functional equations.

It is possible to prove a stability result similar to Hyers functions that do not have bounded Cauchy difference. In the year 1950, Aoki [1] first generalized the Hyers theorem for unbounded Cauchy difference having sum of norms $(\|x\|^p+\|y\|^p)$.

The same result was rediscovered by Rassias [21] in 1978 and proved a generalization of Hyers theorem for additive mappings. This stability result is named *Hyers-Ulam-Rassias stability* or *Hyers-Ulam-Aoki-Rassias stability* for the functional equation.

In 1982, Rassias [18] followed the innovative approach of Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \|y\|^q$ with $p+q \neq 1$. Later, this stability result was called *Ulam-Gavruta-Rassias stability* of functional equation.

In 1991, Gajda [6] provided an affirmative solution to Th. M. Rassias' question for p strictly greater than one.

In 1994, Gavruta [7] provided a further generalization of Rassias [21] theorem in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x,y)$. This stability result is called *Generalized Hyers-Ulam-Rassias stability* of functional equation.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi *et al*. [22] by considering the summation of both the sum and the product of two p-norms in the sprite of Rassias approach and is named *J. M. Rassias Stability* of functional equation. Last seven decades the stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see Gavruta [7], Karthikeyan *et al*. [9–11], Rassias *et al*. [20], and Ravi *et al*. [22]).

In this paper, we prove the generalized Hyers-Ulam stability for a new simple quadratic reciprocal functional equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}} \tag{1.4}$$

in fuzzy Banach algebras by using direct method. It is easily verified that the quadratic reciprocal function $f(x) = \frac{1}{x^2}x$ is a solution of the functional equation (1.4). We apply the definition of fuzzy normed spaces and fuzzy Banach algebras briefly as given by Katsaras [12], Kramosil and Michálek [13], Mirmostafaee and Moslehian [14], and Park [17].

Throughout this paper, assume that $(\mathcal{A}, N)$ is a fuzzy normed algebra and that $(\mathcal{B}, N')$ is a fuzzy Banach algebra.

2. Fuzzy Homomorphism Stability Results

In this section, using Hyers direct method, we establish the stability of fuzzy homomorphisms connected to quadratic reciprocal functional equation (1.4) in fuzzy Banach algebras.

Definition 2.1. A C-linear mapping $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ is called a quadratic reciprocal fuzzy homomorphism if $\mathcal{H}(xy) = \mathcal{H}(x)\mathcal{H}(y)$ for all $x, y \in \mathcal{A}$.

Theorem 2.2. *Let $\alpha : \mathcal{A}^2 \to [0,\infty)$ be a mapping such that for some d with $0 < \left(\frac{d}{2^2}\right) < 1$,*

$$N'\left(\alpha\left(\frac{x}{2^n}, \frac{x}{2^n}\right), r\right) \geq N'(d^n \alpha(x,x), r), \tag{2.1}$$

for all $x \in \mathcal{A}$ and all $r > 0$, $d > 0$, and

$$\lim_{n\to\infty} N'\left(\alpha\left(\frac{x}{2^n}, \frac{y}{2^n}\right), r\right) = 1, \tag{2.2}$$

for all $x, y \in \mathcal{A}$ and all $r > 0$. Suppose that a function $f : \mathcal{A} \to \mathcal{B}$ satisfies the inequality

$$N\left(f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}}, r\right) \geq N'(\alpha(x,y), r) \tag{2.3}$$

and

$$N(f(xy)-f(x)f(y),r) \geq N'(\alpha(x,y),r), \tag{2.4}$$

for all $x,y \in \mathcal{A}$ and all $r>0$. Then, there exists a unique quadratic reciprocal homomorphism $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ such that

$$N(f(x)-\mathcal{H}(x),r) \geq N'\left(\alpha(x,x), \frac{r|2^2-d|}{2^2}\right) \tag{2.5}$$

for all $x \in \mathcal{A}$ and all $r>0$, $d>0$. The mapping $\mathcal{H}(x)$ is defined by

$$\mathcal{H}(x) = N - \lim_{n\to\infty} \frac{f\left(\frac{x}{2^n}\right)}{2^{2n}} \tag{2.6}$$

for all $x \in \mathcal{A}$.

Proof. Replacing (x,y) by $\left(\frac{x}{2},\frac{x}{2}\right)$ in (2.3), we get

$$N\left(f(x)-\frac{1}{4}f\left(\frac{x}{2}\right),r\right) \geq N'\left(\alpha\left(\frac{x}{2},\frac{x}{2}\right),r\right) \tag{2.7}$$

for all $x \in \mathcal{A}$ and all $r>0$. Again replacing x by $\frac{x}{2^n}$ and r by $d^n r$, and using (2.1) in (2.7), we reach

$$N\left(\frac{f\left(\frac{x}{2^n}\right)}{2^{2n}} - \frac{f\left(\frac{x}{2^{n+1}}\right)}{2^{2(n+1)}}, \frac{d^n r}{2^{2n}}\right) \geq N'(\alpha(x,x),r) \tag{2.8}$$

for all $x \in \mathcal{A}$ and all $r>0$. It is easy to see that

$$f(x) - \frac{f\left(\frac{x}{2^n}\right)}{2^{2n}} = \sum_{i=0}^{n-1}\left[\frac{f\left(\frac{x}{2^i}\right)}{2^{2i}} - \frac{f\left(\frac{x}{2^{i+1}}\right)}{2^{2(i+1)}}\right] \tag{2.9}$$

for all $x \in \mathcal{A}$. From equations (2.8) and (2.9), we arrive

$$\begin{aligned} N\left(f(x)-\frac{f\left(\frac{x}{2^n}\right)}{2^{2n}}, \sum_{i=0}^{n-1}\frac{d^i r}{2^{2i}}\right) &\geq \min \bigcup_{i=0}^{n-1}\left\{\frac{f\left(\frac{x}{2^i}\right)}{2^{2i}} - \frac{f\left(\frac{x}{2^{i+1}}\right)}{2^{2(i+1)}}, \frac{d^i\, r}{2^{2i}}\right\} \\ &\geq \min \bigcup_{i=0}^{n-1}\left\{N'\left(\alpha\left(\frac{x}{2^i},\frac{x}{2^i}\right),r\right)\right\} \\ &= N'(\alpha(x,x),r) \end{aligned} \tag{2.10}$$

for all $x \in \mathcal{A}$ and all $r>0$. Replacing x by $\frac{x}{2^m}$ in (2.10) and using (2.1), we obtain

$$N\left(\frac{f\left(\frac{x}{2^m}\right)}{2^{2m}} - \frac{f\left(\frac{x}{2^{m+n}}\right)}{2^{2(m+n)}}, \sum_{i=0}^{n-1}\frac{d^i\, r}{2^{2(i+m)}}\right) \geq N'\left(\alpha(x,x),\frac{r}{d^m}\right) \tag{2.11}$$

for all $x \in \mathcal{A}$ and all $r>0$ and all $m,n \geq 0$. Replacing r by $d^m r$ in (2.11), we get

$$N\left(\frac{f\left(\frac{x}{2^m}\right)}{2^{2m}} - \frac{f\left(\frac{x}{2^{m+n}}\right)}{2^{2(m+n)}}, \sum_{i=m}^{m+n-1}\frac{d^i\, r}{2^{2i}}\right) \geq N'(\alpha(x,x),r) \tag{2.12}$$

for all $x \in \mathcal{A}$ and all $r>0$ and all $m,n \geq 0$. From (2.12), we reach

$$N\left(\frac{f\left(\frac{x}{2^m}\right)}{2^{2m}} - \frac{f\left(\frac{x}{2^{m+n}}\right)}{2^{2(m+n)}}, r\right) \geq N'\left(\alpha(x,x), \frac{r}{\sum_{i=m}^{m+n-1}\frac{d^i}{2^{2i}}}\right) \tag{2.13}$$

for all $x \in \mathcal{A}$ and all $r > 0$ and all $m,n \geq 0$. Since $0 < d < 2^2$ and $\sum_{i=0}^{n}\left(\frac{d}{2^2}\right)^i < \infty$, the Cauchy criterion for convergence and implies that $\{\frac{f\left(\frac{x}{2^n}\right)}{2^{2n}}\}$ is a Cauchy sequence in $(\mathcal{B},N)$. Since $(\mathcal{B},N)$ is a fuzzy Banach algebra, this sequence converges to some point $\mathcal{H}(x) \in \mathcal{B}$. So one can we define the mapping $\mathcal{H}:\mathcal{A} \to \mathcal{B}$ by $\mathcal{H}(x) = N - \lim_{n\to\infty}\left(\frac{f\left(\frac{x}{2^n}\right)}{2^{2n}}\right)$ for all $x \in \mathcal{A}$. Letting $m = 0$ in (2.13), we get

$$N\left(f(x) - \frac{f\left(\frac{x}{2^n}\right)}{2^{2n}}, r\right) \geq N'\left(\alpha(x,x), \frac{r}{\sum_{i=0}^{n-1}\frac{d^i}{2^{2i}}}\right) \tag{2.14}$$

for all $x \in \mathcal{A}$ and all $r > 0$. Letting $n \to \infty$ in (2.14), we arrive $N(f(x) - \mathcal{H}(x), r) \geq N'\left(\alpha(x,x), \frac{r(4-d)}{4}\right)$ for all $x \in \mathcal{A}$ and all $r > 0$. Now, we need to prove $\mathcal{H}$ satisfies the functional equation (1.4), replacing (x,y) by $\left(\frac{x}{2^n}, \frac{y}{2^n}\right)$ in (2.3), we obtain

$$N\left(\frac{1}{2^{2n}}\left(f\left(\frac{x}{2^n}+\frac{y}{2^n}\right) - \frac{f\left(\frac{x}{2^n}\right)f\left(\frac{y}{2^n}\right)}{f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)+2\sqrt{f\left(\frac{x}{2^n}\right)f\left(\frac{y}{2^n}\right)}}\right), r\right) \geq N'\left(\alpha\left(\frac{x}{2^n},\frac{y}{2^n}\right), 2^{2n}r\right) \tag{2.15}$$

for all $r > 0$ and all $x,y \in \mathcal{A}$. Now,

$$\begin{aligned}
&N\left(\mathcal{H}(x+y) - \frac{\mathcal{H}(x)\mathcal{H}(y)}{\mathcal{H}(x)+\mathcal{H}(y)+2\sqrt{\mathcal{H}(x)\mathcal{H}(y)}}, r\right)\\
&\geq \min\left\{N\left(\mathcal{H}(x+y) - \frac{1}{2^{2n}}f\left(\frac{2^n(x+y)}{2^2n}\right), \frac{r}{3}\right),\right.\\
&\qquad N\left(-\frac{\mathcal{H}(x)\mathcal{H}(y)}{\mathcal{H}(x)+\mathcal{H}(y)+2\sqrt{\mathcal{H}(x)\mathcal{H}(y)}} + \frac{\frac{1}{2^{2n}}f\left(\frac{x}{2^n}\right)f\left(\frac{y}{2^n}\right)}{f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)+2\sqrt{f\left(\frac{x}{2^n}\right)f\left(\frac{y}{2^n}\right))}}, \frac{r}{3}\right),\\
&\qquad \left.N\left(-\frac{\frac{1}{2^{2n}}f\left(\frac{x}{2^n}\right)f\left(\frac{y}{2^n}\right)}{f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)+2\sqrt{f\left(\frac{x}{2^n}\right)f\left(\frac{y}{2^n}\right)}}, \frac{r}{3}\right)\right\}
\end{aligned} \tag{2.16}$$

for all $x,y \in \mathcal{A}$ and all $r > 0$. Using (2.15) in (2.16), we arrive

$$\begin{aligned}
N\left(\mathcal{H}(x+y) - \frac{\mathcal{H}(x)\mathcal{H}(y)}{\mathcal{H}(x)+\mathcal{H}(y)+2\sqrt{\mathcal{H}(x)\mathcal{H}(y)}}, r\right) &\geq \min\left\{1, 1, N'\left(\alpha\left(\frac{x}{2^n},\frac{x}{2^n}\right), 2^{2n}r\right)\right\}\\
&\geq N'\left(\alpha\left(\frac{x}{2^n},\frac{x}{2^n}\right), 2^{2n}r\right)
\end{aligned} \tag{2.17}$$

for all $x,y \in \mathcal{A}$ and all $r > 0$. Letting $n \to \infty$ in (2.17) and using (2.2), we see that

$$N\left(\mathcal{H}(x+y) - \frac{\mathcal{H}(x)\mathcal{H}(y)}{\mathcal{H}(x)+\mathcal{H}(y)+2\sqrt{\mathcal{H}(x)\mathcal{H}(y)}}, r\right) = 1 \tag{2.18}$$

for all $x,y \in \mathcal{A}$ and all $r > 0$. From (2.18), we get

$$\mathcal{H}(x+y) = \frac{\mathcal{H}(x)\mathcal{H}(y)}{\mathcal{H}(x)+\mathcal{H}(y)+2\sqrt{\mathcal{H}(x)\mathcal{H}(y)}}$$

for all $x,y \in \mathcal{A}$. Hence $\mathcal{H}$ satisfies the quadratic reciprocal functional equation (1.4). This shows that $\mathcal{H}$ is quadratic reciprocal. So, it follows that

$$N\left(\mathcal{H}(xy) - \mathcal{H}(x)\mathcal{H}(y), r\right) = N\left(\frac{1}{2^{4n}}(f(2^{2n}xy) - f(2^n x)f(2^n y)), \frac{r}{2^{4n}}\right)$$

$$\geq N'\left(\alpha\left(\frac{x}{2^n},\frac{x}{2^n}\right),r\right) \tag{2.19}$$

for all $x,y \in \mathcal{A}$ and all $r>0$. Letting $n\to\infty$ in (2.19) and using (2.2), we obtain

$$N(\mathcal{H}(xy)-\mathcal{H}(x)\mathcal{H}(y),r)=1$$

for all $x,y\in\mathcal{A}$ and all $r>0$. Hence, we have $\mathcal{H}(xy)=\mathcal{H}(x)\mathcal{H}(y)$ for all $x,y\in\mathcal{A}$. Therefore, $\mathcal{H}$ is a quadratic reciprocal homomorphism. In order to prove $\mathcal{H}(x)$ is unique, let $\mathcal{H}'(x)$ be another quadratic reciprocal function satisfying (1.4) and (2.6). Hence,

$$\begin{aligned}
N(\mathcal{H}(x)-\mathcal{H}'(x),r) &= N\left(\frac{\mathcal{H}\left(\frac{x}{2^n}\right)}{2^{2n}}-\frac{\mathcal{H}'\left(\frac{x}{2^n}\right)}{2^{2n}},r\right)\\
&\geq \min\left\{N\left(\frac{\mathcal{H}\left(\frac{x}{2^n}\right)}{2^{2n}}-\frac{f\left(\frac{x}{2^n}\right)}{2^{2n}},\frac{r}{2}\right),N\left(\frac{f\left(\frac{x}{2^n}\right)}{2^{2n}}-\frac{\mathcal{H}'\left(\frac{x}{2^n}\right)}{2^{2n}},\frac{r}{2}\right)\right\}\\
&\geq N'\left(\alpha\left(\frac{x}{2^n},\frac{x}{2^n}\right),\frac{2^{2n}r(2^2-d)}{2.2^2}\right)\\
&\geq N'\left(\alpha(x,x),\frac{2^{2n}r(2^2-d)}{8d^n}\right)
\end{aligned}$$

for all $x\in\mathcal{A}$ and all $r>0$. Since

$$\lim_{n\to\infty}\frac{2^{2n}r(2^2-d)}{8d^n}=\infty,$$

we obtain

$$\lim_{n\to\infty}N'\left(\alpha(x,x),\frac{2^{2n}r(2^2-d)}{8d^n}\right)=1$$

for all $x\in\mathcal{A}$ and all $r>0$. Thus the mapping $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ is a unique quadratic reciprocal homomorphism. This completes the proof of the theorem. □

From Theorem 2.2, we obtain the following Corollary 2.3 concerning the generalized Ulam-Hyers stability for the functional equation (1.4).

Corollary 2.3. *Suppose that a function $f:\mathcal{A}\to\mathcal{B}$ satisfies the inequalities*

$$N\left(f(x+y)-\frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}},r\right)$$

$$\geq\begin{cases}N'(\varsigma,r), & \\ N'(\varsigma(\|x\|^{\upsilon}+\|y\|^{\upsilon}),r), & \upsilon\neq-2;\\ N'(\varsigma(\|x\|^{\upsilon}\|y\|^{\upsilon}),r), & \upsilon\neq-1;\\ N'(\varsigma\{\|x\|^{\upsilon}\|y\|^{\upsilon}+\|x\|^{2\upsilon}+\|y\|^{2\upsilon}\},r), & \upsilon\neq-1,\end{cases} \tag{2.20}$$

and

$$N(\mathcal{H}(xy)-\mathcal{H}(x)\mathcal{H}(y),r)\geq\begin{cases}N'(\varsigma,r);\\ N'(\varsigma(\|x\|^{\upsilon}+\|y\|^{\upsilon}),r);\\ N'(\varsigma(\|x\|^{\upsilon}\|y\|^{\upsilon}),r);\\ N'(\varsigma\{\|x\|^{\upsilon}\|y\|^{\upsilon}+\|x\|^{2\upsilon}+\|y\|^{2\upsilon}\},r)\end{cases} \tag{2.21}$$

for all $x,y\in\mathcal{A}$ and all $r>0$, where ς,υ are constants with $\varsigma>0$.

Then there exists a unique quadratic reciprocal homomorphism $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ *such that*

$$N(f(x)-\mathcal{H}(x),r)\geq\begin{cases}N'(\varsigma,|2|r);\\ N'(\varsigma\|x\|^{v},2|2^2-2^{-v}|r);\\ N'(4\varsigma\|x\|^{2v},|2^2-2^{-2v}|r);\\ N'\left(\dfrac{9\varsigma}{2}\|x\|^{2v},|2^2-2^{-2v}|r\right),\end{cases} \tag{2.22}$$

for all $x\in\mathcal{A}$ *and all* $r>0$.

3. Fuzzy Derivations Stability Results

In this section, using Hyers direct method, we prove the generalized Hyers-Ulam stability of fuzzy derivations associated with quadratic reciprocal functional equation (1.4) in fuzzy Banach algebras.

Definition 3.1. A C-linear mapping $\mathcal{D}:\mathcal{A}\to\mathcal{A}$ is called a quadratic reciprocal fuzzy derivation if $\mathcal{D}(xy)=\mathcal{D}(x)\frac{1}{y^2}+\frac{1}{x^2}\mathcal{D}(y)$ for all $x,y\in\mathcal{A}$.

The proof of the following theorem is similar to that of Theorem 2.2. Hence the details of the proof is omitted.

Theorem 3.2. *Let* $\alpha:\mathcal{A}^2\to[0,\infty)$ *be a mapping such that for some* d *with* $0<\left(\frac{d}{2^2}\right)^{\beta}<1$ *and the conditions* (2.1) *and* (2.2) *for all* $x,y\in\mathcal{A}$ *and all* $r>0$. *Suppose that a function* $f:\mathcal{A}\to\mathcal{A}$ *satisfies the inequalities* (2.3) *and*

$$N\left(f(xy)-f(x)\frac{1}{y^2}-\frac{1}{x^2}f(y),r\right)\geq N'(\alpha(x,y),r) \tag{3.1}$$

for all $x,y\in\mathcal{A}$ *and all* $r>0$. *Then, there exists a unique quadratic reciprocal derivation* $\mathcal{D}:\mathcal{A}\to\mathcal{A}$ *such that*

$$N(f(x)-\mathcal{D}(x),r)\geq N'\left(\alpha(x,x),\frac{r|2^2-d|}{2^2}\right) \tag{3.2}$$

for all $x\in\mathcal{A}$ *and all* $r>0$. *The mapping* $\mathcal{D}(x)$ *is defined by*

$$\mathcal{D}(x)=N-\lim_{n\to\infty}\left(\frac{f\left(\frac{x}{2^n}\right)}{2^{2n}}\right) \tag{3.3}$$

for all $x\in\mathcal{A}$.

Proof. By the same reasoning as that in the proof of Theorem 2.2, there exists a unique quartic mapping $\mathcal{D}:\mathcal{A}\to\mathcal{A}$ satisfying (3.2) The mapping $\mathcal{D}:\mathcal{A}\to\mathcal{A}$ given by $\mathcal{D}(x)=N-\lim_{n\to\infty}2^{2n}f(2^nx)$ for all $x\in\mathcal{A}$. It follows from (2.3) that

$$\begin{aligned}N\left(\mathcal{D}(xy)-\mathcal{D}(x)\frac{1}{y^2}-\frac{1}{x^2}\mathcal{D}(y),r\right)&=N\left(2^{4n}\left(f(2^{2n}xy)-f(2^nx)\frac{1}{2^{2n}y^2}-\frac{1}{2^{2n}x^2}f(2^ny)\right),\frac{r}{2^{4n}}\right)\\&\geq N'(\alpha(2^nx,2^ny,2^nz),r)\end{aligned} \tag{3.4}$$

for all $r>0$ and all $x,y\in\mathcal{A}$. Letting $n\to\infty$ in (3.4) and using (2.2), we reach

$$N\left(\mathcal{D}(xy)-\mathcal{D}(x)\frac{1}{y^2}-\frac{1}{x^2}\mathcal{D}(y),r\right)=1$$

for all $x, y \in \mathcal{A}$ and $r > 0$. Hence, we have $\mathcal{D}(xy) = \mathcal{D}(x)\frac{1}{y^2} + \frac{1}{x^2}\mathcal{D}(y)$ for all $x, y \in \mathcal{A}$. Therefore, $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ is a quartic reciprocal derivation satisfying (3.2). The rest of the proof is similar to that of Theorem 2.2. □

From Theorem 2.2, we obtain the following Corollary 3.3 concerning the generalized Ulam-Hyers stability for the functional equation (1.4).

Corollary 3.3. *Suppose that a function $f : \mathcal{A} \to \mathcal{A}$ satisfies the inequalities* (2.20) *and*

$$N\left(\mathcal{D}(xy) - \mathcal{D}(x)\frac{1}{y^2} - \frac{1}{x^2}\mathcal{D}(y), r\right) \geq \begin{cases} N'(\varsigma, r), \\ N'(\varsigma(\|x\|^{\upsilon} + \|y\|^{\upsilon}), r); \\ N'(\varsigma(\|x\|^{\upsilon}\|y\|^{\upsilon}), r); \\ N'(\varsigma\{\|x\|^{\upsilon}\|y\|^{\upsilon} + \|x\|^{2\upsilon} + \|y\|^{2\upsilon}\}, r) \end{cases} \tag{3.5}$$

for all $x, y \in \mathcal{A}$ and all $r > 0$, where ς, υ are constants with $\varsigma > 0$. Then, there exists a unique quadratic reciprocal derivation $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ such that

$$N(f(x) - \mathcal{D}(x), r) \geq \begin{cases} N'(\varsigma, |2|r); \\ N'(\varsigma\|x\|^{\upsilon}, 2|2^2 - 2^{-\upsilon}|r); \\ N'(4\varsigma\|x\|^{2\upsilon}, |2^2 - 2^{-2\upsilon}|r); \\ N'\left(\frac{9\varsigma}{2}\|x\|^{2\upsilon}, |2^2 - 2^{-2\upsilon}|r\right) \end{cases} \tag{3.6}$$

for all $x \in \mathcal{A}$ and all $r > 0$.

4. Conclusion

This article has proved the Hyers-Ulam, Hyers-Ulam-Rassias, generalized Hyers-Ulam-Rassias, and Rassias stability results of the quadratic-reciprocal functional equation connected with fuzzy homomorphisms and fuzzy derivations between fuzzy Banach algebras using direct method.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] T. Aoki, On the stability of the linear transformation in Banach spaces, *Journal of the Mathematical Society of Japan* **2**(1-2) (1950), 64 – 66, DOI: 10.2969/jmsj/00210064.

[2] M. Arunkumar and S. Karthikeyan, Solution and stability of n-dimensional additive functional equation, *International Journal of Applied Mathematics* **25**(2) (2012), 163 – 174.

[3] M. Arunkumar and S. Karthikeyan, Solution and intuitionistic fuzzy stability of n-dimensional quadratic functional equation: Direct and fixed point methods, *International Journal of Advanced Mathematical Sciences* **2**(1) (2014), 21 – 33, DOI: 10.14419/ijams.v2i1.1498.

[4] M. Arunkumar, S. Karthikeyan and S. Vijayakumar, A fixed point approach to the stability of reciprocal quadratic functional equation in intuitionistic fuzzy Banach Algebras, *International Journal of Current Advanced Research* **7**(1) (2018), 143 – 150, URL: https://journalijcar.org/sites/default/files/issue-files/A25.pdf.

[5] A. Bahyrycz, J. Brzdęk, E.-S. El-hady and Z. Leśniak, On Ulam stability of functional equations in 2-normed spaces – A survey, *Symmetry* **13**(11) (2021), 2200, DOI: 10.3390/sym13112200.

[6] Z. Gajda, On stability of additive mappings, *International Journal of Mathematics and Mathematical Sciences* **14** (1991), Article ID 817959, DOI: 10.1155/S016117129100056X.

[7] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *Journal of Mathematical Analysis and Applications* **184**(3) (1994), 431 – 436, DOI: 10.1006/jmaa.1994.1211.

[8] D. H. Hyers, On the stability of the linear functional equation, *Proceedings of the National Academy of Sciences* **27**(4) (1941), 222 – 224, DOI: 10.1073/pnas.27.4.222.

[9] S. Karthikeyan, C. Park, P. Palani and T. R. K. Kumar, Stability of an additive-quartic functional equation in modular spaces, *Journal of Mathematics and Computer Science* **26**(1) (2022), 22 – 40, DOI: 10.22436/jmcs.026.01.04.

[10] S. Karthikeyan, J. M. Rassias, S. Vijayakumar and K. Sakthivel, Stability results of additive-quadratic n-dimensional functional equation: Fixed point approach, *Communications in Mathematics and Applications* **13**(2) (2022), 461 – 476, DOI: 10.26713/cma.v13i2.1757

[11] S. Karthikeyan, M. Suresh, G. Ganapathy, V. B. Priya and A. J. Bala, Stability of n-dimensional mixed type quadratic and cubic functional equation, *Journal of Mathematics and Computer Science* **31**(2) (2023), 214 – 224, DOI: 10.22436/jmcs.031.02.07.

[12] A. K. Katsaras, Fuzzy topological vector spaces II, *Fuzzy Sets and Systems* **12**(2) (1984), 143 – 154, DOI: 10.1016/0165-0114(84)90034-4.

[13] I. Kramosil and J. Michálek, Fuzzy metric and statistical metric spaces, *Kybernetica* **11**(5) (1975), 336 – 344, URL: https://www.kybernetika.cz/content/1975/5/336/paper.pdf.

[14] A. K. Mirmostafaee and M. S. Moslehian, Fuzzy versions of Hyers-Ulam-Rassias theorem, *Fuzzy Sets and Systems* **159**(6) (2008), 720 – 729, DOI: 10.1016/j.fss.2007.09.016.

[15] A. K. Mirmostafaee, M. Mirzavaziri and M. S. Moslehian, Fuzzy stability of the Jensen functional equation, *Fuzzy Sets and Systems* **159**(6) (2008), 730 – 738, DOI: 10.1016/j.fss.2007.07.011.

[16] A. K. Mirmostafaee and M. S. Moslehian, Fuzzy almost quadratic functions, *Results in Mathematics* **52** (1-2) (2008), 161 – 177, DOI: 10.1007/s00025-007-0278-9.

[17] C. Park, Fixed points and stability of the Cauchy-Jensen functional inequality in fuzzy Banach algebras, *Applied Mathematics Letters* **24**(12) (2011), 2024 – 2029, DOI: 10.1016/j.aml.2011.05.036.

[18] J. M. Rassias, On approximately of approximately linear mappings by linear mappings, *Journal of Functional Analysis* **46**(1) (1982), 126 – 130, DOI: 10.1016/0022-1236(82)90048-9.

[19] J. M. Rassias, M. Arunkumar and S. Karthikeyan, Lagrange's quadratic functional equation connected with homomorphisms and derivations on Lie C^*-algebras: Direct and fixed point methods, *Malaya Journal of Matematik* **S**(1) (2015), 228 – 241, URL: https://www.malayajournal.org/articles/paper22.pdf.

[20] J. M. Rassias, M. Arunkumar and S. Karthikeyan, Stability of DGC's quadratic functional equation in Quasi-Banach Algebras: Direct and fixed point methods, *International Journal of Pure and Applied Mathematical Sciences* **10** (1) (2017), 41 – 67.

[21] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proceedings of the American Mathematical Society* **72** (1978), 297 – 300, DOI: 10.2307/2042795.

[22] K. Ravi, M. Arunkumar and J. M. Rassias, On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation, *International Journal of Mathematics and Statistics* **3** (2008), 36 – 47.

[23] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, John Wiley & Sons, Inc., New York, xvii + 150 pages (1964).

[24] J.-Z. Xiao and X.-H. Zhu, Fuzzy normed spaces of operators and its completeness, *Fuzzy Sets and Systems* **133**(3) (2003), 389 – 399, DOI: 10.1016/S0165-0114(02)00274-9.