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Research Article

Fourth Hankel and Toeplitz Determinants for Reciprocal of Bounded Turning Functions and Inverse of Reciprocal of Bounded Turning Functions Subordinate to cos *z*

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Abstract. The purpose of the present research article is to find an upper bounds of fourth Hankel and Toeplitz determinants for reciprocal of bounded turning functions subordinate to $\cos z$ and for the inverse of reciprocal of bounded turning functions subordinate to $\cos z$. The Zalcman conjecture is verified for specific values of n for the functions in these classes. The sharp upper bounds for Fekete-Szegö inequalities were obtained.

Keywords. Reciprocal of bounded turning function, Inverse of reciprocal of bounded turning function, Hankel determinant, Toeplitz determinant

Mathematics Subject Classification (2020). Primary 30C45; Secondary 30C50, 30C80

1. Introduction

Let $\mathcal{A}$ denote the family of all holomorphic functions of the Maclaurin series expansion form

$$f(z) = z + \sum_{n\geq 2} a_n z^n, \quad \forall\, z \in \mathcal{U} \tag{1.1}$$

in the open unit disc $\mathcal{U}$ with the normalization property $f(0) = f'(0) - 1 = 0$. The subfamily of $\mathcal{A}$ denoted by $\mathcal{S}$, containing all univalent functions of the form (1.1).

A function $f \in \mathcal{A}$ is said to subordinate to $g \in \mathcal{A}$ if there exists an analytic function w defined on $\mathcal{A}$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathcal{U}$ (such functions are called Schwarz functions) such that $f(z) = g(w(z))$ for all $z \in \mathcal{U}$. In this case, we write $f \prec g$. If g is univalent, then $f \prec g$ if and only if $g(0) = f(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Pommerenke [14] introduced the qth Hankel determinant for the sequence $a_n, a_{n+1}, a_{n+2} \cdots$. For $n, q \geq 1$

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & a_{n+2} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & a_{n+3} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & a_{n+q+1} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

Ali *et al*. [2] studied the symmetric Toeplitz determinant $T_q(n)$ in 2018, it is given by

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix}, \tag{1.3}$$

where $n, q \geq 1$, $a_1 = 1$ and $a_n, a_{n+1}, a_{n+2} \cdots$ are the coefficients of analytic function f given by (1.1).

The pioneering work of Babalola [4], Bansal [5], Zaprawa [17], Orhan and Zaprawa [13], Kowalczyk *et al*. [9], Kwon *et al*. [10], on third hankel determinants for certain subclasses of class $\mathcal{S}$ motivated Arif *et al*. [3] to take up a problem of finding fourth Hankel determinant for bounded turning functions and analytic functions associated with Bernoulli Lemniscate (see [3] and reference therein).

In [11], Libera and Złotkiewicz studied early coefficients of the inverse of a regular convex function as well as coefficient bounds for the inverse of a function with derivatives in $\mathcal{P}$. Inspired by the works of Libera and Złotkiewicz [11], subsequently several other researchers studied inverse of certain subclasses of class $\mathcal{S}$ (see [2], [6] and [7]). Venkateswarlu *et al*. [15] studied the third Hankel determinant for the inverse of reciprocal of bounded turning functions. Yakaiah *et al*. [16], studied fourth Hankel and Toeplitz determinants for a class of analytic univalent functions subordinated to $\cos z$ (see [16]).

Let $\phi(z) = \cos z$. Then it is non-univalent analytic function which maps the unit disc onto the circular region in the right half plane which is symmetric about the real axis with normalization conditions $\phi(0) = 1$ and $\phi'(0) = 0$.

Motivated by the above stated work, in the present paper we introduced two new subclasses $\widehat{RT}_{\cos z}$ and in $\widehat{RT}^*_{\cos z}$ and we study the Hankel and Toeplitz determinants of order four for $f \in \widehat{RT}_{\cos z}$ and $f \in \widehat{RT}^*_{\cos z}$.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be function of the class $f \in \widehat{RT}_{\cos z}$, if and only if

$$\frac{1}{f'(z)} \prec \cos z, \quad \forall\, z \in \mathcal{U}.$$

Definition 1.2. A function g is said to be in the class $\widehat{RT}^*_{\cos z}$ if $f \in \widehat{RT}_{\cos z}$ and

$$g(w) = f^{-1}(w) = w + \sum_{n\geq 2} d_n w^n, \quad \text{for } |w| < r_0(f), r_0(f) \geq \frac{1}{4}. \tag{1.4}$$

2. A Set of Useful Lemmas

Let $\mathcal{P}$ be the family of holomorphic functions p in $\mathcal{U}$ with the conditions $p(0) = 1$ and $\Re\big(p(z)\big) > 0$ for all $z \in \mathcal{U}$ and of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \tag{2.1}$$

are called functions with positive real part.

The following lemmas concerning functions with positive real part of the form (2.1) are useful in the sequel.

Lemma 2.1 ([14]). *If $p \in \mathcal{P}$, then*

$$|p_n| \leq 2, \quad \forall\ n \geq 1. \tag{2.2}$$

Lemma 2.2 ([3]). *If $p \in \mathcal{P}$, then*

$$|Jp_1^3 - Kp_1p_2 + Lp_3| \leq 2(|J| + |K - 2J| + |J - K + L|). \tag{2.3}$$

For any real numbers J, K, L.
For $J = 1, K = 2$ and $L = 1$, we have $|p_1^3 - 2p_1p_2 + p_3| \leq 2$.

Lemma 2.3 ([12]). *If $p \in \mathcal{P}$, then*

$$|p_2 - \mu p_1^2| \leq 2\max\{1, |2\mu - 1|\}. \tag{2.4}$$

For any complex number μ. Sharpness holds for $p(z) = \frac{1+z^2}{1-z^2}$ and $p(z) = \frac{1+z}{1-z}$.

Lemma 2.4 ([17]). *If $p \in \mathcal{P}$, then*

$$|p_n - \mu p_k p_{n-k}| \leq 2, \quad \forall\ k, n = 1, 2, \cdots, n > k, \mu \in [0, 1]. \tag{2.5}$$

In the case when $\mu = 1$, we have $|p_n - p_k p_{n-k}| \leq 2$ holds.

3. Main Results

Let $f \in \widehat{RT}_{\cos z}$. Then by the technique of subordination

$$\frac{1}{f'(z)} = \cos(w(z)), \tag{3.1}$$

where w in B_0 is a Schwarz's function which satisfies $w(0) = 0$ and $|w(z)| - 1 \leq 0$. Let $p \in \mathcal{P}$ be defined as

$$p(z) = [1 - w(z)]^{-1}[1 + w(z)] = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots,$$

$$w(z) = \frac{p(z) - 1}{p(z) + 1}.$$

By simple computations we get

$$\frac{1}{f'(z)} = 1 + (-2b_2)z + (-3b_3 + 4b_2^2)z^2 + (-4b_4 + 12b_2b_3 - 8b_2^3)z^3$$

$$
\begin{aligned}
&+(-5b_5+16b_2b_4+9b_3^2-36b_2^2b_3+16b_2^4)z^4\\
&+(-6b_6-32b_2^5-54b_2b_3^2+96b_2^3b_3-48b_2^2b_4+24b_3b_4+20b_2b_5)z^5\\
&+(-7b_7+24b_2b_6+64b_2^6+216b_2^2b_3^2-240b_2^4b_3+128b_2^3b_4-144b_2b_3b_4\\
&-60b_2^2b_5-27b_3^3+30b_3b_5+16b_4^2)z^6+\cdots, \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
\cos(w(z)) = 1+(0)z&+\left(-\frac{1}{8}p_1^2\right)z^2+\left(-\frac{1}{4}p_1p_2+\frac{1}{8}p_1^3\right)z^3\\
&-\frac{1}{2}\left(\frac{1}{2}p_1p_3+\frac{1}{4}p_2^2-\frac{3}{4}p_1^2p_2+\frac{35}{192}p_1^4\right)z^4\\
&+\left(-\frac{1}{4}p_1p_4+\frac{3}{8}p_1^2p_3+\frac{3}{8}p_1p_2^2+\frac{11}{192}p_1^5-\frac{35}{96}p_1^3p_2-\frac{1}{4}p_2p_3\right)z^5\\
&+\left(-\frac{1501}{46080}p_1^6-\frac{35}{64}p_1^2p_2^2-\frac{1}{8}p_3^2+\frac{55}{192}p_1^4p_2\right.\\
&\left.+\frac{3}{4}p_1p_2p_3-\frac{35}{96}p_1^3p_3-\frac{1}{4}p_1p_5+\frac{3}{8}p_1^2p_4-\frac{1}{4}p_2p_4+\frac{1}{8}p_2^3\right)z^6+\cdots. \qquad (3.3)
\end{aligned}
$$

From relations (3.1), (3.2) and (3.3), we get

$$b_2 = 0, \qquad (3.4)$$

$$b_3 = \frac{p_1^2}{24}, \qquad (3.5)$$

$$b_4 = \frac{p_1p_2}{16}-\frac{p_1^3}{32}, \qquad (3.6)$$

$$b_5 = \frac{1}{20}\left(\frac{41}{96}p_1^4+p_1p_3+\frac{p_2^2}{2}-\frac{3}{2}p_1^2p_2\right), \qquad (3.7)$$

$$b_6 = \frac{1}{6}\left(\frac{41}{96}p_1^3p_2-\frac{17}{192}p_1^5+\frac{1}{4}p_1p_4+\frac{1}{4}p_2p_3-\frac{3}{8}p_1^2p_3-\frac{3}{8}p_1p_2^2\right), \qquad (3.8)$$

$$
\begin{aligned}
b_7 = \frac{1}{7}&\left(\frac{3361}{4608}p_1^6+\frac{41}{64}p_1^2p_2^2+\frac{41}{96}p_1^3p_3-\frac{61}{192}p_1^4p_2+\frac{1}{8}p_3^2-\frac{3}{4}p_1p_2p_3+\frac{1}{4}p_1p_5\right.\\
&\left.-\frac{3}{8}p_1^2p_4+\frac{1}{4}p_2p_4-\frac{1}{8}p_2^3\right). \qquad (3.9)
\end{aligned}
$$

We now obtain the initial coefficient bounds for $f \in \widehat{RT}_{\cos z}$ in the next theorem.

Theorem 3.1. *Let $f \in \widehat{RT}_{\cos z}$. Then $|b_2| = 0$, $|b_3| \le \frac{1}{6}$, $|b_4| \le \frac{1}{4}$, $|b_5| \le \frac{79}{240}$, $|b_6| \le \frac{5}{6}$ and $|b_7| \le \frac{45}{14}$.*

Proof. By using Lemmas 2.1, 2.2, 2.3 and 2.4 to the relations (3.4) to (3.9), we acquire

$$|b_2| = 0, \qquad (3.10)$$

$$|b_3| = \frac{|p_1|^2}{24},$$

$$|b_3| \le \frac{1}{6}, \qquad (3.11)$$

$$|b_4| = \frac{|p_1|}{16}\left|p_2-\frac{p_1^2}{2}\right|,$$

$$|b_4| \le \frac{1}{4}, \tag{3.12}$$

$$|b_5| = \frac{|p_1|}{20}\frac{|41p_1^3 - 144p_1p_2 + 96p_3|}{96} + \frac{|p_2|^2}{40},$$

$$|b_5| \le \frac{79}{240}, \tag{3.13}$$

$$|b_6| = \frac{1}{6}\left[\frac{|p_1|^2}{192}\left|17p_1^3 - 82p_1p_2 + 72p_3\right| + \frac{|p_1|}{4}|p_4 - p_2^2| + \frac{|p_2|}{4}\left|p_3 - \frac{p_1p_2}{2}\right|\right],$$

$$|b_6| \le \frac{5}{6}, \tag{3.14}$$

$$|b_7| = \frac{61}{1344}|p_1|^4\left|p_2 - \frac{3361}{14640}p_1^2\right| + \frac{3}{28}|p_1||p_2|\left|p_3 - \frac{41}{48}p_1p_2\right| + \frac{|p_2|}{28}\left|p_4 - \frac{1}{2}p_2^2\right|$$
$$+ \frac{3}{56}|p_1|^2|p_4 - p_1p_3| + \frac{15}{2016}|p_1|^3|p_3| + \frac{|p_3|^2}{56} + \frac{|p_1||p_5|}{28},$$

$$|b_7| \le \frac{45}{14}. \tag{3.15}$$

□

Theorem 3.2. *If $f \in \widehat{RT}_{\cos z}$, then*

$$|b_3 - \mu b_2^2| \le \frac{1}{6},$$

where μ is any complex number. The function $f_1(z) = z + \frac{1}{6}z^3 + \frac{1}{36}z^5 + \ldots \in \widehat{RT}_{\cos z}$ is an extremal function for this inequality.

Proof. Let $f \in \widehat{RT}_{\cos z}$. Then from the relations (3.4) and (3.5) and using the fact $|p_1| \le 2$, we have

$$|b_3 - \mu b_2^2| = |b_3| = \frac{|p_1|^2}{24} \le \frac{1}{6}.$$

□

Theorem 3.3. *If $f \in \widehat{RT}_{\cos z}$, then $|H_3(1)| \le 0.1219$.*

Proof. By making use of the relations (3.10) to (3.13) we acquire

$$|H_3(1)| \le |b_2b_4 - b_3^2||b_3| + |b_4 - b_3b_2||b_4| + |b_3 - b_2^2||b_5| \tag{3.16}$$
$$\le |b_3||b_3^2| + |b_4||b_4| + |b_5||b_3|$$
$$\le 0.1219. \tag{3.17}$$

□

We now obtain an upper bounds of $|H_4(1)|$, $|T_4(1)|$ and $|T_4(2)|$ for the function f in $\widehat{RT}_{\cos z}$ and $\widehat{RT}^*_{\cos z}$.

Theorem 3.4. *If $f(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \widehat{RT}_{\cos z}$, then $|H_4(1)| \le 0.7148$.*

Proof. Khan *et al.* [8] have given the expansion of the fourth Hankel determinant as below:

$$|H_4(1)| \le |b_7||H_3(1)| + 2|b_4||b_6||b_2b_4 - b_3^2| + 2|b_5||b_6||b_2b_3 - b_4| + |b_6|^2|b_3 - b_2^2|$$
$$+ |b_5|^2|b_2b_4 - b_3^2| + |b_5|^2|b_2b_4 + 2b_3^2| + |b_5|^3 + |b_4|^4 + 3|b_3||b_4|^2|b_5|. \tag{3.18}$$

By making use of the relations (3.10) to (3.17) we acquire

$$|H_4(1)| \leq 0.7148. \tag{3.19}$$

Theorem 3.5. *If* $f \in \widehat{RT}_{\cos z}$, *then*

$$|b_2^2 - b_3| \leq \frac{1}{6}, \tag{3.20}$$

$$|b_3^2 - b_5| \leq \frac{247}{720}, \tag{3.21}$$

$$|b_4^2 - b_7| \leq \frac{687}{168}. \tag{3.22}$$

Proof. Let $f \in \widehat{RT}_{\cos z}$. Then choosing $\mu = 1$ in Theorem 3.2 we get $|b_2^2 - b_3| = |b_3| \leq \frac{1}{6}$. Consider

$$\begin{aligned} |b_3^2 - b_5| &= \left| \frac{p_1^4}{576} - \frac{1}{20}\left(\frac{41}{96}p_1^4 + p_1p_3 + \frac{p_2^2}{2} - \frac{3}{2}p_1^2p_2\right)\right| \\ &\leq \frac{1}{20}\left[\frac{|p_1|}{288}|113p_1^3 - 432p_1p_2 + 288p_3| + \frac{|p_2|^2}{2}\right]. \end{aligned} \tag{3.23}$$

By making use of Lemmas 2.1 and 2.2 to the relation (3.23) we get the relation (3.21). Consider

$$\begin{aligned} |b_4^2 - b_7| &= \Bigg|\left(\frac{p_1p_2}{16} - \frac{p_1^3}{32}\right)^2 - \frac{1}{7}\Bigg(\frac{3361}{4608}p_1^6 + \frac{41}{64}p_1^2p_2^2 + \frac{41}{96}p_1^3p_3 - \frac{61}{192}p_1^4p_2 + \frac{1}{8}p_3^2 - \frac{3}{4}p_1p_2p_3 \\ &\quad + \frac{1}{4}p_1p_5 - \frac{3}{8}p_1^2p_4 + \frac{1}{4}p_2p_4 - \frac{1}{8}p_2^3\Bigg)\Bigg| \\ &\leq \frac{1}{7}\Bigg[\frac{223}{768}|p_1|^4\left|p_2 - \frac{1523}{6690}p_1^2\right| + \frac{3}{4}|p_1||p_2|\left|p_3 - \frac{157}{192}p_1p_2\right| + \frac{|p_2|}{4}\left|p_4 - \frac{1}{2}p_2^2\right| \\ &\quad + \frac{|p_3|}{96}\left|41p_1^3 + 12p_1p_2 + 12p_3\right| + \frac{|p_1|}{4}\left|p_5 - \frac{1}{2}p_2p_3\right| + \frac{3}{8}|p_1|^2|p_4|\Bigg]. \end{aligned} \tag{3.24}$$

By making use of Lemmas 2.1, 2.2, 2.3 and 2.4 to the relation (3.24) we get the relation (3.22). □

Remark 3.1. Zalcman conjectured that the coefficients of $f \in \mathcal{S}$ of the form (1.1) satisfy the inequality

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \quad \text{for } n \geq 2.$$

From the previous theorem we can conclude that Zalcman conjecture is true for $n = 2,3,4$ when $f \in \widehat{RT}_{\cos z}$.

Theorem 3.6. *If* $f(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \widehat{RT}_{\cos z}$, *then*

$$|T_4(1)| \leq \frac{725}{648}. \tag{3.25}$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \widehat{RT}_{\cos z}$. Then by making use of inequalities as in Theorem 3.1, we get

$$|T_4(1)| = |(1 - b_2^2)^2 - (b_2b_3 - b_4)^2 + (b_3^2 - b_2b_4)^2 - (b_2 - b_2b_3)^2 + 2(b_2^2 - b_3)(b_3 - b_2b_4)|$$

$$\leq 1 + |b_4|^2 + |b_3|^4 + 2|b_3|^2$$
$$\leq \frac{725}{648}. \tag{3.26}$$

□

Theorem 3.7. *If $f(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \widehat{RT}_{\cos z}$, then*

$$|T_4(2)| \leq \frac{709}{57600}. \tag{3.27}$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \widehat{RT}_{\cos z}$. Since $|b_2| = 0$, we get

$$\begin{aligned} |T_4(2)| &= |(b_2^2 - b_3^2)^2 - (b_3 b_4 - b_2 b_5)^2 + (b_4^2 - b_3 b_5)^2 - (b_2 b_3 - b_3 b_4)^2 \\ &\quad + 2(b_3^2 - b_2 b_4)(b_2 b_4 - b_3 b_5)| \\ &\leq |b_3|^4 + |b_3|^2 |b_4|^2 + |b_4^2 - b_3 b_5|^2 + |b_3|^2 |b_4|^2 + 2|b_3|^3 |b_5|. \end{aligned} \tag{3.28}$$

Consider

$$b_4^2 - b_3 b_5 = \frac{p_1^2}{32}\left[-\frac{11}{120}\left(p_4 - p_2^2\right) - \frac{p_1^2}{40}\left(p_2 - \frac{1}{9}p_1^2\right) + \frac{11}{120}\left(p_4 - \frac{24}{33}p_1 p_3\right)\right].$$

By an application of modulus on either side of above relation and execution of Lemmas 2.1 to 2.4, we get

$$|b_4^2 - b_3 b_5| \leq \frac{17}{240}. \tag{3.29}$$

Making use of bounds as in the inequality (3.29) along with inequalities as in Theorem 3.1 in (3.28), we get the required result. □

4. Fourth Hankel and Toeplitz Determinants for Inverse of Certain Analytic Functions Related to cos *z*

Let $g \in \widehat{RT}^*_{\cos z}$. Then there exists $f \in \widehat{RT}_{\cos z}$ such that $w = f(g(w)) = f(f^{-1}(w))$. On substituing the expansion of g and f as in (1.4) and (1.1), respectively in $w = f(g(w)) = f(f^{-1}(w))$ and using the coefficient values of b_i's from Equations (3.4)-(3.9), we get

$$d_2 = 0, \tag{4.1}$$

$$d_3 = -\frac{p_1^2}{24}, \tag{4.2}$$

$$d_4 = -\frac{p_1 p_2}{16} + \frac{p_1^3}{32}, \tag{4.3}$$

$$d_5 = \frac{1}{20}\left(-\frac{31}{96}p_1^4 - p_1 p_3 - \frac{p_2^2}{2} + \frac{3}{2}p_1^2 p_2\right). \tag{4.4}$$

$$d_6 = \frac{1}{6}\left(-\frac{61}{192}p_1^3 p_2 + \frac{13}{384}p_1^5 - \frac{1}{4}p_1 p_4 - \frac{1}{4}p_2 p_3 + \frac{3}{8}p_1^2 p_3 + \frac{3}{8}p_1 p_2^2\right), \tag{4.5}$$

$$d_7 = \frac{1}{7}\left(\frac{13}{3072}p_1^6 - \frac{227}{480}p_1^2 p_2^2 - \frac{49}{480}p_1^3 p_3 + \frac{19}{120}p_1^4 p_2 - \frac{1}{8}p_3^2 + \frac{3}{4}p_1 p_2 p_3 - \frac{1}{4}p_1 p_5 + \frac{3}{8}p_1^2 p_4 - \frac{1}{4}p_2 p_4 + \frac{1}{8}p_2^3\right). \tag{4.6}$$

Theorem 4.1. *If* $g(w) = w + \sum_{n=2}^{\infty} d_n w^n \in \widehat{RT}^*_{\cos z}$, *then* $|d_2| = 0$, $|d_3| \le \frac{1}{6}$, $|d_4| \le \frac{1}{4}$, $|d_5| \le \frac{89}{240}$, $|d_6| \le \frac{5}{6}$ *and* $|d_7| \le \frac{497}{210}$.

Proof. By using Lemmas 2.1, 2.2, 2.3 and 2.4 to the relations (4.1) to (4.6), we acquire

$$|d_2| = 0, \tag{4.7}$$

$$|d_3| = \frac{|p_1|^2}{24},$$

$$|d_3| \le \frac{1}{6}, \tag{4.8}$$

$$|d_4| = \frac{|p_1|}{16}\left|p_2 - \frac{p_1^2}{2}\right|,$$

$$|d_4| \le \frac{1}{4}, \tag{4.9}$$

$$|d_5| = \frac{|p_1|}{20}\frac{|31p_1^3 - 144p_1p_2 + 96p_3|}{96} + \frac{|p_2|^2}{40},$$

$$|d_5| \le \frac{89}{240}, \tag{4.10}$$

$$|d_6| = \frac{1}{6}\left[\frac{|p_1|^2}{384}\left|13p_1^3 - 122p_1p_2 + 144p_3\right| + \frac{|p_1|}{4}|p_4 - p_2^2| + \frac{|p_2|}{4}\left|p_3 - \frac{p_1p_2}{2}\right|\right],$$

$$|d_6| \le \frac{5}{6}, \tag{4.11}$$

$$|d_7| = \frac{19}{840}|p_1|^4\left|p_2 - \frac{65}{2432}p_1^2\right| + \frac{3}{28}|p_1||p_2|\left|p_3 - \frac{227}{360}p_1p_2\right| + \frac{|p_2|}{28}\left|p_4 - \frac{1}{2}p_2^2\right|$$
$$+ \frac{3}{56}|p_1|^2|p_4 - \frac{49}{180}p_1p_3| + \frac{|p_3|^2}{56} + \frac{|p_1||p_5|}{28},$$

$$|d_7| \le \frac{497}{210}. \tag{4.12}$$

□

Theorem 4.2. *If* $g \in \widehat{RT}^*_{\cos z}$ *is of the form* (1.4), *then*

$$|d_3 - \mu d_2^2| \le \frac{1}{6},$$

where μ *is any complex number. The function* $g_1(w) = w - \frac{1}{6}w^3 - \frac{1}{24}w^5 + \ldots \in \widehat{RT}^*_{\cos z}$ *is an extremal function for this inequality.*

Proof. Since $g \in \widehat{RT}^*_{\cos z}$, then from relations (4.1) and (4.2), consider

$$|d_3 - \mu d_2^2| = |d_3| \le \frac{1}{6}.$$

□

Theorem 4.3. *If* $g \in \widehat{RT}^*_{\cos z}$, *then* $|H_3(1)| \le 0.1289$.

Proof. By making use of the relations (4.7) to (4.10) we acquire

$$|H_3(1)| \le |d_2d_4 - d_3^2||d_3| + |d_4 - d_3d_2||d_4| + |d_3 - d_2^2||d_5|$$

$$\leq |d_3||d_3^2| + |d_4||d_4| + |d_5||d_3|$$
$$\leq 0.1289. \tag{4.13}$$

□

Theorem 4.4. *If* $g(w) = w + \sum_{n=2}^{\infty} d_n w^n \in \widehat{RT}^*_{\cos z}$, *then*

$$|H_4(1)| \leq 0.5981. \tag{4.14}$$

Proof. Khan *et al*. [8] have given the expansion of the fourth Hankel determinant as below:

$$|H_4(1)| \leq |d_7||H_3(1)| + 2|d_4||d_6||d_2d_4 - d_3^2| + 2|d_5||d_6||d_2d_3 - d_4| + |d_6|^2|d_3 - d_2^2| + |d_5|^2|d_2d_4 - d_3^2| + |d_5|^2|d_2d_4 + 2d_3^2| + |d_5|^3 + |d_4|^4 + 3|d_3||d_4|^2|d_5|. \tag{4.15}$$

By making use of the relations (4.7) to (4.13) we acquire

$$|H_4(1)| \leq 0.5981. \tag{4.16}$$

□

Theorem 4.5. *If* $g \in \widehat{RT}^*_{\cos z}$ *of the form* (1.4)*, then*

$$|d_2^2 - d_3| \leq \frac{1}{6}, \tag{4.17}$$

$$|d_3^2 - d_5| \leq \frac{257}{720}, \tag{4.18}$$

$$|d_4^2 - d_7| \leq \frac{1209}{840}. \tag{4.19}$$

Proof. Since $g \in \widehat{RT}^*_{\cos z}$, then from relations (4.1) and (4.2), consider

$$|d_2^2 - d_3| = |d_3| \leq \frac{1}{6}.$$

Consider

$$|d_3^2 - d_5| = \left|\frac{p_1^4}{576} - \frac{1}{20}\left(-\frac{31}{96}p_1^4 - p_1p_3 - \frac{p_2^2}{2} + \frac{3}{2}p_1^2p_2\right)\right|$$
$$\leq \frac{1}{20}\left[\frac{|p_1|}{288}|103p_1^3 - 432p_1p_2 + 288p_3| + \frac{|p_2|^2}{2}\right]. \tag{4.20}$$

By making use of Lemmas 2.1 and 2.2 to the relation (4.20) we get the relation (4.18).
Consider

$$|d_4^2 - d_7| = \left|\left(\frac{p_1^3}{32} - \frac{p_1p_2}{16}\right)^2 - \frac{1}{7}\left(\frac{13}{3072}p_1^6 - \frac{227}{480}p_1^2p_2^2 - \frac{49}{480}p_1^3p_3 + \frac{19}{120}p_1^4p_2 - \frac{1}{8}p_3^2 + \frac{3}{4}p_1p_2p_3 - \frac{1}{4}p_1p_5 + \frac{3}{8}p_1^2p_4 - \frac{1}{4}p_2p_4 + \frac{1}{8}p_2^3\right)\right|$$
$$\leq \frac{1}{7}\left[\frac{713}{3840}|p_1|^4\left|p_2 - \frac{10}{713}p_1^2\right| + \frac{3}{4}|p_1||p_2|\left|p_3 - \frac{1921}{2880}p_1p_2\right| + \frac{|p_2|}{4}\left|p_4 - \frac{1}{2}p_2^2\right| + \frac{|p_3|}{480}\left|49p_1^3 + 60p_1p_2 + 60p_3\right| + \frac{|p_1|}{4}\left|p_5 - \frac{1}{2}p_2p_3\right| + \frac{3}{8}|p_1|^2|p_4|\right]. \tag{4.21}$$

By making use of Lemmas 2.1, 2.2, 2.3 and 2.4 to the relation (4.21) we get the relation (4.19). □

Remark 4.1. One can conclude from Theorem 4.5 that Zalcman conjecture is true for $n = 2,3,4$ when $g \in \widehat{RT}^*_{\cos z}$.

Theorem 4.6. *If* $g(z) = z + \sum_{n=2}^{\infty} d_n z^n \in \widehat{RT}^*_{\cos z}$, *then*

$$|T_4(1)| \le \frac{725}{648}. \tag{4.22}$$

Proof. Let $g(z) = z + \sum_{n=2}^{\infty} d_n z^n \in \widehat{RT}^*_{\cos z}$. Since $|d_2| = 0$,

$$|T_4(1)| \le 1 + |d_4|^2 + |d_3|^4 + 2|d_3|^2. \tag{4.23}$$

The required inequality follows by using bounds obtained in Theorem 4.1 in the relation (4.23). □

Theorem 4.7. *If* $g(z) = z + \sum_{n=2}^{\infty} d_n z^n \in \widehat{RT}^*_{\cos z}$, *then*

$$|T_4(2)| \le \frac{6581}{518400}. \tag{4.24}$$

Proof. Let $g(z) = z + \sum_{n=2}^{\infty} d_n z^n \in \widehat{RT}^*_{\cos z}$. Since $|d_2| = 0$, we get

$$|T_4(2)| \le |d_3|^4 + |d_3|^2|d_4|^2 + |d_4^2 - d_3 d_5|^2 + |d_3|^2|d_4|^2 + 2|d_3|^3|d_5|. \tag{4.25}$$

Consider

$$d_4^2 - d_3 d_5 = \frac{p_1^2}{32}\left[-\frac{11}{120}\left(p_4 - p_2^2\right) - \frac{p_1^2}{40}\left(p_2 - \frac{7}{18}p_1^2\right) + \frac{11}{120}\left(p_4 - \frac{40}{55}p_1 p_3\right)\right].$$

By an application of modulus on either side of above relation and execution of Lemmas 2.1 to 2.4, we get

$$|d_4^2 - d_3 d_5| \le \frac{17}{240}. \tag{4.26}$$

Substitute the bounds obtained in Theorem 4.1 and relation (4.26) in (4.25) we get the required result. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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