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Research Article

An Analysis of Multi-objective Fuzzy Stochastic Nonlinear Programming Models

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Abstract. A novel method is developed for multi-objective fuzzy stochastic nonlinear programming models with some stochastic constraints in which randomness is described by gamma random variables and fuzziness is expressed by L-R fuzzy numbers. The solution for aforesaid model is obtained by three stages: Defuzzification, conversion of MOSNLPP into MONLLP using chance constrained technique and solving multi objective deterministic nonlinear programming problem. An example is exemplified to validate and strengthen the efficacy of proposed technique.

Keywords. Multi objective fuzzy stochastic nonlinear programming, L-R fuzzy numbers, Gamma distribution

Mathematics Subject Classification (2020). 90C15, 90C30, 90C70

1. Introduction

Fuzzy logic is an amazing tool which helps to find the solution for many real life and engineering problems consisting of imprecise and vagueness information. Numerous research articles have been published on linear and nonlinear mathematical programming involving fuzzy parameters, by several modification and focused on different aspects. In today's frame, it is imperative that to identify the solutions for the problems involving randomness and fuzziness together. Using goal programming approach, Masoud *et al*. [5] investigated a stochastic linear programming with multi-objective functions, in which the probabilistic parameters have been normally distributed. Barik and Biswal [1] formulated probabilistic quadratic programming problems where randomness is characterized by Weibull distribution. Khalifa *et al*. [3] has developed a two phase technique using weighted average approach in which the parameters in right side of the constraints are normally distributed random variables, to obtain the effective solution for stochastic multi-objective programming problem. Rout *et al*. [6] presented a quadratic programming problem with multi-choice and multi-objective fuzzy probabilistic and its solution technique. In this paper, an optimal solution for *multi-objective fuzzy stochastic nonlinear programming problem* (MOFSNLP) is developed. The right side of constraints parameters is supposed to follow gamma distribution and fuzzy parameters are to be taken as L-R fuzzy numbers.

2. Preliminaries

Definiton 2.1 (Generalized Fuzzy Number [8]). A fuzzy set $\widetilde{A}$ which is defined on R is forenamed as a generalized fuzzy number if its membership function $\mu_{\widetilde{A}}$ satisfies the below characteristics:

(i) The memberships function $\mu_{\widetilde{A}}$ from R to $[0,\omega]$ is continuous.

(ii) $\mu_{\widetilde{A}}(x) = 0$, for all $x \in (-\infty, a] \cup [d, \infty)$.

(iii) $\mu_{\widetilde{A}}(x)$ strictly $\uparrow$ function on $[a,b]$ and strictly $\downarrow$ function on $[c,d]$.

(iv) $\mu_{\widetilde{A}}(x) = \omega$, for all $x \in [b,c]$, where $a < \omega \le 1$.

Definiton 2.2 (Reference Function [8]). A function $\vartheta : [0,\infty) \to [0,1]$ is a reference function of a fuzzy number if and only if ϑ satisfies the following properties:

(i) $\vartheta(x) = \vartheta(-x)$.

(ii) $\vartheta(0) = 1$.

(iii) ϑ is decreasing and piecewise continuous function at $[0,\infty)$.

Definiton 2.3 (L-R Type Fuzzy Number [5]). A fuzzy parameter $\widetilde{A}$ is said to be L-R type if there exists decreasing functions $LR : [0,\infty) \to [0,1]$ such that

(i) $L(0) = R(0) = 1$,

(ii) $L(x) = L(-x)$, $R(x) = R(-x)$,

(iii) $L(x)$, $R(x)$ approaches zero when $x \to \infty$,

and its membership function $\mu_{\widetilde{A}}(x)$ satisfies

$$\mu_{\widetilde{A}}(x) = \begin{cases} \omega L\left(\frac{t-x}{\alpha}\right), & t \ge x,\ \alpha > 0, \\ \omega R\left(\frac{x-t}{\beta}\right), & t \le x,\ \beta > 0, \end{cases}$$

where ω is the average value of $\widetilde{A}$ and α and β are the left spread and right spread, respectively.

Definiton 2.4 (L-R type Generalized Trapezoidal Fuzzy Number [5]). An L-R type generalized trapezoidal fuzzy number is a fuzzy number $\widetilde{A}=(a,b,\alpha,\beta;\omega)_{LR}$ whose membership function is defined by

$$\mu_{\widetilde{A}}(x)=\begin{cases}\omega L\left(\frac{a-x}{\alpha}\right), & a\geq x,\ \alpha>0,\\ \omega R\left(\frac{x-b}{\beta}\right), & b\leq x,\ \beta>0,\\ \omega, & \text{elsewhere.}\end{cases}$$

Definiton 2.5 (Ranking Function [8]). The ranking function of an L-R type generalized trapezoidal fuzzy number $\widetilde{A}=(a,b,\alpha,\beta;w)_{LR}$ is expressed as

$$R(\widetilde{A})=\frac{w(2a+2b-\alpha+\beta)}{12}.$$

Definiton 2.6 (Taylor's Theorem). Let $\widetilde{f}(x_1,x_2,x_3,\dots,x_n)$be a nonlinear function which possesses continuous partial derivatives of first order then, the approximate linear function of $\widetilde{f}(x_1,x_2,x_3,\dots,x_n)$ at $\widetilde{P}(p,p_2,p_3,\dots,p_n)$ is given by

$$\widetilde{f}(x_1,x_2,x_3,\dots,x_n)\approx\widetilde{f}(\widetilde{P})+\nabla x_1\left(\frac{\partial\widetilde{f}}{\partial x_1}\right)_{\widetilde{P}}+\nabla x_2\left(\frac{\partial\widetilde{f}}{\partial x_2}\right)_{\widetilde{P}}+\dots+\nabla x_n\left(\frac{\partial\widetilde{f}}{\partial x_n}\right)_{\widetilde{P}},$$

where $\nabla x_1=x_1-p_1$, $\nabla x_2=x_2-p_2$, $\dots$, $\nabla x_n=x_n-p_n$.

Definiton 2.7 (Gamma Distribution). A continuous random variable b is said to follow gamma distribution if its probability density function is given by

$$\Gamma f(b)=\begin{cases}\frac{1}{\Gamma_\alpha\beta^\alpha}b^{\alpha-1}e^{-\frac{b}{\beta}}, & \alpha\geq 0,\ \beta\geq 0,\ b\geq 0,\\ 0, & \text{otherwise,}\end{cases}$$

where $\Gamma_\alpha=\int_0^\infty e^{-b}b^{\alpha-1}db$, $\alpha>0$ and β is known as scale parameter.

3. Mathematical Formulation of Problems

3.1 Multi-Objective Stochastic Nonlinear Programming (MOSNLP)

The mathematical formulation of MOSNLP can be written as

$$\min f^{(k)}=\sum_{j=1}^{n}c_j^{(k)}x_j^{(p)},\quad p=2,3,4,\dots,\ k=1,2,\dots,K \tag{3.1}$$

subject to

$$\text{Prob}\left[\sum_{j=1}^{n}a_{ij}x_j\geq b_i\right]\geq 1-\alpha_i,\quad i=1,2,\dots,m,\ j=1,2,\dots,n \tag{3.2}$$

$$x_j\geq 0. \tag{3.3}$$

3.2 Fuzzy Multi-Objective Stochastic Nonlinear Programming (FMOSNLP)

The mathematical formulation of FMOSNLP can be stated as

$$\min\widetilde{f}^{(k)}=\sum_{j=1}^{n}\widetilde{c}_j^{(k)}x_j^{(p)},\ p=2,3,4,\dots,\ k=1,2,\dots,K \tag{3.4}$$

subject to

$$\text{Prob}\left[\sum_{j=1}^{n} \widetilde{a}_{ij}x_j \geq b_i\right] \geq \theta_i, \quad i=1,2,\dots,m,\ j=1,2,\dots,n \tag{3.5}$$

$$x_j \geq 0 \tag{3.6}$$

where the objective function coefficients $\widetilde{c}_j$ constraint coefficients $\widetilde{a}_{ij}$ are taken to be L-R type generalized trapezoidal fuzzy numbers and right hand of the constraint b_i $(i=1,2,\dots,m)$ are independent random variables which follows gamma distribution with known parameters and $\theta_i \in (0,1)$.

4. Algorithm to Solve FMOSNLP

The proposed solution procedure of solving FMOSNLP (3.4)-(3.6) is done by the following three stages.

Step 1: The fuzzy probabilistic constraint is remodeled into equivalent fuzzy constraint using the following theorem.

Theorem 4.1. $\text{Prob}\left[\sum_{j=1}^{n} \widetilde{a}_{ij}x_j \geq b_i\right] \geq \theta_i \cong \sum_{j=1}^{n} \widetilde{a}_{ij}x_j \geq \frac{-\beta}{\ln\theta_i}$.

Proof. Consider $\text{Prob}\left[\sum_{j=1}^{n} \widetilde{a}_{ij}x_j \geq b_i\right] \geq \theta_i$.

Let $t_i = \sum_{j=1}^{n} \widetilde{a}_{ij}x_j$; $t_i \geq\Rightarrow \text{Prob}[t_i \geq b_i] \geq \theta_i$.

Since the random variable b_i follows gamma distribution, by using Definition 2.7 the constraint (3.3) can be written as

$$\int_{y_i}^{\infty} f(b_i)\,db_i \geq \theta_i$$

$$\cong \int_{y_i}^{\infty} \frac{1}{\Gamma_\alpha \beta^\alpha} b^{\alpha-1} e^{-\frac{b}{\beta}}\,db_i \geq \theta_i.$$

For standard gamma variate $\alpha = 1$,

$$\cong \int_{-\frac{t_i}{\beta}}^{\infty} e^{-y}\,dy \leq \theta_i \cong e^{\frac{t_i}{\beta}} \geq \theta_i$$

$$\cong t_i \geq \frac{-\beta}{\ln\theta_i}$$

$$\cong \sum_{j=1}^{n} \widetilde{a}_{ij}x_j \geq \frac{-\beta}{\ln\theta_i}$$

The equivalent mathematical form of fuzzy multi-objective stochastic nonlinear programming (3.4)-(3.6) is formulated by

$$\min \widetilde{f}^{(k)} = \sum_{j=1}^{n} \widetilde{c}_j^{(k)} x_j^{(p)}, \quad p=2,3,4,\dots,\ k=1,2,\dots,K \tag{4.1}$$

subject to

$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \geq \frac{-\beta}{\ln\theta_i}, \quad i = 1,2,\dots,m,\ j = 1,2,\dots,n \tag{4.2}$$

$$x_j \geq 0 \tag{4.3}$$

Step 2: Make use of ranking function, fuzzy numbers are transformed into its crisp values to obtain deterministic multi-objective nonlinear programming problem, which is stated below.

$$\min \tilde{f}^{(k)} = \sum_{j=1}^{n} R(\tilde{c}_j^{(k)}) x_j^{(p)}, \quad p = 2,3,4,\dots,\ k = 1,2,\dots,K$$

subject to

$$\sum_{j=1}^{n} R(\tilde{a}_{ij}) x_j \geq \frac{-\beta}{\ln\theta_i}, \quad i = 1,2,\dots,m,\ j = 1,2,\dots,n$$

$$x_j \geq 0$$

which is equivalent to

$$\min Z^{(k)} = \sum_{j=1}^{n} d_j^{(k)} x_j^{(p)}, \quad p = 2,3,4,\dots,\ k = 1,2,\dots \tag{4.4}$$

subject to

$$\sum_{j=1}^{n} a'_{ij} x_j \geq \frac{-\beta}{\ln\theta_i}, \quad i = 1,2,\dots,m,\ j = 1,2,\dots,n \tag{4.5}$$

$$x_j \geq 0 \tag{4.6}$$

Step 3: In this step, the nonlinear objective functions (4.4) are reformulated into linear objective functions using Definition 2.6. The equivalent mathematical form is given by

$$\min Z^{(k)} = \sum_{j=1}^{n} d_j^{(k)} x'_j, \quad p = 2,3,4,\dots,\ k = 1,2,\dots \tag{4.7}$$

subject to

$$\sum_{j=1}^{n} a'_{ij} x_j \geq \frac{-\beta}{\ln\theta_i}, \quad i = 1,2,\dots,m,\ j = 1,2,\dots,n \tag{4.8}$$

$$x_j \geq 0 \tag{4.9}$$

This concludes the proof. □

Using LINGO software the optimal solution of MOLPP (4.7)-(4.9) is obtained.

5. Numerical Example

Consider the FMOSNLP

$$\min f^1(x) = \tilde{c}_{11}\tilde{x}_1 + \tilde{c}_{12}\tilde{x}_2 + \tilde{c}_{13}\tilde{x}_1^2 \tag{5.1}$$

$$\min f^2(x) = \tilde{c}_{21}\tilde{x}_1 + \tilde{c}_{22}\tilde{x}_2 + \tilde{c}_{23}\tilde{x}_2{}^2 \tag{5.2}$$

subject to

$$\text{Prob}\left[\tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 \geq b_1\right] \geq 0.90 \tag{5.3}$$

$$\text{Prob}\left[\tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 \geq b_2\right] \geq 0.95 \tag{5.4}$$

$$\tilde{a}_{31}\tilde{x}_1 + \tilde{a}_{32}\tilde{x}_2 \geq \tilde{b}_3 \tag{5.5}$$

$$x_1, x_2 \geq 0 \tag{5.6}$$

where b_1, b_2 are random variables following gamma distribution with known parameter $\beta = 3$ and the and the objective function coefficients $\tilde{c}_{ij}$'s, constraint coefficients $\tilde{a}_{ij}$'s are assumed to be generalized L-R type fuzzy numbers as follows.

$\tilde{c}_{11} : \langle 4, 6, 2, 4; 0.6 \rangle$ $\quad$ $\tilde{a}_{11} : \langle 2, 7, 1, 1; 0.7 \rangle$

$\tilde{c}_{12} : \langle 2, 5, 3, 6; 0.5 \rangle$ $\quad$ $\tilde{a}_{12} : \langle 4, 9, 6, 8; 0.8 \rangle$

$\tilde{c}_{13} : \langle 3, 6, 2, 7; 0.4 \rangle$ $\quad$ $\tilde{a}_{21} : \langle 2, 6, 1, 7; 0.9 \rangle$

$\tilde{c}_{21} : \langle 5, 8, 3, 9; 0.7 \rangle$ $\quad$ $\tilde{a}_{22} : \langle 5, 8, 0, 2; 0.4 \rangle$

$\tilde{c}_{22} : \langle 6, 14, 6, 8; 0.2 \rangle$ $\quad$ $\tilde{a}_{31} : \langle 2, 5, 1, 5; 0.8 \rangle$

$\tilde{c}_{23} : \langle 8, 12, 3, 9; 0.3 \rangle$ $\quad$ $\tilde{a}_{32} : \langle 3, 5, 8, 12, 0.6 \rangle$

$\tilde{b}_3 : \langle 16, 22, 10, 12; 0.2 \rangle$

Using Definition 2.5, the crisp values are determined as

$\tilde{c}_{11} = 1.100, \quad \tilde{c}_{12} = 0.708, \quad \tilde{c}_{13} = 0.767,$

$\tilde{c}_{21} = 1.867, \quad \tilde{c}_{22} = 0.700, \quad \tilde{c}_{23} = 1.150,$

$\tilde{a}_{11} = 1.050, \quad \tilde{a}_{12} = 1.867, \quad \tilde{a}_{21} = 1.650,$

$\tilde{a}_{22} = 0.933, \quad \tilde{a}_{31} = 1.200, \quad \tilde{a}_{32} = 1.000,$

$\tilde{b}_3 = 1.300$. Using Theorem 4.1 the FMOSNLP (5.1)-(5.18) is rewritten as FMONLP given below:

$$\min f^1(x) = 1.100\tilde{x}_1 + 0.708\tilde{x}_2 + 0.767\tilde{x}_1^2 \tag{5.7}$$

$$\min f^2(x) = 1.867\tilde{x}_1 + 0.700\tilde{x}_2 + 1.150\tilde{x}_2{}^2 \tag{5.8}$$

subject to

$$1.050\tilde{x}_1 + 1.867\tilde{x}_2 \geq 28.47 \tag{5.9}$$

$$1.650\tilde{x}_1 + 0.933\tilde{x}_2 \geq 58.49 \tag{5.10}$$

$$1.200\tilde{x}_1 + 1.000\tilde{x}_2 \geq 1.300 \tag{5.11}$$

$$x_1, x_2 \geq 0 \tag{5.12}$$

Using Definition 2.6 and by taking $(1,1)$ as initial point in the Taylor's theorem the multi-objective nonlinear programming (5.7)-(5.12) is converted into multi-objective linear programming (5.13)-(5.18) as given below:

$$\min Z^1 = 2.634\tilde{x}_1 + 0.708\tilde{x}_2 - 0.767 \tag{5.13}$$

$$\min Z^2 = 1.867\tilde{x}_1 + 3.000\tilde{x}_2 - 1.150 \tag{5.14}$$

subject to

$$1.050\tilde{x}_1 + 1.867\tilde{x}_2 \geq 28.47 \tag{5.15}$$

$$1.650\tilde{x}_1 + 0.933\tilde{x}_2 \geq 58.49 \tag{5.16}$$

$$1.200\tilde{x}_1 + 1.000\tilde{x}_2 \geq 1.300 \tag{5.17}$$

$$x_1, x_2 \geq 0 \tag{5.18}$$

Solving the multi-objective linear programming (5.13)-(5.18) as a single objective function linear programming using LINGO, the global optimum solution is

$$X^{(1)} = (0.000, 62.5690), \quad Z^{(1)} = 43.61769,$$

$$X^{(2)} = (35.44848, 0.000), \quad Z^{(2)} = 65.0323.$$

The compromising solution is obtained by solving the LPP (5.19)-(5.23)

$$\min Z = 0.0829\tilde{x}_1 + 0.0623\tilde{x}_2 - 0.0353 \tag{5.19}$$

subject to

$$1.050\tilde{x}_1 + 1.867\tilde{x}_2 \geq 28.47 \tag{5.20}$$

$$1.650\tilde{x}_1 + 0.933\tilde{x}_2 \geq 58.49 \tag{5.21}$$

$$1.200\tilde{x}_1 + 1.000\tilde{x}_2 \geq 1.300 \tag{5.22}$$

$$x_1, x_2 \geq 0 \tag{5.23}$$

The optimal solution to the LPP is

$$X = (35.44848, 0.0000), \quad Z = 2.903379.$$

6. Conclusion

In this study, a novel approach has been developed to obtain the compromise solution for a fuzzy multi-objective stochastic nonlinear programming problem. This technique is very helpful to solve various manufacturing problems and decision making problems in which the fuzziness and randomness are involved together with multiple objectives. This work can be prolonged to solve multi-objective geometric programming where the random variables follow different types of distribution and with many other types of fuzzy parameters.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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