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Some Smooth Compactly Supported Tight Framelets

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Abstract. For any dilation matrix with integer entries, we construct a family of smooth compactly supported tight wavelet frames in $L^2(\mathbb{R}^d)$, $d \geq 1$. Estimates for the degrees of smoothness of these framelets are given. Our construction involves some compactly supported refinable functions, the Oblique Extension Principle and a slight generalization of a theorem of Lai and Stöckler.

1. **Introduction**

Given a dilation matrix with integer entries, we construct smooth compactly supported tight framelets in $L^2(\mathbb{R}^d)$, $d \geq 1$, associated to such a dilation, and with any desired degree of smoothness. Tight wavelet frames have recently become the focus of increased interest because they can be computed and applied just as easily as orthonormal wavelets, but are easier to construct.

We begin with notation and definitions. The sets of strictly positive integers, integers, and real numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ respectively. Given a Lebesgue measurable set $S \subset \mathbb{R}^d$, $|S|$ will denote its Lebesgue measure and χ_S will be its characteristic function. Given a $d \times d$ matrix B, the complex conjugate of its transpose will be denoted by B^*. The $n \times n$ identity matrix will be denoted by $\mathbf{I}_{n\times n}$.

We say that $A \in \mathbb{R}^{d\times d}$ is a dilation matrix preserving the lattice $\mathbb{Z}^d$ if all eigenvalues of A have modulus greater than 1 and $A\mathbb{Z}^d \subset \mathbb{Z}^d$. Note that if $A \in \mathbb{R}^{d\times d}$ is a dilation matrix preserving the lattice $\mathbb{Z}^d$, then $d_A := |\det A|$ is an integer greater than 1. The quotient group $\mathbb{Z}^d/A\mathbb{Z}^d$ is well defined, and by $\mathbf{\Delta}_A \subset \mathbb{Z}^d$ we will denote a full collection of representatives of the cosets of $\mathbb{Z}^d/A\mathbb{Z}^d$. Recall that there are exactly d_A cosets ([6], [17, p. 109]).

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A sequence $\{\phi_n\}_{n=1}^{\infty}$ of elements in a separable Hilbert space $\mathbb{H}$ is a *frame* for $\mathbb{H}$ if there exist constants $C_1, C_2 > 0$ such that

$$C_1\|h\|^2 \le \sum_{n=1}^{\infty} |\langle h, \phi_n\rangle|^2 \le C_2\|h\|^2, \quad \text{for all } h \in \mathbb{H},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbb{H}$. The constants C_1 and C_2 are called *frame bounds*. The definition implies that a frame is a complete sequence of elements of $\mathbb{H}$. A frame $\{\phi_n\}_{n=1}^{\infty}$ is *tight* if we may choose $C_1 = C_2$.

Let $A \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$. A set of functions $\Psi = \{\psi_1, \dots, \psi_N\} \subset L^2(\mathbb{R}^d)$ is called a wavelet frame, if the system

$$\{d_A^{j/2}\psi_\ell(A^j\mathbf{x} + \mathbf{k}); j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, 1 \le \ell \le N\}$$

is a frame for $L^2(\mathbb{R}^d)$. If this system is a tight frame for $L^2(\mathbb{R}^d)$ then Ψ is called a tight wavelet frame. In particular, a tight wavelet frame with frame constant equal to 1 is called a *tight framelet*.

Let $\widehat{f}$ denote the Fourier transform of the function f. Thus, if $f \in L^1(\mathbb{R}^d)$, $\mathbf{t}, \mathbf{x} \in \mathbb{R}^d$,

$$\widehat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} f(\mathbf{t})e^{-2\pi i\mathbf{t}\cdot\mathbf{x}}d\mathbf{t},$$

where $\mathbf{t}\cdot\mathbf{x}$ denotes the usual inner product of vectors $\mathbf{t}$ and $\mathbf{x}$.

Han [9], and independently Ron and Shen [13], found necessary and sufficient conditions for translates and dilates of a set of functions to be a tight framelet. Ron and Shen also formulated what is known as the Unitary Extension Principle (UEP), which, in addition to its other applications, provides a method for constructing compactly supported framelets. In [14] (see also [15]), Ron and Shen describe a method for constructing compactly supported tight affine frames in $L^2(\mathbb{R}^d)$ from box splines. Using Ron and Shen's method, Gröchenig and Ron [7] show how to construct, for any dilation matrix, compactly supported framelets with any desired degree of smoothness. These tight wavelet frames have at most one vanishing moment. Furthermore, based on works by Ron and Shen and by Gröchenig and Ron, Han [8] also constructs compactly supported tight wavelet frames with degree of smoothness and vanishing moments of order as large as desired.

In this paper we construct a family of compactly supported refinable functions and we use these functions, the Oblique Extension Principle and a slight generalization of Theorem 3.4 by Lai and Stöckler [12] to construct smooth compactly supported tight framelets.

The UEP led to the Oblique Extension Principle (OEP), a method based on the UEP; it was developed by Chui, He and Stöckler [2], and independently by Daubechies, Han, Ron and Shen [4], who gave the method its name. With the definition of the Fourier transform that we shall adopt in the next section, the OEP may be formulated as follows:

Theorem A. *Let $A \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$. Let $\phi \in L^2(\mathbb{R}^d)$ be compactly supported and refinable, i.e.*

$$\widehat{\phi}(A^*\mathbf{t}) = P(\mathbf{t})\widehat{\phi}(\mathbf{t}),$$

where $P(\mathbf{x})$ is a trigonometric polynomial and A^* is the transpose of A. Assume moreover that $|\widehat{\phi}(\mathbf{0})| = 1$ and $|\widehat{\phi}(\mathbf{t})| \le C(1+|\mathbf{t}|)^{-\alpha}$ for some $\alpha > d/2$. Let $S(\mathbf{t})$ be another trigonometric polynomial such that $S(\mathbf{t}) \ge 0$ and $S(\mathbf{0}) = 1$. Assume there are trigonometric polynomials or rational functions Q_ℓ, $\ell = 1,\cdots,N$, that satisfy the OEP condition

$$S(A^*\mathbf{t})P(\mathbf{t})\overline{P(\mathbf{t}+\mathbf{j})} + \sum_{\ell=0}^{N} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})} = \begin{cases} S(\mathbf{t}) & \text{if } \mathbf{j} \in \mathbb{Z}^d, \\ 0 & \text{if } \mathbf{j} \in ((A^*)^{-1}(\mathbb{Z}^d)/\mathbb{Z}^d) \setminus \mathbb{Z}^d. \end{cases} \tag{1}$$

If

$$\widehat{\psi_\ell}(\mathbf{t}) := Q_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}), \quad \ell = 1,\ldots,N,$$

then $\Psi = \{\psi_1,\ldots,\psi_N\}$ is a tight framelet in $L^2(\mathbb{R}^d)$.

This version of the OEP is a straightforward consequence of [13, Corollary 6.7].

Using the OEP and the following slight generalization of Theorem 3.4 of Lai and Stöckler [12], we obtain a general method for constructing compactly supported framelets in $L^2(\mathbb{R}^d)$ associated to any fixed dilation matrix. The proof is identical, and we include it for the sake of completeness. We have also included in the statement a generalization of the algorithm implicit in the proof of Theorem 3.4.

Theorem 1. *Let $A \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$ and let $\mathbf{\Delta}_A = \{\mathbf{q}_s\}_{s=0}^{d_A-1}$ and $\mathbf{\Delta}_{A^*} = \{\mathbf{p}_s\}_{s=0}^{d_A-1}$ be full collections of representatives of the cosets of $\mathbb{Z}^d/A\mathbb{Z}^d$ and $\mathbb{Z}^d/A^*\mathbb{Z}^d$ respectively with $\mathbf{q}_0 = \mathbf{p}_0 = \mathbf{0}$. Let $P(\mathbf{t})$ be a trigonometric polynomial defined on $\mathbb{R}^d$ that satisfies the condition*

$$\sum_{s=0}^{d_A-1} |P(\mathbf{t} + (A^*)^{-1}(\mathbf{p}_s))|^2 \le 1,$$

let

$$\mathscr{P}(\mathbf{t}) := \left(P(\mathbf{t} + (A^*)^{-1}(\mathbf{p}_s)); s = 0,\ldots,d_A-1\right)^T,$$

and let

$$\mathscr{M}(\mathbf{t}) := d_A^{-1/2}\left(e^{i2\pi\mathbf{q}_l\cdot(\mathbf{t}+(A^*)^{-1}(\mathbf{p}_s))}; l,s = 0,\ldots,d_A-1\right) \tag{2}$$

be the polyphase matrix, where s denotes the row index and l denotes the column index.

Let the $d_A \times 1$ matrix function $G(\mathbf{t})$ be defined by

$$G(\mathbf{t}) := \mathscr{M}^*(\mathbf{t})\mathscr{P}(\mathbf{t}) = \left(L_k(A^*\mathbf{t}); k = 0,\ldots,d_A-1\right)^T, \tag{3}$$

where $\mathscr{M}^*(\mathbf{t})$ *denotes the complex conjugate transpose of* $\mathscr{M}(\mathbf{t})$*. Suppose that there exist trigonometric polynomials* $\widetilde{P}_1, \dots, \widetilde{P}_M$ *such that*

$$\sum_{k=0}^{d_A-1} |L_k(\mathbf{t})|^2 + \sum_{j=1}^{M} |\widetilde{P}_j(\mathbf{t})|^2 = 1. \tag{4}$$

Let $N := d_A + M$ *and let the* $N \times 1$ *matrix function* $\mathscr{G}(\mathbf{t})$ *be defined by*

$$\mathscr{G}(\mathbf{t}) := \big(L_k(A^*\mathbf{t}); k = 0, \dots, d_A - 1, \widetilde{P}_j(A^*\mathbf{t}); 1 \le j \le M\big)^T,$$

and

$$\widetilde{\mathscr{Q}}(\mathbf{t}) := I_{N\times N} - \mathscr{G}(\mathbf{t})\mathscr{G}^*(\mathbf{t}).$$

Let $H(\mathbf{t})$ *denote the first* $d_A \times N$ *block matrix of* $\widetilde{\mathscr{Q}(\mathbf{t})}$*,*

$$\mathscr{Q}(\mathbf{t}) := \mathscr{M}(\mathbf{t})H(\mathbf{t}),$$

and let $[Q_1(\mathbf{t}), \dots, Q_N(\mathbf{t})]$ *denote the first row of* $\mathscr{Q}(\mathbf{t})$*. Then the trigonometric polynomials* P *and* Q_ℓ*,* $\ell = 1, \dots, N$*, satisfy the identity* (1) *with* $S(\mathbf{t}) = 1$*.*

Proof. Using (4), we have $\mathscr{G}^*(\mathbf{t})\mathscr{G}(\mathbf{t}) = 1$; then

$$\begin{aligned}\widetilde{\mathscr{Q}}(\mathbf{t})\widetilde{\mathscr{Q}}^*(\mathbf{t}) &= I_{N\times N} - 2\mathscr{G}(\mathbf{t})\mathscr{G}^*(\mathbf{t}) + \mathscr{G}(\mathbf{t})\mathscr{G}^*(\mathbf{t})\mathscr{G}(\mathbf{t})\mathscr{G}^*(\mathbf{t}) \\ &= I_{N\times N} - \mathscr{G}(\mathbf{t})\mathscr{G}^*(\mathbf{t}) = \widetilde{\mathscr{Q}}(\mathbf{t}).\end{aligned}$$

Thus

$$\mathscr{G}(\mathbf{t})\mathscr{G}^*(\mathbf{t}) + \widetilde{\mathscr{Q}}(\mathbf{t})\widetilde{\mathscr{Q}}^*(\mathbf{t}) = I_{N\times N}.$$

Restricting to the first principal $d_A \times d_A$ blocks in the above matrices, we have

$$G(\mathbf{t})G^*(\mathbf{t}) + H(\mathbf{t})H^*(\mathbf{t}) = I_{d_A\times d_A}. \tag{5}$$

From Lemma 5.1 of [5], we know that $\mathscr{M}(\mathbf{t})$ is unitary. Hence from (5) we conclude that

$$\mathscr{M}(\mathbf{t})G(\mathbf{t})G^*(\mathbf{t})\mathscr{M}^*(\mathbf{t}) + \mathscr{M}(\mathbf{t})H(\mathbf{t})H^*(\mathbf{t})\mathscr{M}^*(\mathbf{t}) = I_{d_A\times d_A}.$$

Thus,

$$\mathscr{P}(\mathbf{t})\mathscr{P}^*(\mathbf{t}) + \mathscr{Q}(\mathbf{t})\mathscr{Q}^*(\mathbf{t}) = I_{d_A\times d_A}$$

which is equivalent to saying that the $P(\mathbf{t})$ and $Q_1(\mathbf{t}), \dots, Q_N(\mathbf{t})$ satisfy the equality (1) with $S(\mathbf{x}) = 1$ and dilation matrix A. □

2. A Family of Tight Framelets

We begin by constructing a family of smooth compactly supported refinable functions in $L^2(\mathbb{R}^d)$, $d > 1$, associated to a dilation matrix A.

We will use the following theorem of Gröchenig and Madych (see [6, Theorem 2] and remark following the statement of the theorem).

Theorem B. *Let $A \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, let $\mathbf{\Delta}_A = \{\mathbf{q}_s\}_{s=0}^{d_A-1}$ be a full collection of representatives of the cosets of $\mathbb{Z}^d/A\mathbb{Z}^d$ with $\mathbf{q}_0 = \mathbf{0}$. Then the characteristic function χ_E, where the set E is defined by*

$$E := \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} = \sum_{j=1}^{\infty} A^{-j}\mathbf{k}^{(j)}, \mathbf{k}^{(j)} \in \mathbf{\Delta}_A\right\}, \tag{6}$$

is a non null compactly supported measurable function such that $\|\chi_E\|_{L^1(\mathbb{R}^d)} \geq 1$ and satisfies the refinement equation

$$\widehat{\chi}_E(A^*(\mathbf{t})) = H(\mathbf{t})\widehat{\chi}_E(\mathbf{t}) \tag{7}$$

where

$$H(\mathbf{t}) := \frac{1}{d_A}\sum_{s=0}^{d_A-1} e^{-2\pi i \mathbf{t}\cdot\mathbf{q}_s} \tag{8}$$

We will also need the following

Lemma 1. *Let $A \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, let $\mathbf{\Delta}_A = \{\mathbf{q}_s\}_{s=0}^{d_A-1}$ and $\mathbf{\Delta}_{A^*} = \{\mathbf{p}_s\}_{s=0}^{d_A-1}$ be full collections of representatives of the cosets of $\mathbb{Z}^d/A\mathbb{Z}^d$ and $\mathbb{Z}^d/A^*\mathbb{Z}^d$ respectively with $\mathbf{q}_0 = \mathbf{p}_0 = \mathbf{0}$, and let $H(\mathbf{t})$ de given by* (8)*. Then*

$$\sum_{s=0}^{d_A-1} |H(\mathbf{t} + (A^*)^{-1}(\mathbf{p}_s))|^2 = 1.$$

Proof. We have

$$|H(\mathbf{t})|^2 = \frac{1}{d_A^2}\left(\sum_{s=0}^{d_A-1} e^{-2\pi i \mathbf{t}\cdot\mathbf{q}_s}\right)\left(\sum_{r=0}^{d_A-1} e^{2\pi i \mathbf{t}\cdot\mathbf{q}_r}\right) = \frac{1}{d_A^2}\sum_{s,r=0}^{d_A-1} e^{-2\pi i \mathbf{t}\cdot(\mathbf{q}_s-\mathbf{q}_r)}$$
$$= \frac{1}{d_A^2}\left(d_A + \sum_{s,r=0,s\neq r}^{d_A-1} e^{-2\pi i \mathbf{t}\cdot(\mathbf{q}_s-\mathbf{q}_r)}\right).$$

Thus

$$\begin{aligned}
&\sum_{s=0}^{d_A-1} |H(\mathbf{t} + (A^*)^{-1}(\mathbf{p}_s))|^2 \\
&\quad= \sum_{s=0}^{d_A-1} \frac{1}{d_A^2}\left(d_A + \sum_{j,r=0,j\neq r}^{d_A-1} e^{-2\pi i(\mathbf{t}+(A^*)^{-1}(\mathbf{p}_s))\cdot(\mathbf{q}_j-\mathbf{q}_r)}\right) \\
&\quad= 1 + \frac{1}{d_A^2}\sum_{j,r=0,j\neq r}^{d_A-1} e^{-2\pi i \mathbf{t}\cdot(\mathbf{q}_j-\mathbf{q}_r)} \sum_{s=0}^{d_A-1} e^{-2\pi i (A^*)^{-1}(\mathbf{p}_s)\cdot(\mathbf{q}_j-\mathbf{q}_r)}.
\end{aligned}$$

However, from e.g. [5] or [1, Lemma 3], we see that if $\mathbf{k} \in \mathbb{Z}^d \setminus A(\mathbb{Z}^d)$, then

$$\sum_{s=0}^{d_A-1} e^{-2\pi i (A^*)^{-1}(\mathbf{p}_s)\cdot\mathbf{k}} = 0,$$

and the assertion follows. □

The following statement may be found in ([17, Appendix A.2]). The proof is straightforward and will be omitted.

Lemma C. *Let C^0 be the class of continuous functions in $L^2(\mathbb{R}^d)$, and let C^r, $r = 1, 2, \ldots$ be the class of functions f such that all partial derivatives of f of order not greater than r are continuous and in $L^2(\mathbb{R}^d)$. If*

$$[\widehat{f}(\mathbf{t})| \leq C(1 + |\mathbf{t}|)^{-N-\varepsilon}$$

for some integer $N \geq d$ and $\varepsilon > 0$, then f is in C^{N-d}.

Proofs of the following proposition may be found in [16, Lemma 3.1], [17, Proposition 5.23] or [7, Result 2.6].

Proposition D. *Let $A \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, let $\mathbf{\Delta}_A = \{\mathbf{q}_s\}_{s=0}^{d_A-1}$ be a full collection of representatives of the cosets of $\mathbb{Z}^d/A\mathbb{Z}^d$ with $\mathbf{q}_0 = \mathbf{0}$, and let $E \subset \mathbb{R}^d$ be the set defined by (6). Then there exist two positive constants ϵ and C such that*

$$|\widehat{\chi}_E(\mathbf{t})| \leq C|\mathbf{t}|^{-\epsilon}.$$

We can now prove the following.

Proposition 1. *Let $A \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, let $\mathbf{\Delta}_A = \{\mathbf{q}_s\}_{s=0}^{d_A-1}$ be a full collection of representatives of the cosets of $\mathbb{Z}^d/A\mathbb{Z}^d$ with $\mathbf{q}_0 = \mathbf{0}$, let $\phi_1 := |E|^{-1}\chi_E$ and for $n \in \mathbb{N}$ let ϕ_n denote the n-fold convolution of ϕ_1 with itself. Let H_n be defined by (8). Then ϕ_n is non null, compactly supported, and square-integrable on $\mathbb{R}^d$, $\|\phi_n\|_{L2(\mathbb{R}^d)} \leq 1$, $\widehat{\phi}_n(\mathbf{0}) = 1$, and, setting $P(\mathbf{t}) = H^n(\mathbf{t})$, the refinement equation*

$$\widehat{\phi}_n(A^*(\mathbf{t})) = P(\mathbf{t})\widehat{\phi}_n(\mathbf{t}) \tag{9}$$

holds. Moreover, if $\epsilon\, n - d > r > 1$, where ϵ is defined in Proposition D, ϕ_n is in continuity class C^r.

Proof. By Theorem B, χ_E is a non null compactly supported function, which implies that also ϕ_n is a non-null compactly supported function. Moreover, since $\widehat{\phi}_n(\mathbf{t}) = \widehat{\phi}_1^n(\mathbf{t})$, it follows that $\widehat{\phi}_n(\mathbf{0}) = 1$. Further, by Young's inequality for convolutions and bearing in mind that $|E| \geq 1$,

$$\|\phi_n\|_{L^2(\mathbb{R}^d)} \leq \|\phi_{n-1}\|_{L^2(\mathbb{R}^d)}\||E|^{-1}\chi_E\|_{L^1(\mathbb{R}^d)} \leq \|\phi_1\|_{L^2(\mathbb{R}^d)} \leq 1.$$

We now verify that the refinement equation (9) is satisfied. Taking the Fourier transform of ϕ_n and applying (7), we have

$$\widehat{\phi}_n(A^*(\mathbf{t})) = |E|^{-n}[\widehat{\chi}_E(A^*(\mathbf{t}))]^n = H(\mathbf{t})^n|E|^{-n}[\widehat{\chi}_E(\mathbf{t})]^n = P(\mathbf{t})\widehat{\phi}_n(\mathbf{t}).$$

We now prove the estimates on the degree of smoothness of ϕ_n. By Proposition D, we have

$$|\widehat{\phi}_n(\mathbf{t})| = |\widehat{\chi}_E(\mathbf{t})|^n \leq C|\mathbf{t}|^{-\epsilon n}.$$

Moreover, since $\widehat{\phi}_n$ is continuous,

$$|\widehat{\phi}_n(\mathbf{t})| \le K(1+|\mathbf{t}|)^{-\epsilon n}.$$

Hence, if $\epsilon n - d > r > 1$, Lemma C implies that ϕ is in continuity class C^r. $\square$

We now use Theorem 1 and the refinable functions obtained in Proposition 1, to construct a family of tight framelets $\Psi = \{\psi_1, \ldots, \psi_N\}$ in $L^2(\mathbb{R}^d)$, $d > 1$, such that the functions ψ_ℓ, $\ell = 1, \ldots, n$, are smooth and compactly supported. First we prove

Lemma 2. *Let $A \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, let $\boldsymbol{\Delta}_{A^*} = \{\mathbf{p}_s\}_{s=0}^{d_A-1}$ be a full collection of representatives of the coset $\mathbb{Z}^d/A^*\mathbb{Z}^d$ with $\mathbf{p}_0 = \mathbf{0}$, let $P(\mathbf{t})$ a trigonometric polynomial with real coefficients. Then there are numbers α_k such that*

$$1 - \sum_{s=0}^{d_A-1} |P(\mathbf{t} + (A^*)^{-1}(\mathbf{p}_s))|^2 = \sum_{\mathbf{k}\in\mathbb{Z}^d} \alpha_{\mathbf{k}} e^{2\pi i \mathbf{k}\cdot A^*(\mathbf{t})}, \quad \alpha_{\mathbf{k}} \in \mathbb{R}. \tag{10}$$

Moreover, a finite number of $\alpha_{\mathbf{k}}$ are nonzero, and $\alpha_{\mathbf{k}} = \alpha_{-\mathbf{k}}$.

Proof. Clearly $P(\mathbf{t})$ may be written in the form

$$P(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}} e^{2\pi i \mathbf{k}\cdot\mathbf{t}},$$

where $a_{\mathbf{k}} \in \mathbb{R}$ and a finite number of the $a_{\mathbf{k}}$ are nonzero. We have

$$|P(\mathbf{t})|^2 = \left(\sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}} e^{2\pi i \mathbf{k}\cdot\mathbf{t}}\right)\left(\sum_{\mathbf{r}\in\mathbb{Z}^d} a_{\mathbf{r}} e^{-2\pi i \mathbf{r}\cdot\mathbf{t}}\right) = \sum_{\mathbf{k},\mathbf{r}} a_{\mathbf{k}} a_{\mathbf{r}} e^{2\pi i (\mathbf{k}-\mathbf{r})\cdot\mathbf{t}}.$$

Thus,

$$\sum_{s=0}^{d_A-1} |P(\mathbf{t} + (A^*)^{-1}(\mathbf{p}_s))|^2 = \sum_{s=0}^{d_A-1} \sum_{\mathbf{k},\mathbf{r}\in\mathbb{Z}^d} a_{\mathbf{k}} a_{\mathbf{r}} e^{2\pi i (\mathbf{k}-\mathbf{r})\cdot(\mathbf{t}+(A^*)^{-1}(\mathbf{p}_s))}.$$

Let $\boldsymbol{\Delta}_A = \{\mathbf{q}_s\}_{s=0}^{d_A-1}$ be a full collection of representatives of the coset $\mathbb{Z}^d/A\mathbb{Z}^d$ with $\mathbf{q}_0 = \mathbf{0}$. Since $\mathbf{k} \in \mathbb{Z}^d$ may be written in the form $\mathbf{k} = \mathbf{q}_s + A(\mathbf{k}')$ for some $s \in \{0, \cdots, d_A - 1\}$ and $\mathbf{k}' \in \mathbb{Z}^d$, we have

$$\begin{aligned}
&\sum_{s=0}^{d_A-1} |P(\mathbf{t} + (A^*)^{-1}(\mathbf{p}_s))|^2 \\
&= \sum_{s=0}^{d_A-1} \sum_{m,l=0}^{d_A-1} \sum_{\mathbf{k},\mathbf{r}\in\mathbb{Z}^d} b_{m,\mathbf{k}} b_{l,\mathbf{r}} e^{2\pi i (\mathbf{q}_m + A(\mathbf{k}) - \mathbf{q}_l + A(\mathbf{r}))\cdot(\mathbf{t}+(A^*)^{-1}(\mathbf{p}_s))} \\
&= \sum_{m,l=0}^{d_A-1} \sum_{\mathbf{k},\mathbf{r}\in\mathbb{Z}^d} b_{m,\mathbf{k}} b_{l,\mathbf{r}} e^{2\pi i (\mathbf{q}_m + A(\mathbf{k}) - \mathbf{q}_l - A(\mathbf{r}))\cdot\mathbf{t}} \sum_{s=0}^{d_A-1} e^{2\pi i (\mathbf{q}_m + A(\mathbf{k}) - \mathbf{q}_l - A(\mathbf{r}))\cdot(A^*)^{-1}(\mathbf{p}_s)}.
\end{aligned}$$

Since $\mathbf{k}, \mathbf{r}, \mathbf{p}_s \in \mathbb{Z}^d$, it follows that

$$\sum_{s=0}^{d_A-1} |P(\mathbf{t}+(A^*)^{-1}(\mathbf{p}_s))|^2$$
$$= \sum_{m,l=0}^{d_A-1} \sum_{\mathbf{k},\mathbf{r}\in\mathbb{Z}^d} b_{m,\mathbf{k}} b_{l,\mathbf{r}} e^{2\pi i(\mathbf{q}_m + A(\mathbf{k}) - \mathbf{q}_l - A(\mathbf{r}))\cdot\mathbf{t}} \sum_{s=0}^{d_A-1} e^{2\pi i(\mathbf{q}_m-\mathbf{q}_l)\cdot(A^*)^{-1}(\mathbf{p}_s)}.$$

If $l \neq m$, then $\mathbf{q}_m - \mathbf{q}_l \notin A(\mathbb{Z}^d)$ because $\mathbf{q}_m$ and $\mathbf{q}_l$ are representatives of different cosets of $\mathbb{Z}^d/A(\mathbb{Z}^d)$. Using Lemma 5.1 in [5] we have

$$\sum_{s=0}^{d_A-1} |P(\mathbf{t}+(A^*)^{-1}(\mathbf{p}_s))|^2 = d_A \sum_{m=0}^{d_A-1} \sum_{\mathbf{k},\mathbf{r}\in\mathbb{Z}^d} b_{m,\mathbf{k}} b_{m,\mathbf{r}} e^{2\pi i(A(\mathbf{k})-A(\mathbf{r}))\cdot\mathbf{t}}$$
$$= d_A \sum_{m=0}^{d_A-1} \sum_{\mathbf{k},\mathbf{r}\in\mathbb{Z}^d} b_{m,\mathbf{k}} b_{m,\mathbf{r}} e^{2\pi i(\mathbf{k}-\mathbf{r})\cdot A^*\mathbf{t}}.$$

Hence the identity (10) is satisfied, and only a finite number of coefficients are non zero. Finally, since $1 - \sum_{s=0}^{d_A-1} |P(\mathbf{t}+(A^*)^{-1}(\mathbf{p}_s))|^2$ is a real trigonometric polynomial with real coefficients, it follows that $\alpha_{-\mathbf{k}} = \alpha_{\mathbf{k}}$. □

We need the following generalization of [12, Theorem 4.2].

Lemma 3. *Let $A \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, let $\mathbf{\Delta}_A = \{\mathbf{q}_s\}_{s=0}^{d_A-1}$ and $\mathbf{\Delta}_{A^*} = \{\mathbf{p}_s\}_{s=0}^{d_A-1}$ be full collections of representatives of the cosets $\mathbb{Z}^d/A\mathbb{Z}^d$ and $\mathbb{Z}^d/A^*\mathbb{Z}^d$ respectively with $\mathbf{q}_0 = \mathbf{p}_0 = \mathbf{0}$, let the trigonometric polynomial $H(\mathbf{t})$ be defined by* (8)*, and let $P(\mathbf{t}) := H^n(\mathbf{t})$. Let the numbers α_k be such that the identity* (10) *is satisfied, let Γ denote the set of nonzero α_k, let the trigonometric polynomials $\widetilde{P}_\mathbf{k}$ be defined by*

$$\widetilde{P}_\mathbf{0}(\mathbf{t}) = 0, \quad \widetilde{P}_\mathbf{k}(\mathbf{t}) := \sqrt{\frac{|\alpha_\mathbf{k}|}{2}}(1 - e^{2\pi i\mathbf{k}\cdot\mathbf{t}}), \qquad \textit{if } \mathbf{k} \in \Gamma \setminus \{\mathbf{0}\}, \tag{11}$$

and let the trigonometric polynomials $L_j(A^*\mathbf{t})$, $j \in \{0, \ldots, d_A - 1\}$ be defined by (3)*. Then*

$$\sum_{j=0}^{d_A-1} |L_j(\mathbf{t})|^2 + \sum_{\mathbf{k}\in\Gamma} |\widetilde{P}_\mathbf{k}(\mathbf{t})|^2 = 1.$$

Proof. Observe that $\alpha_\mathbf{k} = \alpha_{-\mathbf{k}}$. Since the coefficients of the trigonometric polynomial H defined by (8) are positive so are the coefficients of $P(\mathbf{t})$, and it follows that $\alpha_\mathbf{k} < 0$ if $\mathbf{k} \in \Gamma \setminus \mathbf{0}$. Moreover, we know that $P(\mathbf{0}) = 1$; therefore $\alpha_\mathbf{0} = -\sum_{\mathbf{k}\in\Gamma\setminus\mathbf{0}} \alpha_\mathbf{k}$.

Using the elementary formula $1 - \cos(2\pi\mathbf{k}\cdot\mathbf{t}) = \frac{1}{2}|1 - e^{-2\pi i\mathbf{k}\cdot\mathbf{t}}|^2$, we have:

$$\sum_{\mathbf{k}\in\Gamma} |\widetilde{P}_\mathbf{k}(A^*\mathbf{t})|^2 = \sum_{\mathbf{k}\in\Gamma\setminus\mathbf{0}} \frac{|\alpha_\mathbf{k}|}{2} |1 - e^{2\pi i\mathbf{k}\cdot A^*\mathbf{t}}|^2$$

$$
\begin{aligned}
&= \sum_{\mathbf{k}\in\Gamma\setminus\mathbf{0}} |\alpha_{\mathbf{k}}|\big(1-\cos(2\pi\mathbf{k}\cdot A^*\mathbf{t})\big) \\
&= -\sum_{\mathbf{k}\in\Gamma\setminus\mathbf{0}} \alpha_{\mathbf{k}} + \sum_{\mathbf{k}\in\Gamma\setminus\mathbf{0}} \alpha_{\mathbf{k}} \cos(2\pi\mathbf{k}\cdot A^*\mathbf{t}) \\
&= \alpha_{\mathbf{0}} + \sum_{\mathbf{k}\in\Gamma\setminus\mathbf{0}} \alpha_{\mathbf{k}} \cos(2\pi\mathbf{k}\cdot A^*\mathbf{t}) \\
&= \sum_{\mathbf{k}\in\Gamma} \frac{\alpha_{\mathbf{k}}}{2}\big(e^{2\pi i\mathbf{k}\cdot A^*\mathbf{t}} + e^{-2\pi i\mathbf{k}\cdot A^*\mathbf{t}}\big).
\end{aligned}
$$

Bearing in mind that $\alpha_{\mathbf{k}} = \alpha_{-\mathbf{k}}$, we obtain

$$
\sum_{s=0}^{d_A-1} |P(\mathbf{t}+(A^*)^{-1}(\mathbf{p}_s))|^2 + \sum_{\mathbf{k}\in\Gamma} |\widetilde{P}_{\mathbf{k}}(A^*\mathbf{t})|^2 = 1. \tag{12}
$$

On the other hand, note that, since $\mathscr{M}(\mathbf{t})$ defined by (2) is unitary,

$$
\sum_{j=0}^{d_A-1} |L_j(A^*\mathbf{t})|^2 = \mathscr{P}^*(\mathbf{t})\mathscr{M}(\mathbf{t})\mathscr{M}^*(\mathbf{t})\mathscr{P}(\mathbf{t}) = \sum_{s=0}^{d_A-1} |P(\mathbf{t}+(A^*)^{-1}(\mathbf{p}_s))|^2, \tag{13}
$$

and the proof follows from (12) and (13). □

The following theorem describes a construction of a tight smooth framelet of compact support associated to a fixed dilation matrix A preserving the lattice $\mathbb{Z}^d$.

Theorem 2. *Let $A \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, let $\mathbf{\Delta}_A = \{\mathbf{q}_s\}_{s=0}^{d_A-1}$ be a full collection of representatives of the cosets of $\mathbb{Z}^d/A\mathbb{Z}^d$ with $\mathbf{q}_0 = \mathbf{0}$, let $E \subset \mathbb{R}^d$ be defined by* (6) *and let H be the trigonometric polynomial defined by* (8)*. Let $n \in \mathbb{N}$ and $P(\mathbf{t}) = H^n(\mathbf{t})$. Let $\widetilde{P}_j(\mathbf{t})$, $j = 1,\ldots,M$ be the trigonometric polynomials defined as in Lemma* 3*, let $N := d_A + M$, and let $Q_1(\mathbf{t}),\ldots,Q_N(\mathbf{t})$ be the trigonometric polynomials obtained by the algorithm described in Theorem* 1*. If*

$$
\begin{aligned}
&\widehat{\phi}_n(\mathbf{t}) = |E|^{-n}[\widehat{\chi_E}(\mathbf{t})]^n, \\
&\widehat{\psi}_\ell(\mathbf{t}) := Q_\ell(\mathbf{t})\widehat{\phi}_n(\mathbf{t}), \quad \ell = 1,\ldots,N.
\end{aligned}
$$

and $\Psi = \{\psi_1,\ldots,\psi_N\}$ is the set of inverse Fourier transforms of the functions $\widehat{\psi}_\ell$ defined in the preceding displayed identity, then Ψ is a tight framelet in $L^2(\mathbb{R}^d)$ with dilation matrix A, and the functions $\psi_\ell(\mathbf{t})$ are square-integrable on $\mathbb{R}^d$ and compactly supported. Moreover, if $\epsilon n - d > r > 1$ where ϵ is defined in Proposition D, then the functions $\psi_\ell(\mathbf{t})$ are in continuity class C^r.

Proof. That $\Psi = \{\psi_1,\ldots,\psi_N\}$ is a tight framelet follows from Lemma 3, Theorem 1 and Theorem A.

Since the functions Q_ℓ are trigonometric polynomials and therefore bounded on $\mathbb{R}^d$, the smoothness of the functions ψ_ℓ follows from Proposition 1 and Lemma C.

Finally, note that the functions ψ_ℓ are compactly supported because ϕ is compactly supported and the Q_ℓ are trigonometric polynomials. □

References

[1] M. Bownik, Tight frames of multidimensional wavelets, *J. Fourier Anal. Appl.* **3**(5) (1997), 525–542.

[2] C.K. Chui, W. He and J. Stöckler, Compactly supported tight and sibling frames with maximum vanishing moments, *Appl. Comput. Harmonic Anal.* **13** (2002), 224–262.

[3] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

[4] I. Daubechies, B. Han, A. Ron and Z.W. Shen, Framelets: MRA-based constructions of wavelet frames, *Appl. Comput. Harmon. Anal.* **14** (2003), 1–46.

[5] K. Gröchening and A. Haas, Self-similar lattice tilings, *J. Fourier Analysis and Appl.* **1** (1994), 131–170.

[6] K. Gröchening and W.R. Madych, Multiresolution analysis, Haar bases and self-similar tillings of R^n, *IEEE Trans. Inform. Theory* **38**(2) (1992), 556–568.

[7] K. Gröchenig and A. Ron, Tight compactly supported wavelet frames of arbitrarily high smoothness, *Proc. Amer. Math. Soc.* **126** (1998), 1101–1107.

[8] B. Han, Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix, *Approximation Theory, Wavelets and Numerical Analysis*, (Chattanooga, TN, 2001), *J. Comput. Appl. Math.* **155**(1) (2003), 43–67.

[9] B. Han, On dual wavelet tight frames, *Applied Comput. Harmon. Anal.* **4** (1997), 380–413.

[10] E. Hernández and G. Weiss, *A First Course on Wavelets*, CRC Press Inc., 1996.

[11] M.-J. Lai, Construction of multivariate compactly supported orthonormal wavelets, *Adv. Comput. Math.* **25** (2006) 41–56.

[12] M.-J. Lai and J. Stöckler, Construction of multivariate compactly supported tight wavelet frames, *Appl. Comput. Harmon. Anal.* **21** (2006), 324–348.

[13] A. Ron and Z.W. Shen, Affine systems in $L^2(\mathbb{R}^d)$: the analysis of the analysis operator, *J. Funct. Anal.* **148** (1997), 408–447.

[14] A. Ron and Z.W. Shen, Compactly supported tight affine frames in $L^2(\mathbb{R}^d)$, *Math. Comp.* **67** (1998), 191–207.

[15] A. Ron and Z.W. Shen, Construction of compactly supported afine frames in $L^2(\mathbb{R}^d)$, in *Advances in Wavelets*, K.S. Lau (editor), Springer-Verlag, New York, 1998, pp. 27–49.

[16] R.S. Strichartz, Wavelets and self-affine tilings, *Constructive Approximation* **9**(2-3) (1993), 327–346.

[17] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, London Math. Soc., Student Texts **37**, 1997.

[18] Y. Shouzhi and X. Yanmei, Construction of compactly supported conjugate symmetric complex tight wavelet frames, *Int. J. Wavelets Multiresolut. Inf. Process.* **8** (2010), 861–874.

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