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Research Article

Analysis of Variance and Standard Deviation of FM/M/1/k Interdependent Stochastic Feedback Arrival Model with Finite Capacity and Single Server

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Abstract. In this paper, analysis of variance and standard deviation of interdependent stochastic feedback arrival rate and single server with service rate are considered. The steady-state solution and system characteristics are derived for this model. The effect of the nodal parameters on the system characteristics is established and using Maple software the analytical results are numerically illustrated and relevant conclusions are presented.

Keywords. Interdependent stochastic feedback arrival rate, Single-server with service rate, Finite capacity, Variance, Standard deviation

Mathematics Subject Classification (2020). Primary 60K25; Secondary 68M20, 90B22

1. Introduction

Queueing systems provide a solid foundation for the development and analysis of real-world applications. Queueing models anticipate system behaviour such as waiting time, the average number of people in line and so on. It is utilised in industrial engineering, computer engineering and other academic disciplines as well as telecommunication and computer science degrees. These forecasts enable us to foresee problems and take appropriate steps to reduce line lengths. Nisha and Thiagarajan obtained results for the M/M/1/K interdependent retrial queueing model with impatient customers [5]. Vacation queueing models with immediate Bernoulli feedback were examined by Kalyanaraman and Ranganathan [2]. Sasikala and Thiagarajan investigated the steady-state behaviour of an M/M/c/N/K interdependent queueing model [8]. In this paper, the Mathematical model for FM/M/1 single server with identical service rate and feedback arrival rates are considered, the steady-state equations and the steady-state probabilities are obtained. Moreover, the analytical expressions for the queueing model and certain performance measures are derived. The analytical results are numerically illustrated and relevant conclusions are presented.

2. Description of the Model

Consider a FM/M/1 finite queueing system with *First In First Out* (FIFO) discipline with interdependent stochastic feedback arrival model. The feedback customers arrive at a counter in accordance with a feedback Poisson process with mean feedback arrival rate $p(\lambda-\varepsilon)>0$. The service times are independently and identically distributed random variables and exponential distribution with service rate $(\mu-\varepsilon)$. The feedback arrival process $\{X_1(t)\}$ and the service process $\{X_2(t)\}$ of the system are correlated and follow bivariate feedback Poisson distribution, is given by

$$\Pr\{X_1(t)=px_1, X_2(t)=x_2\}=e^{-(p\lambda_i+\mu_i-\varepsilon)t}\sum_{j=0}^{\min(px_1,x_2)}\frac{(\varepsilon t)^j(p(\lambda_i-\varepsilon)t)^{px_1-j}((\mu_i-\varepsilon)t)^{x_2-j}}{j!(px_1-j)!(x_2-j)!},$$

$x_1,x_2=0,1,2,\dots\lambda_i\geq 0$, $i=0,1$ and $\mu_i>0$, $i=1$ with parameters $\lambda_0,\lambda_1,\mu,\varepsilon$, $0\leq p\leq 1$, for mean faster and slower rate of feedback arrivals, mean service rate for single-server with mean dependence rate, respectively.

3. Postulates of the Model

(1) When the system is in faster rate of feedback arrivals, the probability of no feedback arrivals and no service completion during a small interval of time h is

$$1-(p(\lambda_0-\varepsilon)+(\mu-\varepsilon))h+O(h).$$

(2) When the system is in faster rate of feedback arrivals, the probability of one feedback arrival and no service completion during a small interval of time h is

$$(p(\lambda_0-\varepsilon))h+O(h).$$

(3) When the system is in slower rate of feedback arrivals, the probability of no feedback arrivals and no service completion during a small interval of time h is

$$1-(p(\lambda_1-\varepsilon)+(\mu-\varepsilon))h+O(h).$$

(4) When the system is in slower rate of feedback arrivals, the probability of one feedback arrival and no service completion during a small interval of time h is

$$(p(\lambda_1-\varepsilon))h+O(h).$$

(5) When the system is in faster or slower rate of feedback arrivals, the probability of no feedback arrivals and one service completion during a small interval of time h is

$$(\mu-\varepsilon)h+O(h).$$

(6) When the system is in faster or slower rate of feedback arrivals, the probability of one feedback arrivals and one service completion during a small interval of time h is

$$(p(\lambda_0-\varepsilon)+p(\lambda_1-\varepsilon)+(\mu-\varepsilon))h+O(h).$$

4. Steady State Equations

Here, only $P_n(0)$ exists when $n=0,1,2,\dots r-1,r$; both $P_n(0)$ and $P_n(1)$ exist when $n=r+1,r+2,\dots R-2,R-1$; only $P_n(1)$ exists when $n=R,R+1,\dots k-1,k$.

Further, $P_n(0)=P_n(1)=0$, if $n>k$.

The steady-state equations are

$$p(\lambda_0-\varepsilon)P_0(0)=(\mu-\varepsilon)P_1(0), \tag{4.1}$$

$$(p(\lambda_0-\varepsilon)+(\mu-\varepsilon))P_n(0)=p(\lambda_0-\varepsilon)P_{n-1}(0)+(\mu-\varepsilon)P_{n+1}(0),\quad n=1,2,3,\dots r-1, \tag{4.2}$$

$$(p(\lambda_0-\varepsilon)+(\mu-\varepsilon))P_r(0)=p(\lambda_0-\varepsilon)P_{r-1}(0)+(\mu-\varepsilon)P_{r+1}(0)+(\mu-\varepsilon)P_{r+1}(1), \tag{4.3}$$

$$(p(\lambda_0-\varepsilon)+(\mu-\varepsilon))P_n(0)=p(\lambda_0-\varepsilon)P_{n-1}(0)+(\mu-\varepsilon)P_{n+1}(0),\quad n=r+1,r+2,\dots R-2, \tag{4.4}$$

$$(p(\lambda_0-\varepsilon)+(\mu-\varepsilon))P_{R-1}(0)=p(\lambda_0-\varepsilon)P_{R-2}(0), \tag{4.5}$$

$$(p(\lambda_1-\varepsilon)+(\mu-\varepsilon))P_{r+1}(1)=(\mu-\varepsilon)P_{r+2}(1), \tag{4.6}$$

$$(p(\lambda_1-\varepsilon)+(\mu-\varepsilon))P_n(1)=p(\lambda_1-\varepsilon)P_{n-1}(1)+(\mu-\varepsilon)P_{n+1}(1),\quad n=r+2,r+3,\dots R-2,R-1, \tag{4.7}$$

$$(p(\lambda_1-\varepsilon)+(\mu-\varepsilon))P_R(1)=p(\lambda_1-\varepsilon)P_{R-1}(1)+(\mu-\varepsilon)P_{R+1}(1)+p(\lambda_0-\varepsilon)P_{R-1}(0), \tag{4.8}$$

$$(p(\lambda_1-\varepsilon)+(\mu-\varepsilon))P_n(1)=p(\lambda_1-\varepsilon)P_{n-1}(1)+(\mu-\varepsilon)P_{n+1}(1),\ n=R+1,R+2,\dots k-1, \tag{4.9}$$

$$(p(\lambda_1-\varepsilon))P_{k-1}(1)=(\mu-\varepsilon)P_k(1). \tag{4.10}$$

Let $\rho_0=\frac{p(\lambda_0-\varepsilon)}{\mu-\varepsilon}$ and $\rho_1=\frac{p(\lambda_1-\varepsilon)}{\mu-\varepsilon}$ where ρ_0 and ρ_1 denote the faster and slower rate of feedback arrivals.

Using the results (4.1) and (4.2), we get

$$P_n(0)=\begin{cases}(\rho_0)^nP_0(0), & \text{if } \rho_0\neq 1,\\ P_0(0), & \text{if } \rho_0=1,\end{cases}\quad n=0,1,2,\dots r. \tag{4.11}$$

Using the results (4.3) and (4.4), we get

$$P_n(0)=\begin{cases}(\rho_0)^nP_0(0)-\frac{1-(\rho_0)^{n-r}}{1-\rho_0}P_{r+1}(1), & \text{if } \rho_0\neq 1,\\ P_0(0)-\left(\frac{R-r-1}{2}\right)P_0(0), & \text{if } \rho_0=1,\end{cases}\quad n=r+1,r+2,\dots R-1. \tag{4.12}$$

Using the results (4.5), we get

$$P_{r+1}(1)=\begin{cases}\frac{(1-\rho_0)(\rho_0)^{R+r}}{(\rho_0)^r-(\rho_0)^R}P_0(0), & \text{if } \rho_0\neq 1,\\ \frac{1}{R-r}P_0(0), & \text{if } \rho_0=1.\end{cases} \tag{4.13}$$

Using the results (4.6), (4.7) and (4.8), we get

$$P_n(1)=\begin{cases}\frac{1-(\rho_1)^{n-r}}{1-\rho_1}P_{r+1}(1), & \text{if } \rho_0\neq 1, \rho_1\neq 1,\\ \left(\frac{R-r-1}{2}\right)P_0(0), & \text{if } \rho_0=1, \rho_1=1,\end{cases} \quad n=r+1,r+2,\dots R-1 \tag{4.14}$$

Using the results (4.9) and (4.10), we get

$$P_n(1)=\begin{cases}\frac{(\rho_1)^{n-R}(1-(\rho_1)^{R-r})}{1-\rho_1}P_{r+1}(1), & \text{if } \rho_0\neq 1, \rho_1\neq 1,\\ P_0(0), & \text{if } \rho_0=1, \rho_1=1,\end{cases} \quad n=R,R+1,\dots k. \tag{4.15}$$

5. Characteristics of the Model

In this section, several analytical expression for the system characteristics are derived.

Thus, from the results (4.11) to (4.15), we found that all the steady state probabilities are expressed in terms of $P_0(0)$.

$P(0)$ denotes that the system is in faster rate of feedback arrivals

$$P(0)=\sum_{n=0}^{r}P_n(0)+\sum_{n=r+1}^{R-1}P_n(0). \tag{5.1}$$

Using the results (4.11) and (4.12), we get

$$P(0)=\begin{cases}\frac{1-(\rho_0)^R}{1-\rho_0}P_0(0)-\left(\frac{1-2\rho_0+(\rho_0)^{R-r}}{1-\rho_0}\right)\left(\frac{(\rho_0)^{R+r}}{(\rho_0)^r-(\rho_0)^R}\right)P_0(0), & \text{if } \rho_0\neq 1,\\ 2P_0(0)-\frac{R-r-1}{2}P_0(0), & \text{if } \rho_0=1.\end{cases} \tag{5.2}$$

$P(1)$ denotes that the system is in slower rate of feedback arrivals

$$P(1)=\sum_{n=r+1}^{R-1}P_n(1)+\sum_{n=R}^{k}P_n(1). \tag{5.3}$$

Using the results (4.13), (4.14) and (4.15), we get

$$P(1)=\begin{cases}\left(\frac{1-2\rho_1+(\rho_1)^{R-r}}{(1-\rho_1)^2}\right)P_{r+1}(1)+\left(\frac{(1-(\rho_1)^{k-R+1})(1-(\rho_1)^{R-r})}{(1-\rho_1)^2}\right)P_{r+1}(1), & \text{if } \rho_0\neq 1, \rho_1\neq 1,\\ P_0(0)+\frac{R-r-1}{2}P_0(0), & \text{if } \rho_0=1, \rho_1=1.\end{cases} \tag{5.4}$$

$P_0(0)$ denotes that the system is in empty and can be calculated from the normalizing condition $P(0)+P(1)=1$.

Therefore, we get

$$P_0(0)=\begin{cases}\left(\left(\frac{1-(\rho_0)^R}{1-\rho_0}\right)-\left(\frac{1-2\rho_0+(\rho_0)^{R-r}}{1-\rho_0}\right)\left(\frac{(\rho_0)^{R+r}}{(\rho_0)^r-(\rho_0)^R}\right)+\left(\left(\frac{1-2\rho_1+(\rho_1)^{R-r}}{(1-\rho_1)^2}\right)\right.\right.\\ \left.\left.\quad+\left(\frac{(1-(\rho_1)^{k-R+1})(1-(\rho_1)^{R-r})}{(1-\rho_1)^2}\right)\right)\left(\frac{(1-\rho_0)(\rho_0)^{R+r}}{(\rho_0)^r-(\rho_0)^R}\right)\right)^{-1}, & \text{if } \rho_0\neq 1, \rho_1\neq 1,\\ \frac{1}{3}, & \text{if } \rho_0=1, \rho_1=1.\end{cases} \tag{5.5}$$

Expected number of customers in the system $L_{s(P_0)}$, when the system is in faster rate of feedback arrivals is

$$L_{s(P_0)}=\sum_{n=0}^{r}nP_n(0)+\sum_{n=r+1}^{R-1}nP_n(0). \tag{5.6}$$

From (5.6), (4.11) and (4.12), we get

$$L_{s(P_0)}=\begin{cases}\left(\frac{\rho_0-(R+1)(\rho_0)^{R+1}+R(\rho_0)^{R+2}}{(1-\rho_0)^2}\right)P_0(0)-\left((R-1)r+\frac{(R-r-1)(R-r)}{2}\right)\frac{P_{r+1}(1)}{1-\rho_0}\\ \quad +\left(r\left(\frac{\rho_0-(\rho_0)^{R-r}}{1-\rho_0}\right)+\left(\frac{\rho_0-(R-r+1)(\rho_0)^{R-r+1}+(R-r)(\rho_0)^{R-r+2}}{(1-\rho_0)^2}\right)\right)\frac{P_{r+1}(1)}{1-\rho_0}, & \text{if } \rho_0\neq 1,\\ \left(\frac{R(R-1)}{2}\right)P_0(0)-\left(\left(r(R-1)+\frac{(R-r-1)(R-r)}{2}\right)\left(\frac{R-r+1}{2}\right)P_0(0)\right), & \text{if } \rho_0=1.\end{cases} \tag{5.7}$$

Expected number of customers in the system $L_{s(P_1)}$, when the system is in slower rate of feedback arrivals is

$$L_{s(P_1)}=\sum_{n=r+1}^{R-1}nP_n(1)+\sum_{n=R}^{k}nP_n(1). \tag{5.8}$$

From (5.8), (4.13), (4.14) and (4.15), we get

$$L_{s(P_1)}=\begin{cases}\left(\left((R-1)r+\frac{(R-r-1)(R-r)}{2}\right)-\left(r\left(\frac{\rho_1-(\rho_1)^{R-r}}{1-\rho_1}\right)+\left(\frac{\rho_1-(R-r+1)(\rho_1)^{R-r+1}+(R-r)(\rho_1)^{R-r+2}}{(1-\rho_1)^2}\right)\right)\right)\\ \quad\cdot\frac{P_{r+1}(1)}{1-\rho_1}+\left(R\left(\frac{1-(\rho_1)^{k-R+1}}{1-\rho_1}\right)+\left(\frac{\rho_1-(k-R+1)(\rho_1)^{k-R+1}+(k-R)(\rho_1)^{k-R+2}}{(1-\rho_1)^2}\right)\right)\\ \quad\cdot\left(\frac{1-(\rho_1)^{R-r}}{1-\rho_1}\right)P_{r+1}(1), & \text{if } \rho_0\neq 1,\rho_1\neq 1,\\ \left((R-1)r+\frac{(R-r-1)(R-r)}{2}\right)\left(\frac{R-r-1}{2}\right)P_0(0)+\left(kR+\frac{(k-R)(k-R+1)}{2}\right)P_0(0), & \text{if } \rho_0=1,\rho_1=1,\end{cases} \tag{5.9}$$

$$W_{s(P_0)}=\frac{L_{s(P_0)}}{p(\lambda_0-\varepsilon)}\quad\text{and}\quad W_{s(P_1)}=\frac{L_{s(P_1)}}{p(\lambda_1-\varepsilon)}\qquad\text{(by using Little's formula).}$$

6. Analysis of Variance and Standard Deviation

Expected number of customers at the first moment $E(N_{(P_0)})$, when the system is in faster rate of feedback arrivals is

$$E(N_{(P_0)})=\sum_{n=0}^{r}nP_n(0)+\sum_{n=r+1}^{R-1}nP_n(0). \tag{6.1}$$

Using the results (4.11), (4.12) and (6.1), we get

$$E(N_{(P_0)})=\begin{cases}\left(\frac{\rho_0-(R+1)(\rho_0)^{R+1}+R(\rho_0)^{R+2}}{(1-\rho_0)^2}\right)P_0(0)-\left((R-1)r+\frac{(R-r-1)(R-r)}{2}\right)\frac{P_{r+1}(1)}{1-\rho_0}\\ \quad +\left(r\left(\frac{\rho_0-(\rho_0)^{R-r}}{1-\rho_0}\right)+\left(\frac{\rho_0-(R-r+1)(\rho_0)^{R-r+1}+(R-r)(\rho_0)^{R-r+2}}{(1-\rho_0)^2}\right)\right)\frac{P_{r+1}(1)}{1-\rho_0}, & \text{if } \rho_0\neq 1,\\ \left(\frac{R(R-1)}{2}\right)P_0(0)-\left(\left(r(R-1)+\frac{(R-r-1)(R-r)}{2}\right)\left(\frac{R-r-1}{2}\right)P_0(0)\right), & \text{if } \rho_0=1.\end{cases} \tag{6.2}$$

Expected number of customers at the first moment $E(N_{(P_1)})$, when the system is in slower rate of feedback arrivals is

$$E(N_{(P_1)})=\sum_{n=r+1}^{R-1}nP_n(1)+\sum_{n=R}^{k}nP_n(1). \tag{6.3}$$

Using the results (4.13), (4.14), (4.15) and (6.3), we get

$$E(N_{(P_1)})=\begin{cases}\left(\left((R-1)r+\frac{(R-r-1)(R-r)}{2}\right)-\left(r\left(\frac{\rho_1-(\rho_1)^{R-r}}{1-\rho_1}\right)+\left(\frac{\rho_1-(R-r+1)(\rho_1)^{R-r+1}+(R-r)(\rho_1)^{R-r+2}}{(1-\rho_1)^2}\right)\right)\right)\\ \quad\cdot\frac{P_{r+1}(1)}{1-\rho_1}+\left(R\left(\frac{1-(\rho_1)^{k-R+1}}{1-\rho_1}\right)+\left(\frac{\rho_1-(k-R+1)(\rho_1)^{k-R+1}+(k-R)(\rho_1)^{k-R+2}}{(1-\rho_1)^2}\right)\right)\\ \quad\cdot\left(\frac{1-(\rho_1)^{R-r}}{1-\rho_1}\right)P_{r+1}(1), & \text{if } \rho_0\neq 1,\rho_1\neq 1,\\ \left((R-1)r+\frac{(R-r-1)(R-r)}{2}\right)\left(\frac{R-r-1}{2}\right)P_0(0)+\left(kR+\frac{(k-R)(k-R+1)}{2}\right)P_0(0), & \text{if } \rho_0=1,\rho_1=1.\end{cases} \tag{6.4}$$

Expected number of customers at the second moment $E(N^2_{(P_0)})$, when the system is in faster rate of feedback arrivals is

$$E(N^2_{(P_0)}) = \sum_{n=0}^{r} n(n-1)P_n(0) + \sum_{n=r+1}^{R-1} n(n-1)P_n(0) + E(N_{(P_0)}). \tag{6.5}$$

Using the results (4.11), (4.12) and (6.5), we get

$$E(N^2_{(P_0)}) = \begin{cases} \left(\frac{\rho_0(1+\rho_0)-R^2(\rho_0)^R(1-\rho_0)^3}{(1-\rho_0)^3}\right)P_0(0) \\ \quad -\left(\left(\frac{R(R-1)(2R-1)}{6}\right)\left(\frac{1-2\rho_0+(\rho_0)^{R-r}}{(1-\rho_0)^2}\right)\left(\frac{(1-\rho_0)(\rho_0)^{R+r}}{(\rho_0)^r-(\rho_0)^R}\right)P_0(0)\right), & \text{if } \rho_0 \neq 1, \\ \left(\frac{R(R-1)(2R-1)}{6}\right)P_0(0) - \left(\frac{R(R-1)(2R-1)}{6}\right)\left(\frac{R-r-1}{2}\right)P_0(0), & \text{if } \rho_0 = 1. \end{cases} \tag{6.6}$$

Expected number of customers at the second moment $E(N^2_{(P_1)})$, when the system is in slower rate of feedback arrivals is

$$E(N^2_{(P_1)}) = \sum_{n=r+1}^{R-1} n(n-1)P_n(1) + \sum_{n=R}^{k} n(n-1)P_n(1) + E\left(N_{(P_1)}\right). \tag{6.7}$$

Using the results (4.13), (4.14), (4.15) and (6.7), we get

$$E(N^2_{(P_1)}) = \begin{cases} \left(\frac{R(R-1)(2R-1)}{6}\right)\left(\frac{1-2\rho_1+(\rho_1)^{R-r}}{(1-\rho_1)^2}\right)P_{r+1}(1) \\ \quad +\left(\left(\frac{k(k+1)(2k+1)}{6}\right)\left(\frac{1-(\rho_1)^{k-R+1}}{1-\rho_1}\right)\left(\frac{1-(\rho_1)^{R-r}}{1-\rho_1}\right)P_{r+1}(1)\right), & \text{if } \rho_0 \neq 1, \rho_1 \neq 1, \\ \left(\frac{R(R-1)(2R-1)}{6}\right)\left(\frac{R-r-1}{2}\right)P_0(0) + \left(\frac{k(k+1)(2k+1)}{6}\right)P_0(0), & \text{if } \rho_0 = 1, \rho_1 = 1. \end{cases} \tag{6.8}$$

Variance of faster rate of feedback arrivals is

$$\text{Var}(N_{(P_0)}) = E(N^2_{(P_0)}) - (E(N_{(P_0)}))^2, \tag{6.9}$$

$$\text{Var}(N_{(P_0)}) = \begin{cases} A_2 - (A_1)^2, & \text{if } \rho_0 \neq 1, \\ A_4 - (A_3)^2, & \text{if } \rho_0 = 1, \end{cases} \tag{6.10}$$

where A_1, A_2, A_3, A_4 are

$$A_1 = \left(\left(\frac{\rho_0 - (R+1)(\rho_0)^{R+1} + R(\rho_0)^{R+2}}{(1-\rho_0)^2}\right)P_0(0) - \left((R-1)r + \frac{(R-r-1)(R-r)}{2}\right)\frac{P_{r+1}(1)}{1-\rho_0}\right.$$
$$\left. + \left(r\left(\frac{\rho_0-(\rho_0)^{R-r}}{1-\rho_0}\right) + \left(\frac{\rho_0-(R-r+1)(\rho_0)^{R-r+1}+(R-r)(\rho_0)^{R-r+2}}{(1-\rho_0)^2}\right)\right)\frac{P_{r+1}(1)}{1-\rho_0}\right),$$

$$A_2 = \left(\left(\frac{\rho_0\left(1+\rho_0\right) - R^2(\rho_0)^R(1-\rho_0)^3}{(1-\rho_0)^3}\right)P_0(0)\right.$$
$$\left. - \left(\left(\frac{R(R-1)(2R-1)}{6}\right)\left(\frac{1-2\rho_0+(\rho_0)^{R-r}}{(1-\rho_0)^2}\right)\left(\frac{(1-\rho_0)(\rho_0)^{R+r}}{(\rho_0)^r-(\rho_0)^R}\right)P_0(0)\right)\right),$$

$$A_3 = \left(\left(\frac{R(R-1)}{2}\right)P_0(0) - \left(\left(r(R-1) + \frac{(R-r-1)(R-r)}{2}\right)\left(\frac{R-r-1}{2}\right)P_0(0)\right)\right),$$

$$A_4 = \left(\left(\frac{R(R-1)(2R-1)}{6}\right)P_0(0) - \left(\frac{R(R-1)(2R-1)}{6}\right)\left(\frac{R-r-1}{2}\right)P_0(0)\right).$$

Standard deviation of Faster rate of Feedback arrivals is $\sqrt{\text{Var}(N_{(P_0)})}$.

Variance of slower rate of feedback arrivals is

$$\text{Var}(N_{(P_1)}) = E(N^2_{(P_1)}) - (E(N_{(P_1)}))^2, \tag{6.11}$$

$$\mathrm{Var}(N_{(P_1)}) = \begin{cases} A_6 - (A_5)^2, & \text{if } \rho_0 \neq 1, \rho_1 \neq 1, \\ A_8 - (A_7)^2, & \text{if } \rho_0 = 1, \rho_1 = 1, \end{cases} \tag{6.12}$$

where A_5, A_6, A_7, A_8 are

$$A_5 = \left(\left((R-1)r + \frac{(R-r-1)(R-r)}{2} \right) - \left(r \left(\frac{\rho_1 - (\rho_1)^{R-r}}{1-\rho_1} \right. \right. \right.$$
$$\left. \left. \left. + \left(\frac{\rho_1 - (R-r+1)(\rho_1)^{R-r+1} + (R-r)(\rho_1)^{R-r+2}}{(1-\rho_1)^2} \right) \right) \right) \frac{P_{r+1}(1)}{1-\rho_1}$$
$$\left. + \left(R \left(\frac{1-(\rho_1)^{k-R+1}}{1-\rho_1} \right) + \left(\frac{\rho_1 - (k-R+1)(\rho_1)^{k-R+1} + (k-R)(\rho_1)^{k-R+2}}{(1-\rho_1)^2} \right) \right) \left(\frac{1-(\rho_1)^{R-r}}{1-\rho_1} \right) P_{r+1}(1) \right),$$

$$A_6 = \left(\left(\frac{R(R-1)(2R-1)}{6} \right) \left(\frac{1-2\rho_1 + (\rho_1)^{R-r}}{(1-\rho_1)^2} \right) P_{r+1}(1) \right.$$
$$\left. + \left(\left(\frac{k(k+1)(2k+1)}{6} \right) \left(\frac{1-(\rho_1)^{k-R+1}}{1-\rho_1} \right) \left(\frac{1-(\rho_1)^{R-r}}{1-\rho_1} \right) P_{r+1}(1) \right) \right),$$

$$A_7 = \left(\left((R-1)r + \frac{(R-r-1)(R-r)}{2} \right) \left(\frac{R-r-1}{2} \right) P_0(0) + \left(kR + \frac{(k-R)(k-R+1)}{2} \right) P_0(0) \right),$$

$$A_8 = \left(\left(\frac{R(R-1)(2R-1)}{6} \right) \left(\frac{R-r-1}{2} \right) P_0(0) + \left(\frac{k(k+1)(2k+1)}{6} \right) P_0(0) \right).$$

Standard deviation of slower rate of feedback arrivals is $\sqrt{\mathrm{Var}(N_{(P_1)})}$.

7. Numerical Illustrations

Table 1. $\mathrm{p} = 0.8$; $\varepsilon = 0.5$

λ_0	μ	$P_0(0)$	P(0)	$L_{s(P_0)}$	$W_{s(P_0)}$	$E(N_{(P_0)})$	$E(N^2_{(P_0)})$	$\mathrm{Var}(N_{(P_0)})$	Standard deviation
3	4	0.429	0.996	1.226	0.613	1.226	4.824	3.321	1.822
4	5	0.378	0.993	1.407	0.503	1.407	7.188	5.208	2.282
5	6	0.344	0.988	1.527	0.424	1.527	9.619	7.287	2.699
6	7	0.323	0.985	1.587	0.361	1.587	11.724	9.205	3.034
7	8	0.308	0.984	1.642	0.316	1.642	13.598	10.901	3.302
8	9	0.294	0.978	1.661	0.277	1.661	15.553	12.794	3.577
9	10	0.285	0.978	1.673	0.246	1.673	17.225	14.426	3.798
10	11	0.276	0.973	1.685	0.222	1.685	18.894	16.055	4.007
11	12	0.270	0.971	1.688	0.201	1.688	20.164	17.315	4.161
12	13	0.264	0.968	1.692	0.184	1.692	21.524	18.661	4.320

Table 2. $p = 0.8$; $\varepsilon = 0.5$

λ_1	μ	$P_0(0)$	P(1)	$L_{s(P_1)}$	$W_{s(P_1)}$	$E(N_{(P_1)})$	$E(N^2_{(P_1)})$	$Var(N_{(P_1)})$	Standard deviation
2	3	0.429	0.003	0.097	0.080	0.097	8.928	8.919	2.986
3	4	0.378	0.007	0.228	0.114	0.228	21.915	21.863	4.676
4	5	0.344	0.011	0.382	0.136	0.382	36.768	36.622	6.052
5	6	0.323	0.015	0.533	0.148	0.533	53.344	53.060	7.284
6	7	0.308	0.017	0.650	0.148	0.650	62.928	62.506	7.906
7	8	0.294	0.021	0.786	0.151	0.786	79.080	78.462	8.858
8	9	0.285	0.023	0.883	0.147	0.883	88.932	88.152	9.389
9	10	0.276	0.026	1.005	0.148	1.005	105.484	104.474	10.221
10	11	0.270	0.028	1.088	0.143	1.088	114.927	113.743	10.665
11	12	0.264	0.029	1.164	0.139	1.164	123.534	122.179	11.053

Table 3. $p = 0.8$; $\varepsilon = 1$

λ_0	μ	$P_0(0)$	P(0)	$L_{s(P_0)}$	$W_{s(P_0)}$	$E(N_{(P_0)})$	$E(N^2_{(P_0)})$	$Var(N_{(P_0)})$	Standard deviation
3	4	0.467	0.998	1.085	0.678	1.085	3.673	2.496	1.580
4	5	0.40	0.995	1.327	0.553	1.327	6.019	4.258	2.063
5	6	0.360	0.991	1.477	0.462	1.477	8.367	6.185	2.487
6	7	0.333	0.987	1.565	0.391	1.565	10.657	8.208	2.865
7	8	0.314	0.983	1.618	0.337	1.618	12.804	10.186	3.192
8	9	0.30	0.980	1.645	0.294	1.645	14.578	11.872	3.446
9	10	0.289	0.977	1.669	0.261	1.669	16.455	13.669	3.697
10	11	0.281	0.977	1.687	0.234	1.687	18.073	15.227	3.902
11	12	0.274	0.975	1.687	0.211	1.687	19.459	16.613	4.076
12	13	0.268	0.973	1.692	0.192	1.692	20.775	17.912	4.232

Table 4. p = 0.8; $\varepsilon = 1$

λ_1	μ	$P_0(0)$	P(1)	$L_{s(P_1)}$	$W_{s(P_1)}$	$E(N_{(P_1)})$	$E(N^2_{(P_1)})$	$\text{Var}(N_{(P_1)})$	Standard deviation
2	3	0.467	0.0014	0.059	0.074	0.059	3.388	3.385	1.840
3	4	0.40	0.005	0.173	0.108	0.173	15.058	15.028	3.877
4	5	0.360	0.009	0.315	0.131	0.315	29.283	29.184	5.402
5	6	0.333	0.013	0.465	0.145	0.465	44.857	44.641	6.681
6	7	0.314	0.017	0.612	0.153	0.612	61.290	60.915	7.805
7	8	0.30	0.019	0.711	0.148	0.711	70.975	70.469	8.395
8	9	0.289	0.022	0.850	0.152	0.850	87.623	86.901	9.322
9	10	0.281	0.024	0.946	0.148	0.946	97.295	96.40	9.818
10	11	0.274	0.026	1.028	0.143	1.028	106.744	105.687	10.280
11	12	0.268	0.028	1.119	0.140	1.119	115.799	114.547	10.703

Table 5. $p = 0.85$; $\varepsilon = 0.5$

λ_0	μ	$P_0(0)$	P(0)	$L_{s(P_0)}$	$W_{s(P_0)}$	$E(N_{(P_0)})$	$E(N^2_{(P_0)})$	$\text{Var}(N_{(P_0)})$	Standard deviation
3	4	0.393	0.994	1.369	0.644	1.369	6.335	4.461	2.112
4	5	0.340	0.991	1.554	0.522	1.554	10.032	7.617	2.760
5	6	0.306	0.984	1.646	0.430	1.646	14.036	11.327	3.366
6	7	0.282	0.978	1.679	0.359	1.679	17.955	15.136	3.891
7	8	0.264	0.972	1.699	0.308	1.699	21.795	18.908	4.348
8	9	0.251	0.967	1.690	0.265	1.690	25.083	22.227	4.715
9	10	0.240	0.962	1.686	0.233	1.686	28.704	25.861	5.085
10	11	0.232	0.958	1.668	0.207	1.668	31.595	28.813	5.368
11	12	0.225	0.955	1.654	0.185	1.654	34.454	31.718	5.632
12	13	0.220	0.956	1.641	0.168	1.641	37.025	34.332	5.859

Table 6. p = 0.85; $\varepsilon = 0.5$

λ_1	μ	$P_0(0)$	P(1)	$L_{s(P_1)}$	$W_{s(P_1)}$	$E(N_{(P_1)})$	$E(N^2_{(P_1)})$	$Var(N_{(P_1)})$	Standard deviation
2	3	0.393	0.005	0.166	0.130	0.166	13.619	13.591	3.687
3	4	0.340	0.010	0.386	0.182	0.386	34.147	33.998	5.831
4	5	0.306	0.018	0.643	0.216	0.643	64.602	64.189	8.012
5	6	0.282	0.023	0.887	0.232	0.887	90.478	89.691	9.471
6	7	0.264	0.029	1.129	0.241	1.129	116.983	115.708	10.757
7	8	0.251	0.033	1.312	0.237	1.312	136.845	135.124	11.624
8	9	0.240	0.038	1.526	0.239	1.526	163.033	160.704	12.677
9	10	0.232	0.041	1.693	0.234	1.693	182.678	179.812	13.409
10	11	0.225	0.044	1.841	0.228	1.841	200.791	197.402	14.050
11	12	0.220	0.046	1.959	0.219	1.959	211.869	208.031	14.423

Table 7. p = 0.85; $\varepsilon = 1$

λ_0	μ	$P_0(0)$	P(0)	$L_{s(P_0)}$	$W_{s(P_0)}$	$E(N_{(P_0)})$	$E(N^2_{(P_0)})$	$Var(N_{(P_0)})$	Standard deviation
3	4	0.433	0.997	1.205	0.709	1.205	4.689	3.237	1.799
4	5	0.362	0.991	1.479	0.580	1.479	8.241	6.054	2.460
5	6	0.321	0.987	1.601	0.471	1.601	12.046	9.483	3.079
6	7	0.293	0.981	1.672	0.393	1.672	15.870	13.074	3.616
7	8	0.272	0.975	1.695	0.332	1.695	19.799	16.926	4.114
8	9	0.257	0.969	1.698	0.285	1.698	23.481	20.598	4.539
9	10	0.245	0.964	1.688	0.248	1.688	27.112	24.263	4.926
10	11	0.236	0.960	1.676	0.219	1.676	30.115	27.306	5.226
11	12	0.228	0.956	1.661	0.195	1.661	33.150	30.391	5.513
12	13	0.222	0.953	1.647	0.176	1.647	35.808	33.095	5.753

Table 8. p = 0.85; $\varepsilon = 1$

λ_1	μ	$P_0(0)$	P(1)	$L_{s(P_1)}$	$W_{s(P_1)}$	$E(N_{(P_1)})$	$E(N^2_{(P_1)})$	$Var(N_{(P_1)})$	Standard deviation
2	3	0.433	0.003	0.093	0.109	0.093	7.534	7.525	2.743
3	4	0.362	0.008	0.271	0.159	0.271	25.824	25.751	5.075
4	5	0.321	0.014	0.515	0.202	0.515	48.983	48.718	6.980
5	6	0.293	0.020	0.753	0.221	0.753	74.172	73.605	8.579
6	7	0.272	0.025	0.998	0.235	0.998	99.970	98.974	9.949
7	8	0.257	0.031	1.227	0.241	1.227	127.105	125.599	11.207
8	9	0.245	0.036	1.437	0.242	1.437	153.724	151.659	12.315
9	10	0.236	0.040	1.612	0.237	1.612	173.166	170.567	13.060
10	11	0.228	0.043	1.768	0.231	1.768	191.260	188.134	13.716
11	12	0.222	0.046	1.912	0.225	1.912	210.404	206.748	14.379

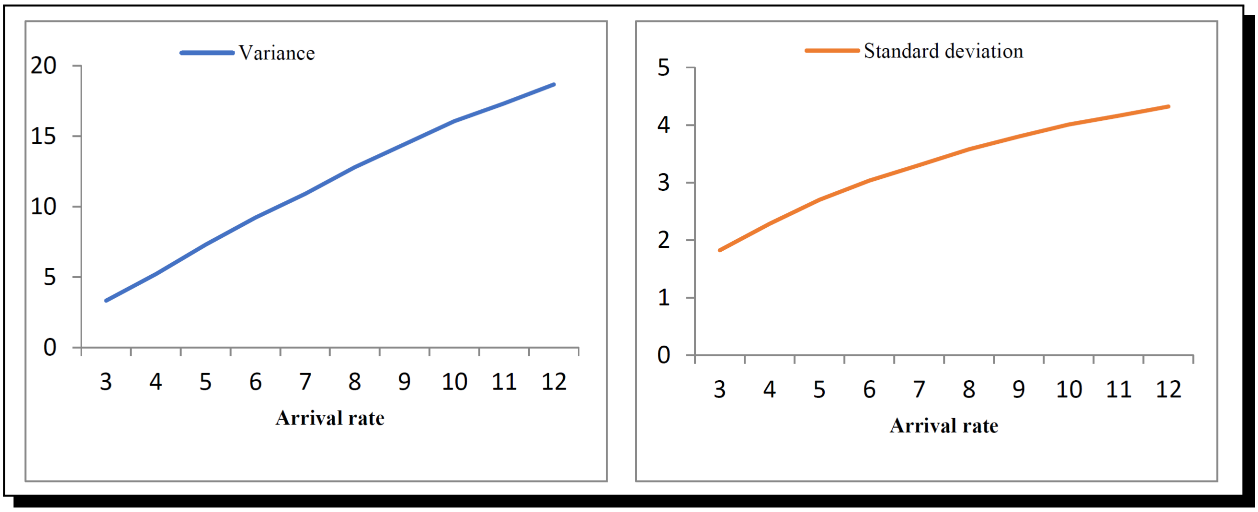

Figure 1. Faster rate of feedback arrivals when $p = 0.8$; $\varepsilon = 0.5$

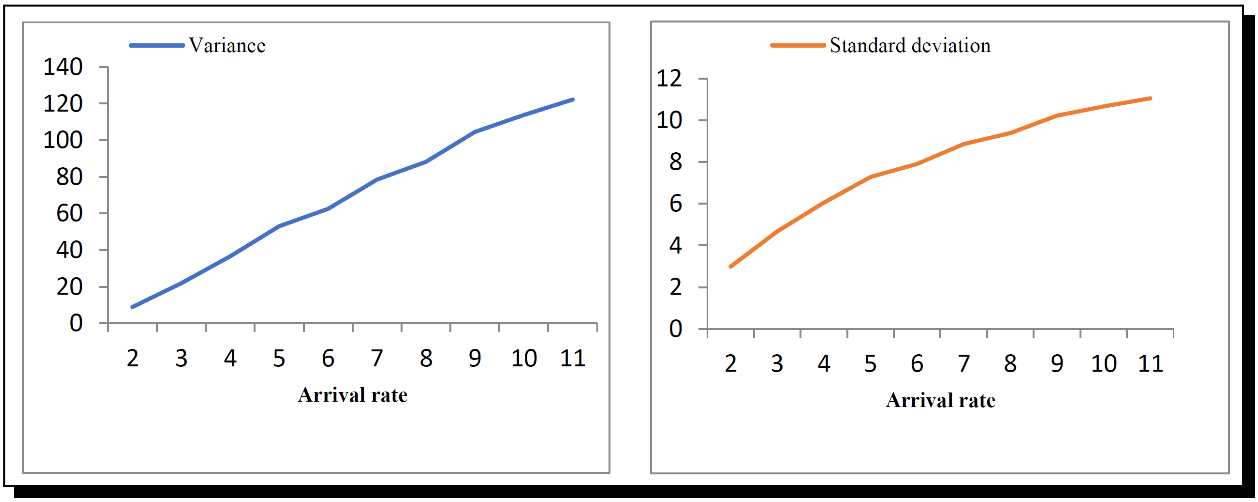

Figure 2. Slower rate of feedback arrivals when $p = 0.8$; $\varepsilon = 0.5$

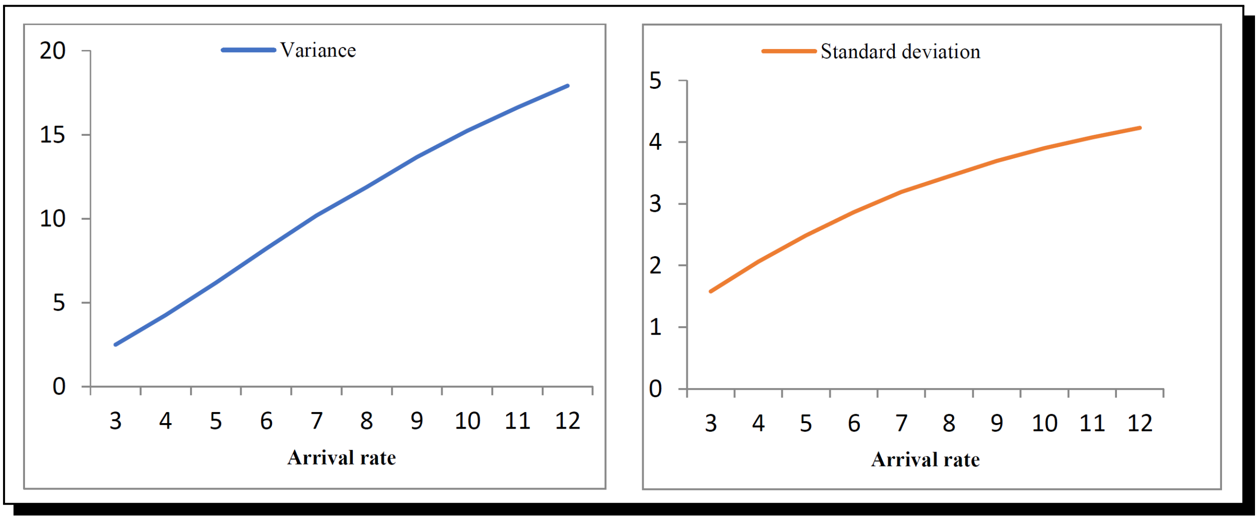

Figure 3. Faster rate of feedback arrivals when $p = 0.8$; $\varepsilon = 1$

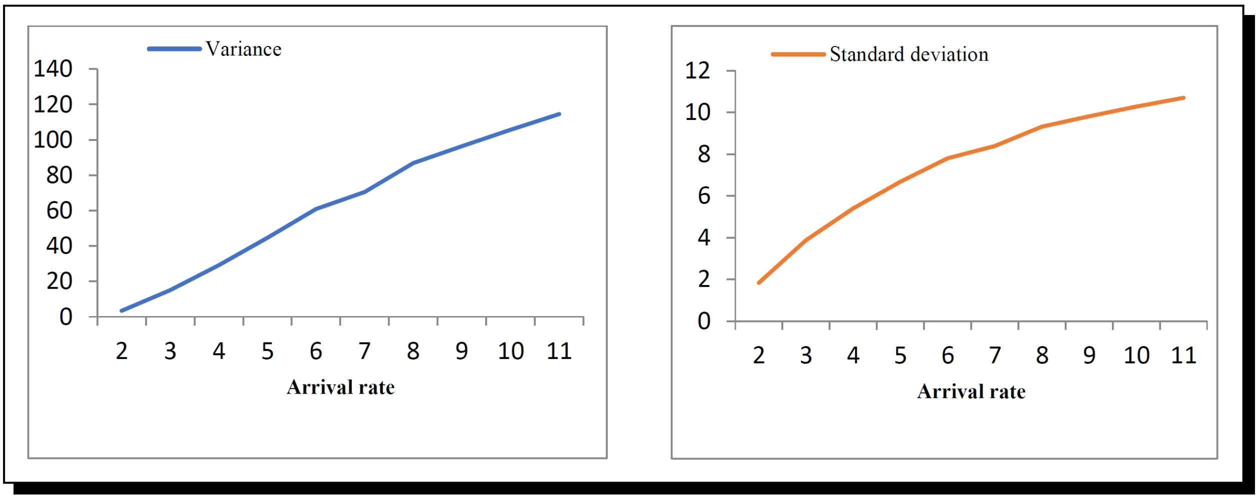

Figure 4. Slower rate of feedback arrivals when $p = 0.8$; $\varepsilon = 1$

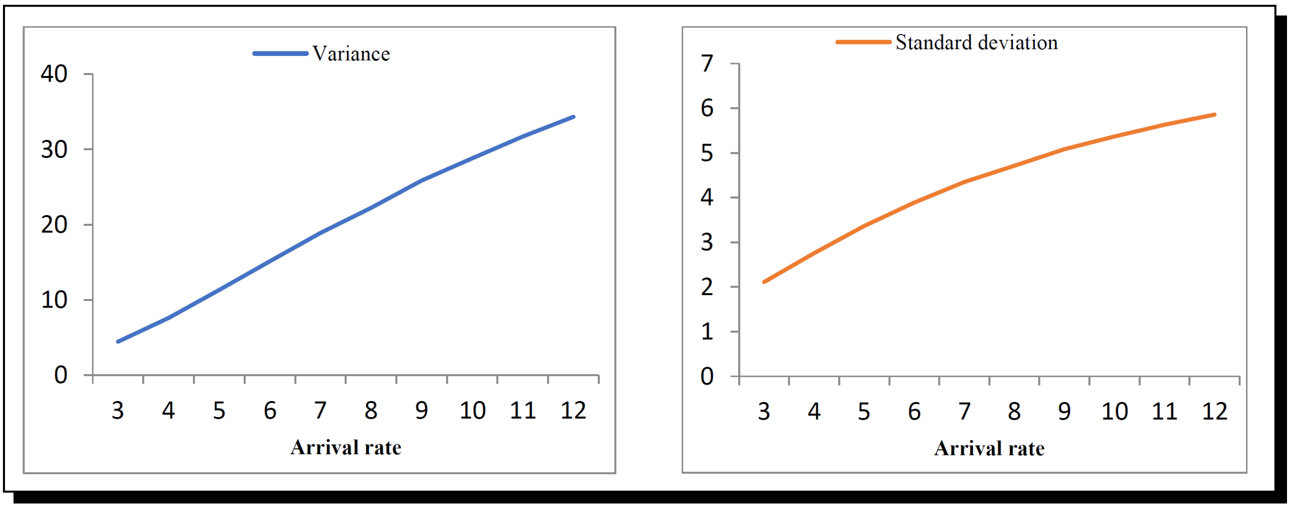

Figure 5. Faster rate of feedback arrivals when $p = 0.85$; $\varepsilon = 0.5$

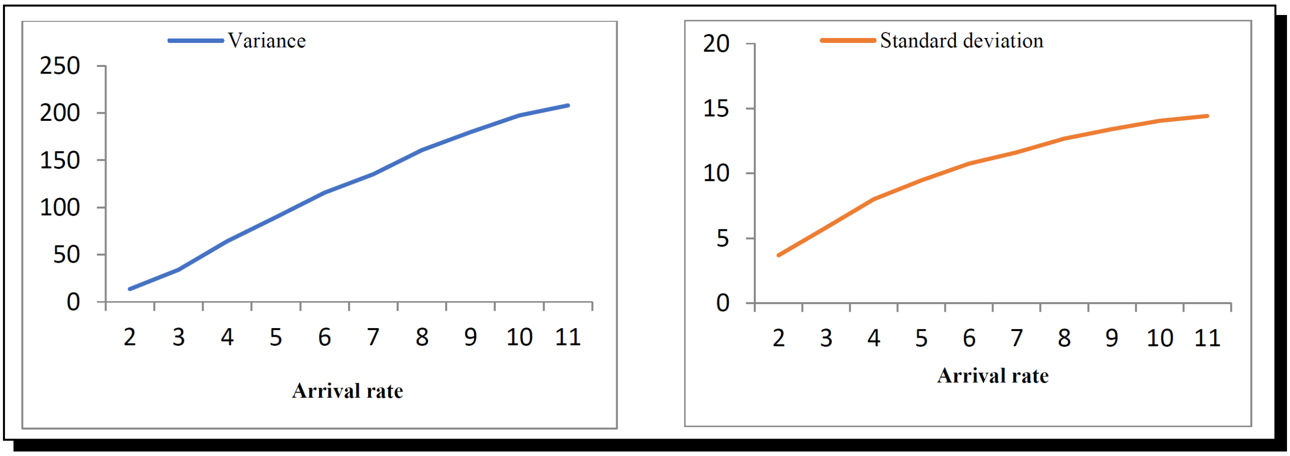

Figure 6. Slower rate of feedback arrivals when $p = 0.85$; $\varepsilon = 0.5$

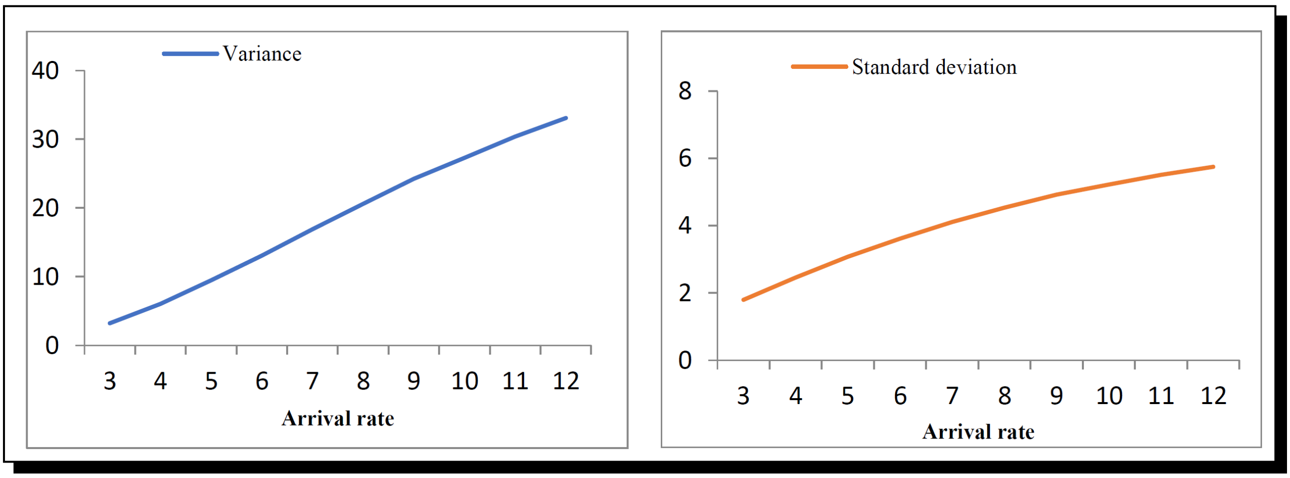

Figure 7. Faster rate of feedback arrivals when $p = 0.85$; $\varepsilon = 1$

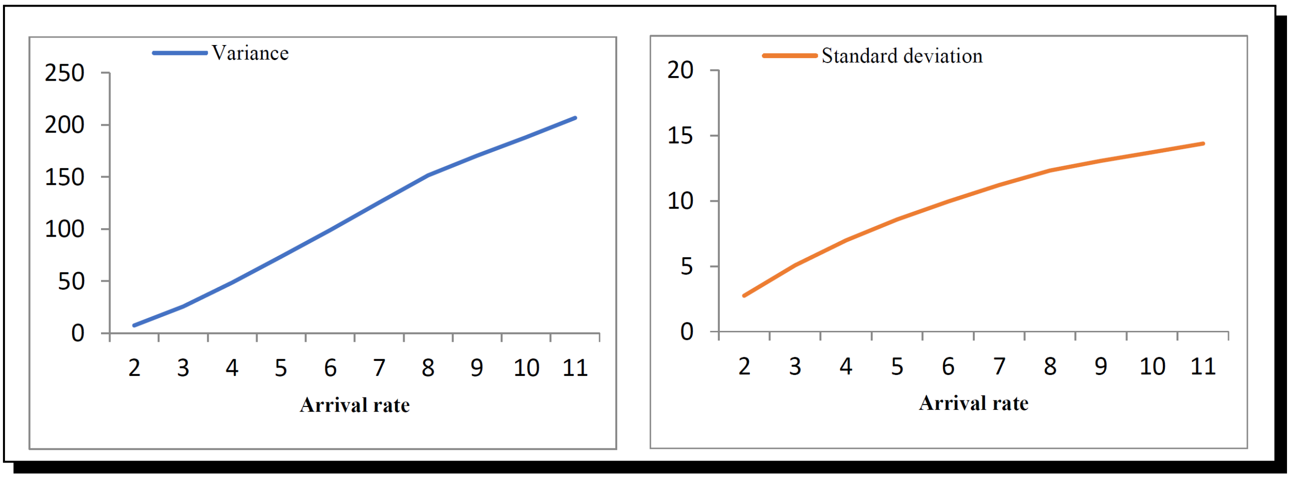

Figure 8. Slower rate of feedback arrivals when $p = 0.85$; $\varepsilon = 1$

8. Conclusion

(1) In faster rate of feedback arrivals, it is observed from the Tables 1, 3, 5 and 7, while increasing the mean dependence rate and the other parameters are kept fixed, the values of $L_{s(P_0)}$, $E(N_{(P_0)})$, $E(N^2_{(P_0)})$, $\mathrm{Var}(N_{(P_0)})$ and Standard deviation are increasing, $W_{s(P_0)}$ is decreasing.

(2) In slower rate of feedback arrivals, it is observed from the Tables 2, 4, 6 and 8, while increasing the mean dependence rate and the other parameters are kept fixed, the values of $L_{s(P_1)}$, $E(N_{(P_1)})$, $E(N^2_{(P_1)})$, $\mathrm{Var}(N_{(P_1)})$ Standard deviation and $W_{s(P_1)}$ are increasing.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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