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Research Article

Common Fixed Point Theorems of Integral Type Contraction on Cone Metric Spaces and Applications

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Abstract. In this paper, we establish the existence and uniqueness of a common fixed point for two pairs of weakly compatible mappings satisfying integral type contractive condition and property (EA) on cone metric spaces, and we prove the existence and uniqueness of a solution for an ordinary differential equation with periodic boundary conditions. The derived results generalized some well-known results that exist in the literature.

Keywords. Common fixed point, Cone metric space, Property (EA), Contractive mappings of integral type, Weakly compatible mappings

Mathematics Subject Classification (2020). Primary 54H25; Secondary 47H10

1. Introduction

The Banach contraction mapping principle [6] is truly the most popular result of the metric fixed point theory. Branciari [7] gave an integral version of the Banach contraction principles and proved a fixed point theorem for a single-valued contractive mapping of integral type contraction in cone metric spaces. Pant [13] initially investigated common fixed points of non-compatible mappings defined on metric spaces. Aamri and Moutawakil [1] defined a property (E.A) that generalizes the concept of non-compatible mappings and gave some common fixed point theorem under strict contractive conditions.

Long-Guang and Xian [11] reintroduced the concept of a cone metric space, replacing the set of real numbers with an ordered Banach space, further generalizations of the result were obtained by Abbas and Jungck [2]. Recently, Rezapour and Hamlbarani [14] generalized the result of Long-Guang and Xian [11] by omitting the assumption of normality. In 2019, Meng [12] also proved some new "fixed point theorems" concerning generalized algebraic cone metric spaces. Bakr [5] introduces the concept of theta cone metric spaces, and gives further generalizations of some well-known fixed point theorems in 2020.

The main purpose of this paper is to give common fixed point theorems for mapping satisfying integral type contractive condition and property (E.A) on cone metric spaces. As an application, we show the existence and uniqueness of the solution for a first order ordinary differential equation with periodic boundary conditions. Our result is inspired by the result of Cho and Bae [8].

2. Preliminaries

Following definitions and properties required in the sequel:

Definition 2.1 ([11])**.** For a non-empty set X and real Banach space E and $d : X \times X \to E$ be a mapping fulfills all the conditions of metric space. Thus d is called a cone metric space as define by Long-Guang and Xian [11].

Definition 2.2 ([11])**.** For a non-empty set X and the mapping $d : X \times X \to E$, a sequence $\{x_n\}$ is called

(i) convergent sequence for $c \gg 0$ there exist an positive integer n_0 so as to $d(x_n, x) \ll c$ for all $n > n_0$ and for certain x in X and any c in E.

(ii) A Cauchy sequence if for any $c \gg 0$, there is n_0 be a natural number, so as to $d(x_n, x_m) \ll c$ for all $n, m > n_0$, for some x in X and every c in E.

For the completeness of cone metric space (X, d) every Cauchy sequence should be convergent in X.

Definition 2.3 ([11])**.** In a cone metric space (X, d) the point $z \in X$ is called the point of coincidence for the mappings $f, g : X \to X$ if $w = fz = gz$.

Definition 2.4 ([11])**.** Suppose f and g be two self-mappings on cone metric space X, then for weak compatibility of f, g the required condition is $fgz = gfz$ when fz is equal to gz for certain z in X.

Definition 2.5 ([11])**.** In a cone metric space (X, d), the pair (S, T) fulfills property (E.A) if $\exists$ $\{x_n\}$ in X and a point t in X so as to $\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0$.

Lemma 2.6 ([11])**.** *Suppose that φ in Φ and $\{r_n\}$ (where n is natural number) with $r_n = a$ (as $\lim_{n\to\infty}$). Then $\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt$.*

Remark 2.1 ([8]). Let P,Q,R and S be four self-mappings in X, such that $P(X)$ is the sub set of $S(X)$ and $Q(X)$ is the subset of $R(X)$, and (Q,S) satisfies "property (E.A)" , and $S(X)$ or $B(X)$ is complete subspace of X. Then (A,B,S,T) is BC's 4-tuple.

Long-Guang and Xian [11] prove the following:

Example 2.1. Let $E=R^2$, $P=\{(x,y)\in E: x,y\geq 0\}\subset R^2$, $X=R^2$ and $d: X\times X\to E$ defined by $d(x,y)=d((x_1,x_2),(y_1,y_2))=(\max\{|x_1-y_1|,|x_2-y_2|,\ \alpha\max|x_1-y_1|,|x_2-y_2|\})$, where $\alpha\geq 0$ is a constant. Then (X,d) is a cone metric space.

3. Main Result

In this section, we give a result related to common fixed point theorems for BC's 4-tuple.

Theorem 3.1. *Supposing that K,M,N,L is a BC's 4-tuple sustaining*

$$\int_0^{d(Kx,Ly)}\phi(t)dt\leq\psi\int_0^{\Delta_1(x,y)}\phi(t)dt, \tag{3.1}$$

where $(\phi,\psi)\in\Phi\times\Psi$ and

$$\Delta_1(x,y)=k_1d(Nx,My)+k_2d(Nx,Ly)+k_3d(My,Ly) \tag{3.2}$$

for each $x,y\in X$ where $k_1,k_2,k_3\geq 0$ and $k_1+k_2+k_3<1$. Then (K,N) and (L,M) have a point of coincidence in X. As well as if they have weak compatibility, then K,M,N,L have a unique common fixed point in X.

Proof. Since (L,M) fulfills (E.A) property, thus there exist $\{x_n\}$ in X and $t\in X$ so as to $Lx_n=Mx_n=t$ (as $\lim\limits_{n\to\infty}$). Since $L(X)\subset N(X)$, there exist a sequence $\{y_n\}$ in X such that $Lx_n=Ny_n$. Hence $\lim\limits_{n\to\infty}Ny_n=t$.
By the completeness of $N(X)$ in X $\exists$ u in X so as to $t=Nu$.
Thus $d(Ny_n,Nu)=d(Lx_n,Nu)=d(Mx_n,Nu)=0$ (as $\lim\limits_{n\to\infty}$).
Putting $x=u$ and $y=x_n$ in contractive conditions of main result thus,

$$\int_0^{d(Ku,Lx_n)}\phi(t)dt\leq\psi\int_0^{\Delta_1(u,x_n)}\phi(t)dt, \tag{3.3}$$

where

$$\Delta_1(u,x_n)=k_1d(Nu,Mx_n)+k_2d(Nu,Lx_n)+k_3d(Mx_n,Lx_n) \tag{3.4}$$

Taking $n\to\infty$ as upper limit in both the equations, thus

$$\lim_{n\to\infty}\int_0^{d(Ku,Lx_n)}\phi(t)dt\leq\psi\lim_{n\to\infty}\int_0^{\Delta_1(u,x_n)}\phi(t)dt,$$

$$\int_0^{d(Ku,Nu)}\phi(t)dt\leq\psi\lim_{n\to\infty}\int_0^{\Delta_1(u,x_n)}\phi(t)dt,$$

$$\int_0^{d(Ku,Nu)}\phi(t)dt \text{ will be less and equal then } 0.$$

Probable only when $Ku=Nu$.

Hence

$$Ku = N. \tag{3.5}$$

Since $M(X) \supset K(X)$, then $\exists\ v$ in X so as to

$$Ku = Mv. \tag{3.6}$$

Interchanging x with u and y with v in (3.1) of Theorem 3.1

$$\int_0^{d(Ku,Lv)} \phi(t)dt \le \psi \int_0^{\Delta_1(u,v)} \phi(t)dt \tag{3.7}$$

where

$$\begin{aligned}\Delta_1(u,v) &= k_1 d(Nu,Mv) + k_2 d(Nu,Lv) + k_3 d(Mv,Lv)\\ &= k_1 d(Mv,Mv) + k_2 d(Mv,Lv) + k_3 d(Mv,Lv)\\ &= (k_2 + k_3) d(Mv,Lv).\end{aligned} \tag{3.8}$$

Since $Ku = Mv$, from (3.6), (3.7) and (3.8)

$$\int_0^{d(Mv,Lv)} \phi(t)dt \le \psi \int_0^{\Delta_1(u,v)} \phi(t)dt$$

which is possible only when $d(Mv,Lv) = 0$ (as sum of k_1, k_2, k_3 is less than 1 and all are greater and equal than zero).

Thus $Mv = Lv$.

Hence

$$Ku = Mv = Nu = Lv. \tag{3.9}$$

Since (K,N) is weakly compatible, $KNu = NKu$. Thus, we have

$$KKu = KNu = NKu = N, \tag{3.10}$$

$$MMv = MLv = LMv = LLv, \tag{3.11}$$

Put $x = Ku$ and $y = v$ in (3.1) and (3.2)

$$\int_0^{d(KKu,Lv)} \phi(t)dt \le \psi \int_0^{\Delta_1(Ku,v)} \phi(t)dt \tag{3.12}$$

where

$$\begin{aligned}\Delta_1(Ku,v) &= k_1 d(NKu,Mv) + k_2 d(NKu,Lv) + k_3 d(Mv,Lv)\\ &= (k_1 + k_2) d(KKu,Ku).\end{aligned} \tag{3.13}$$

From (3.12) and (3.13)

$$\int_0^{d(KKu,Lv)} \phi(t)dt \le \psi \int_0^{(k_1+k_2)d(KKu,Ku)} \phi(t)dt.$$

Thus, from (3.9)

$$\int_0^{d(KKu,Ku)} \phi(t)dt \le \psi \int_0^{(k_1+k_2)d(KKu,Ku)} \phi(t)dt$$

$d(KKu,Ku) = 0$ (as sum of k_1, k_2, k_3 is less than 1 and all are greater and equal than zero)

$$\Rightarrow \quad KKu = Ku$$

$$\Rightarrow \qquad Ku = Kku = Nku$$

Similarly, we have Lv is the common fixed point of L and M.
Hence $LLv = MLv = Lv$.
Since $Ku = Lv$

$$\Rightarrow \qquad KKu = NKu = MKu = LKu$$

Thus, Ku is common fixed point for all the four mappings.

Uniqueness. Suppose that another common fixed point of all four mappings is w.
Put x is equal to z and y is equal to w in (3.1) and (3.2)

$$\int_0^{d(Kz,Lw)} \phi(t)dt \le \psi \int_0^{\Delta_1(z,w)} \phi(t)dt \tag{3.14}$$

here

$$\begin{aligned}\Delta_1(z,w) &= k_1 d(Nz,Mw) + k_2 d(Nz,Lw) + k_3 d(Mw,Lw)\\ &= k_1 d(z,w) + k_2 d(z,w)\\ &= (k_1+k_2)d(z,w)\end{aligned} \tag{3.15}$$

From (3.14) and (3.15)

$$\int_0^{d(Kz,Lw)} \phi(t)dt < \int_0^{(k_1+k_2)d(z,w)} \phi(t)dt$$

$d(z,w) = 0$ (as $k_1 + k_2 + k_3 < 1$ and $k_1, k_2, k_3 \ge 0$).
Hence $z = w$. □

Corollary 3.2. *If* (L,M) *is non-compatible* (*resp.,* (K,N) *is non-compatible*) *instead* (L,M) *fulfills (E.A) property in main result, thus the similar deduction holds.*

4. Application

In this section, we will determine presence and uniqueness of a common solution for the following problem:

$$\left\{\begin{array}{l} v'(t) = h(t,v(t)),\ t \in [0,1] \\ v(0) = v(1) \end{array}\right\} \tag{4.1}$$

where h is a continuous function define on cross product of ($[0,1]$ and R) to R.

Theorem 4.1. *Suppose the problem* (4.1) *with* h *continuous and let* $\exists$ λ *and* μ *both greater than* 0 *such that* μ *is greater than* $1/3(\lambda)$ $\forall$ x, y *in* R *such that*

$$|h(t,y) + \lambda y - [h(t,x) + \lambda x]| \le \mu |y - x|. \tag{4.2}$$

Thus $\exists$ *solution of uniqueness of* (4.1)

Proof. From (4.1)

$$\left\{\begin{array}{l} v'(t) + \lambda v(t) = h(t,v(t)) + \lambda v(t),\ t \in [0,1] \\ v(0) = v(1) \end{array}\right\}$$

Convert the following problem in integral equation such that

$$v(t)=\int_0^1 F(t,s)[h(s,v(s)+\lambda v(s)]ds,$$

where $F(t,s)=\left\{\begin{array}{ll}\dfrac{e^{\lambda(1+s-t)}}{e^{\lambda}-1} & (0\le s\le t\le 1)\\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda}-1} & (0\le t\le t\le 1)\end{array}\right\}$.

We define mappings K and L such that $K:C_R^1[0,1]\to C_R^1[0,1]$ and $L:C_R^1[0,1]\to C_R^1[0,1]$ by

$$(Kv)t=\int_0^1 F(t,s)[g(s,v(s)+\lambda v(s)]d,\quad t\in[0,1],$$

$$(Lv)t=\int_0^1 F(t,s)[g(s,v(s)+\lambda v(s)]ds,\quad t\in[0,1].$$

Note that if $v\in C_0^1[0,1]$ is a fixed point of K as well as L, then $v\in C_R^1[0,1]$ is a solution of (4.1).
Let $X=E=C_R^1[0,1]$ and $\|v\|=\|v\|_\infty+\|v'\|_\infty$ with $P=\{v\in E: v$ is greater and equal to $0\}$.
Let $d:X\times X\to E$, define as $d(x,y)=(\sup_{t\in[0,1]}(|x(t)-y(t)|)f$ when f is a mapping from $[0,1]$ to R and $f(t)=e^t$. Then d is a "cone metric space" on X.
(K,L,N,M) is BC's 4-tuple satisfying property (E.A), and (K,N) and (L,M) is weakly compatible where N and M both the identity mappings.
Let

$$\begin{aligned}d(Ku,Lv)&=\left(\sup_{t\in[0,1]}|Ku(t)-Lv(t)|\right)f\quad\forall\ u,v\in X\\ &\le\mu\left(\sup_{s\in[0,1]}\int_0^1|u(s)-v(s)|\right)f\sup_{t\in[0,1]}\int_0^1 F(t,s)ds\\ &\le\mu d(u,v)\sup_{t\in[0,1]}\frac{1}{e^{\lambda}-1}\left(\frac{1}{\lambda}[e^{\lambda(1+s-t)}]_0^t+\frac{1}{\lambda}[e^{\lambda(s-t)}]_0^1\right)\\ &=\frac{\mu}{\lambda}d(Nu,Mv).\end{aligned}$$

Thus,

$$d(Ku,Lv)\le k_1d(Nu,Mv)+k_2d(Nu,Lv)+K_3d(Mv,Lv),$$

where $k_1=k_2=k_3=\frac{\mu}{\lambda}$.
Let $\Delta_1(u,v)=k_1d(Nu,Mv)+k_2d(Nu,Lv)+K_3d(Mv,Lv)$.
Then $d(Ku,Lv)\le\Delta_1(u,v)$

$$\int_0^{d(Ku,Lv)}\phi(t)dt\le\psi\int_0^{\Delta_1(u,v)}\phi(t)dt\quad\forall\ x,y\in X,$$

where $(\emptyset,\psi)\in\Phi X\times\Psi$.
Thus the mapping K,L,N,M fulfill all the conditions of our main result.
Thus $\exists$ a fixed point k of uniqueness of $K,L,N,$ and M in X.
Hence, the unique solution of (4.1) will be the fixed point k. □

5. Conclusion

We revised the result of Cho and Bae [8] and attested common fixed point theorems for new generalization contraction mapping. We believe that these outcomes will motivate others researcher through generalizations of our work in various ways, to be an appropriate tool for their applications.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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