Communications in Mathematics and Applications
Vol. 14, No. 2, pp. 937–948, 2023
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v14i2.2126
http://www.rgnpublications.com

Research Article

RF-Matrices of Arf Numerical Semigroups With Small Multiplicity

Ö. Belgin* and B. Gülşah

Departments of Mathematics, Faculty of Arts and Sciences, Gaziantep, Gaziantep University, 27310, Gaziantep, Turkey
***Corresponding author:** emirhan@gantep.edu.tr

Received: November 22, 2022 **Accepted:** March 24, 2023

Abstract. In this paper, multiplicity of Arf numerical semigroups and RF-matrices are given. We study Arf numerical semigroups with multiplicity of two, three, four, five and six with RF-matrices.

Keywords. ARF numerical semigroups, Frobenius number, Pseudo-Frobenius number, Multiplicity, RF-matrices

Mathematics Subject Classification (2020). 13H10, 20M14, 20M2

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers and $G \subseteq \mathbb{N}$. If, G is closed under the addition in $\mathbb{N}$ and $0 \in G$ and $\mathbb{N} \backslash G$ is finite then G is called a numerical semigroup. For all $n_1, n_2, \dots, n_e \in G$ it is denoted by

$$G = \langle n_1, n_2, \dots, n_e \rangle = \left\{ \sum_{i=1}^{e} a_i n_i : a_i \in \mathbb{N} \right\}$$

and

$$(n_1, n_2, \dots, n_e) = 1 \Leftrightarrow \mathbb{N} \backslash G \text{ is finite.}$$

Numerical semigroups are also called numerical monoids because numerical semigroups are commutative monoids.

The numerical semigroup concept raises problems that are easy to understand but not obvious to solve. This situation caught the attention of 19th century mathematicians Ferdinand

Frobenius and James Joseph Sylvester [13]. Historically known as the "Frobenius coin problem", the problem was originally "what is the largest amount of money that cannot be obtained using coins that do not have a common divisor?" expressed by the question. In short, "a and b are natural numbers, p and q are greater than 1 and prime between them; What is the largest integer that cannot be expressed as a linear combination $ap+bq$?" The question is known as the frobenius problem. The solution of this problem for two numbers such as p and q. It is designated as $pq-p-q$. This number is represented by $F(G)$ (Rosales and Branco [10]). For two such numbers, it is stated that exactly how many numbers can be written and unwritten as the linear combination of p and q from the numbers in the interval $[0,F(G)]$, and the proof of the theorem related to this is again done by Sylvester [13].

Numerical semigroups have gained importance again in algebra and have found application in algebraic geometry in particular.

Du Val [5] asked if there was an algebraic expression of the geometric approach he had made while presenting his article on the Jacobian algorithm and the multiplicity sequence of an algebraic branchat Istanbul University. Arf, who was attending Du Val's lecture said that the computation of Du Val's characters could be calculated by algebraic means, and after a week he showed how to do this and the results were published in [1], and later these characters were called Arf Characters of a curve. Arf's idea was to calculate what Limpan called later in [9], the Arf ring closure of the coordinate ring of the curve, and then its value semigroup (which is an Arf numerical semigroup). The minimal generators of this semigroup are the Arf characters. Today's mathematicians are interested in the properties of Arf numerical semigroups.

Arf numerical semigroups are always of maximal embedding dimension (Barucci *et al*. [3]). Having a maximal embedding size means that the smallest element in the generator and the number of elements in the generator are equal. The smallest element in the generator gives us the multiplicity of the numerical semigroup. García-Sánchez *et al*. [7] mentioned small multiplicity Arf numerical semigroups in their article.

In this article, we will talk about small multiplicity of Arf numerical semigroups and RF-matrices. Then, we will examine small multiplicity of Arf numerical semigroups via RF-matrices.

2. Basic Definitions

Most of the definitions are found in [2], [11] and [8].

Definition 2.1. Let $\mathbb{N}$ denote the set of naturals number. The number of elements of a set B will be denoted by $|B|$. A subset $G \subseteq \mathbb{N}$ satisfying

(i) $0 \in G$,

(ii) $x, y \in G \Rightarrow x+y \in G$,

(iii) $|\mathbb{N} \backslash G| < \infty$

is called a *numerical semigroup*.

Definition 2.2. The $\mathbb{N} \backslash G$ set in option (iii) we mentioned is called the *gaps* of the numerical semigroup. After that the gaps set will be denoted by $H(G)$. The number of elements of the set $H(G)$ tells us the *genus* of the numerical semigroup G and is denoted by $h(G)$.

Definition 2.3. Let G be a semigroup and $B \subset G$. For every $g \in G$, if we can write it as a linear combination of the elements of the set $B=\{b_1, b_2, \dots, b_n\}$, then the set B is called the set of *generators* of the set G and is denoted by $G = \langle B \rangle$. If G cannot be generated by any proper subset of B, then B is called a *minimal generator system*.

Definition 2.4. The number of elements of the minimal generator system mentioned in Definition 2.3 is called the *embedding dimension* of the G numerical semigroup and is denoted by $e(G)$.

Definition 2.5. Let G be a numerical semigroup. The smallest element of the generator set of the numerical semigroup G is called the *multiplicity* of G and is denoted by $m(G)$.

Definition 2.6. Let G be a numerical semigroup. If $m(G) = e(G)$, the numerical semigroup G is called the semigroup with *maximal embedded dimension*.

Definition 2.7. Let G be a numerical semigroup. The largest element of the gaps set of the numerical semigroup G is called the *frobenius number*.

Definition 2.8. Let G be a numerical semigroup. The smallest integer x given as $x+n \in G$ and $n \in \mathbb{N}$ for the numerical semigroup G is called the *conductor* of G and is denoted by $I(G)$, i.e.,

$$I(G) = F(G) + 1.$$

Definition 2.9. Let G be a numerical semigroup. Let's take an integer x such that $x \;\in G$. If $g \in G \backslash \{0\}$ and $x+g \in G$ are, then the integer x is called a *pseudo-frobenius number* of G. The set of all pseudo-frobenius numbers of the numerical semigroup G is denoted by $PF(G)$.

Example 2.1. $G = \{0,4,7,8,9,11,\dots\}$ is a numerical semigroup,

(i) $0 \in G$

(ii) G is closed according to the addition operation, $x, y \in G \Rightarrow x+y \in G$

(iii) $\mathbb{N} \backslash G = \{1,2,3,5,6,10\}$. The set is finite and from Definition 2.1, G is a numerical semigroup.

Example 2.2. The numerical semigroup given in Example 2.1,

(a) Gaps set and genus: $H(G) = \{1,2,3,5,6,10\}$ and $h(G) = 6$

(b) Minimal generator set and embedding dimension: $G = \langle 4,7,9, \rangle$ and $e(G) = 3$

(c) Multiplicity: $m(G) = 4$

(d) Since $m(G) \neq e(G)$, it is not a maximal embedded dimension

(e) Frobenius number: $F(G) = 10$

(f) Conductor: $I(G) = 11$

(g) Pseudo-frobenius number and type: $PF(G) = \{5,10\}$ and $t(G) = 2$

Definition 2.10. Let G be a numerical semigroup. If the numerical semigroup G satisfies the following property, then G is called Arf numerical semigroup [12],

$$\forall\; x, y, z \in G : x \geq y \geq z \Longrightarrow x+y-z \in G.$$

Lemma 2.1. *Let G be a numerical semigroup,*

$$\forall\; x, y \in G : x \geq y \Longrightarrow 2x - y \in G.$$

If the numerical semigroup G satisfies the above property, then G is called Arf numerical semigroup [4].

Proof. See [4]. □

Example 2.3. $G = \langle 3,10,11\rangle = \{0,3,6,9,\dots\}$ an Arf is a numerical semigroup,

$$x = 3,\ y = 3 \Longrightarrow 2.3-3 = 3 \in G$$
$$x = 6,\ y = 3 \Longrightarrow 2.6-3 = 9 \in G$$
$$x = 6,\ y = 6 \Longrightarrow 2.6-6 = 6 \in G$$
$$x = 9,\ y = 3 \Longrightarrow 2.9-3 = 15 \in G$$
$$x = 9,\ y = 6 \Longrightarrow 2.9-6 = 12 \in G$$
$$x = 9,\ y = 9 \Longrightarrow 2.9-9 = 9 \in G$$

When $x \ge 9$, $2x - y \in G$ will always be $x - y \ge 8$. In this case G is an Arf numerical semigroup.

Lemma 2.2. *Every Arf numerical semigroup is of maximal embedding dimension* [11]. *But the reverse is not true.*

Example 2.4. $G = \langle 3,4,8\rangle = \{0,3,4,7,8,9,10,\dots\}$ $m(G) = e(G) = 3$ maximal embedding dimension but not Arf numerical semigroup,

$$x = 3,\ y = 3 \Longrightarrow 2.3-3 = 3 \in G$$
$$x = 4,\ y = 3 \Longrightarrow 2.4-3 = 5 \notin G$$

G Arf is not a numerical semigroup because 5 is not an element of G.

3. *RF*-Matrices of Arf Numerical Semigroups With Small Multiplicity

We study Arf numerical semigroups with multiplicity of two, three, four, five and six with RF-matrices. The RF-matrices of Arf numerical semigroups were calculated by means of the GAP program [6].

Definition 3.1. Let $f \in PF(G)$. An $e \times e$ matrix $A = (a_{ij})$ is an RF-matrices of f, if $a_{ii} = -1$ for every i, $a_{ij} \in \mathbb{N}$ if $i \neq j$ and for every $i = 1,\dots,e$,

$$\sum_{j=1}^{e} a_{ij} n_j = f\,.$$

Lemma 3.1. *The size of the* RF*-matrices is determined by the number of elements in the minimal generator system of the G numerical semigroup.*

Example 3.1.

$$G = \langle 4,21,22,23\rangle = \{0,4,8,12,16,20,\dots\}.$$

Let's find the RF-matrices of the Arf numerical semigroup G.
To find the RF-matrices, we need to find the elements of the pseudo-frobenius set. In this case,

$$PF(G) = \{x \in G | x + g \in G\}\,.$$

Set of gaps of G;

$$H(G) = \{1,2,3,5,6,7,9,10,11,13,14,15,17,18,19\} \text{ and } F(G) = 19$$

$$\Longrightarrow \quad PF(G) = \{17,18,19\}$$

for $f = 19 \in PF(G)$,

$$19 = a_{11}.4 + a_{12}.21 + a_{13}.22 + a_{14}.23 \text{ and for } a_{11} = -1$$

$$19 = -1.4 + a_{12}.21 + a_{13}.22 + a_{14}.23$$

Thus, the first row of the RF-matrices: $\begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}$.

If $f = 19 \in PF(G)$,

$$19 = a_{21}.4 + a_{22}.21 + a_{23}.22 + a_{24}.23 \text{ and for } a_{22} = -1$$

$$19 = -1.4 + a_{22}.21 + a_{23}.22 + a_{24}.23$$

Thus, the second row of the RF-matrices: $\begin{bmatrix} 10 & -1 & 0 & 0 \end{bmatrix}$.

If $f = 19 \in PF(G)$, we obtain

$$19 = a_{31}.4 + a_{32}.21 + a_{33}.22 + a_{34}.23 \text{ and for } a_{33} = -1$$

$$19 = a_{31}.4 + a_{32}.21 + -1.22 + a_{34}.23$$

Thus, the third row of the RF-matrices: $\begin{bmatrix} 5 & 1 & -1 & 0 \end{bmatrix}$.

If $f = 19 \in PF(G)$, we get

$$19 = a_{41}.4 + a_{42}.21 + a_{43}.22 + a_{44}.23 \text{ and for } a_{44} = -1$$

$$19 = a_{41}.4 + a_{42}.21 + a_{43}.22 + -1.23$$

Thus, the fourth row of the RF-matrices: $\begin{bmatrix} 0 & 2 & 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} 5 & 0 & 1 & -1 \end{bmatrix}$.

Since the element number of the minimal generator set of the G numerical semigroup is 4, the RF-matrices will be a 4×4 type matrix,

$$RF(19) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 10 & -1 & 0 & 0 \\ 5 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{bmatrix}, \quad RF(19) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 10 & -1 & 0 & 0 \\ 5 & 1 & -1 & 0 \\ 5 & 0 & 1 & -2 \end{bmatrix}$$

Note. An RF-matrices can be written for each element in the pseudo-Frobenius set. These written matrices are not usually unique.

Definition 3.2. Let G be an Arf numerical semigroup with multiplicity m. Then G is the minimal generator set $(Ap(G,m)\backslash\{0\} \cup \{m\})$.

Arf numerical semigroup with multiplicity one.
The Arf numerical semigroup with a multiplicity of 1 is just the set of $\mathbb{N}$ numbers.

Arf numerical semigroups of multiplicity two:

Propositon 3.1. *Any numerical semigroup with a multiplicity of* 2 *is also an Arf numerical semigroup. The conductor* $I(G) = I$ *and the numerical semigroup Arf with multiplicity of* 2 *is expressed by* $G = \langle 2, I+1 \rangle$ [7].

Example 3.2. Arf numerical semigroup with multiplicity two,
Frobenius number 5 (conductor 6)

$$G_1 = \langle 2,7 \rangle = \{0,2,4,6,\ldots\}$$

for RF-matrices $PF(G_1) = \{5\}$,
$f = 5 \in PF(G_1)$, we obtain

$$RF(5) = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Frobenius number 7 (conductor 8)

$$G_2 = \langle 2,9 \rangle = \{0,2,4,6,8,\ldots\}$$

for RF-matrices $PF(G_2) = \{7\}$,
$f = 7 \in PF(G_2)$, we get

$$RF(7) = \begin{bmatrix} -1 & 1 \\ 8 & -1 \end{bmatrix}$$

Note. Arf numerical semigroups with multiplicity greater than 3 and 3 cannot be written explicitly by the conductor alone. The genus is needed to completely determine them [7]. If the Arf property is asuumed, then the Arf numerical semigorup is fully determined by the multiplicity and the conductor.

Arf numerical semigroups of multiplicity three:

Propositon 3.2. *Let I be an integer so that $I \geq 3$ and $I \not\equiv 1 \pmod 3$. Then, the Arf numerical semigroup G with a multiplicity of 3 and a conductor I can be written as one of the following: $I \equiv 0$ or $2 \pmod 3$,*

(i) $G = \langle 3, I+1, I+2 \rangle$ *if* $I \equiv 0 \pmod 3$.

(ii) $G = \langle 3, I, I+2 \rangle$ *if* $I \equiv 2 \pmod 3$ [7].

Example 3.3. Arf numerical semigroup with multiplicity three,
Frobenius number 10 (conductor 11)

$$G_3 = \langle 3,11,13 \rangle = \{0,3,6,9,11,\ldots\}$$

for RF-matrices $PF(G_3) = \{8,10\}$,
$f = 8 \in PF(G_3)$, we get

$$RF(8) = \begin{bmatrix} -1 & 0 & 1 \\ 7 & -1 & 0 \\ 4 & 1 & -1 \end{bmatrix}.$$

Frobenius number 11 (conductor 12)

$$G_4 = \langle 3,13,14 \rangle = \{0,3,6,9,12,\ldots\}$$

for RF-matrices $PF(G_4) = \{10,11\}$,
$f = 10 \in PF(G_4)$, we obtain

$$RF(10) = \begin{bmatrix} -1 & 1 & 0 \\ 3 & -1 & 1 \\ 8 & 0 & -1 \end{bmatrix}.$$

Arf numerical semigroups of multiplicity four:

Propositon 3.3. *The Arf numerical semigroup with conductor I and multiplicity 4 can be written as one of the following: $I \equiv 0,2$ or $3 (\text{mod} 4)$,*

(i) $G = \langle 4, 4a+2, I+1, I+3\rangle$ *if* $I \equiv 0 (\text{mod} 4)$, *for* $\left(a \in \{1,\dots,\frac{I}{4}\}\right)$,

(ii) $G = \langle 4, 4a+2, I+1, I+3\rangle$ *if* $I \equiv 2 (\text{mod} 4)$, *for* $\left(a \in \{1,\dots,\frac{I-2}{4}\}\right)$,

(iii) $G = \langle 4, I, I+2, I+3\rangle$ *if* $I \equiv 3 (\text{mod} 4)$ [7].

Example 3.4. Arf numerical semigroup with multiplicity four,

Frobenius number 21 (conductor 22)

$$G_5 = \langle 4,6,23,25\rangle = \{0,4,6,8,10,12,14,16,18,20,22,\dots\}$$

for RF-matrices $PF(G_5) = \{2,19,21\}$,

for $f = 2 \in PF(G_5)$,

$$RF(2) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}.$$

Frobenius number 21 (conductor 22)

$$G_6 = \langle 4,10,23,25\rangle = \{0,4,8,10,12,14,16,18,20,22,\dots\}$$

for RF-matrices $PF(G_6) = \{6,19,21\}$,

$f = 6 \in PF(G_6)$, we get

$$RF(6) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 4 & -1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 2 & 0 & 1 & -1 \end{bmatrix}.$$

Frobenius number 21 (conductor 22)

$$G_7 = \langle 4,14,23,25\rangle = \{0,4,8,12,14,16,18,20,22,\dots\}$$

for RF-matrices $PF(G_7) = \{10,19,21\}$,

$f = 10 \in PF(G_7)$, we obtain

$$RF(10) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 6 & -1 & 0 & 0 \\ 2 & 0 & -1 & 1 \\ 3 & 0 & 1 & -1 \end{bmatrix}$$

Frobenius number 22 (conductor 23)

$$G_8 = \langle 4,23,25,26\rangle = \{0,4,8,12,16,20,23,\dots\}$$

for RF-matrices $PF(G_8) = \{19,21,22\}$,

for $f = 19 \in PF(G_8)$,

$$RF(19) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 4 & -1 & 0 & 1 \\ 11 & 0 & -1 & 0 \\ 5 & 0 & 1 & -1 \end{bmatrix}.$$

Arf numerical semigroups of multiplicity five:

Propositon 3.4. *The Arf numerical semigroup with conductor I and multiplicity 5 can be written as one of the following, $I \equiv 0,2,3$ or $4 \pmod 5$,*

(i) *If* $I \equiv 0 \pmod 5$

  (a) $G = \langle 5, I-2, I+1, I+2, I+4 \rangle$, *or*

  (b) $G = \langle 5, I+1, I+2, I+3, I+4 \rangle$

(ii) $G = \langle 5, I, I+1, I+2, I+4 \rangle$ *if* $I \equiv 2 \pmod 5$.

(iii) $G = \langle 5, I, I+1, I+3, I+4 \rangle$ *if* $I \equiv 3 \pmod 5$.

(iv) *If* $I \equiv 4 \pmod 5$

  (a) $G = \langle 5, I-2, I, I+2, I+4 \rangle$, *or*

  (b) $G = \langle 5, I, I+2, I+3, I+4 \rangle$ [7].

Example 3.5. Arf numerical semigroup with multiplicity five,

Frobenius number 29 (conductor 30)

$$G_9 = \langle 5, 28, 31, 32, 34 \rangle = \{0, 5, 10, 15, 20, 25, 28, 30, \dots\}$$

for RF-matrices $PF(G_9) = \{23, 26, 27, 29\}$,

if $f = 23 \in PF(G_9)$,

$$RF(23) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 4 & -1 & 1 & 0 & 0 \\ 4 & 0 & -1 & 0 & 1 \\ 11 & 0 & 0 & -1 & 0 \\ 5 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Frobenius number 31 (conductor 32)

$$G_{10} = \langle 5, 32, 33, 34, 36 \rangle = \{0, 5, 10, 15, 20, 25, 30, 32, \dots\}$$

for RF-matrices $PF(G_{10}) = \{27, 28, 29, 31\}$,

$f = 27 \in PF(G_{10})$, we get

$$RF(27) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 5 & -1 & 0 & 1 & 0 \\ 12 & 0 & -1 & 0 & 0 \\ 5 & 0 & 0 & -1 & 1 \\ 6 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Frobenius number 32 (conductor 33)

$$G_{11} = \langle 5, 33, 34, 36, 37 \rangle = \{0, 5, 10, 15, 20, 25, 30, 33, \dots\}$$

for RF-matrices $PF(G_{11}) = \{28, 29, 31, 32\}$,

for $f = 28 \in PF(G_{11})$,

$$RF(28) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 5 & -1 & 0 & 1 & 0 \\ 5 & 0 & -1 & 0 & 1 \\ 6 & 0 & 1 & -1 & 1 \\ 13 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Frobenius number 33 (conductor 34)

$$G_{12} = \langle 5,32,34,36,38\rangle = \{0,5,10,15,20,25,30,32,34,\ldots\}$$

for RF-matrices $PF(G_{12}) = \{27,29,31,33\}$,

$f = 27 \in PF(G_{12})$, we obtain

$$RF(27) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 5 & -1 & 1 & 0 & 0 \\ 5 & 0 & -1 & 1 & 0 \\ 5 & 0 & 0 & -1 & 1 \\ 13 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Arf numerical semigroups of multiplicity six:

Propositon 3.5. *The Arf numerical semigroup with conductor I and multiplicity 6 can be written as one of the following, $I \equiv 0,2,3,4$ or $5\,(\mathrm{mod}\,6)$,*

(i) *If* $I \equiv 0(\mathrm{mod}\,6)$

(a) $G = \langle 6, I+1, I+2, I+3, I+4, I+5\rangle$

(b) $G = \langle 6, 6m+2, 6m+4, I+1, I+3, I+5\rangle$

(c) $G = \langle 6, 6m+3, I+1, I+2, I+4, I+5\rangle$

(d) $G = \langle 6, 6m+4, 6m+8, I+1, I+3, I+5\rangle$

For $\left(m = 1,2,\ldots,\frac{I}{6}-1\right)$

(ii) *If* $I \equiv 2(\mathrm{mod}\,6)$

(a) $G = \langle 6, 6n+2, 6n+4, I+1, I+3, I+5\rangle$

(b) $G = \langle 6, 6k+3, I, I+2, I+3, I+5\rangle$

(c) $G = \langle 6, 6k+4, 6k+8, I+1, I+3, I+5\rangle$

For $\left(n = 1,2,\ldots,\frac{I-2}{6},\ k = 1,2,\ldots,\frac{I-2}{6}-1\right)$

(iii) *If* $I \equiv 3\,(\mathrm{mod}\,6)$

$G = \langle 6, 6x+3, I+1, I+2, I+4, I+5\rangle$ *for* $\left(x = 1,2,\ldots,\frac{I-3}{6}\right)$

(iv) *If* $I \equiv 4(\mathrm{mod}\,6)$

(a) $G = \langle 6, 6y+2, 6y+4, I+1, I+3, I+5\rangle$

(b) $G = \langle 6, 6y+4, 6y+8, I+1, I+3, I+5\rangle$

For $\left(y = 1,2,\ldots,\frac{I-4}{6}\right)$

(v) *If* $I \equiv 2(\mathrm{mod}\,6)$

(a) $G = \langle 6, I, I+2, I+3, I+4, I+5\rangle$

(b) $G = \langle 6, 6t+3, I, I+2, I+3, I+5\rangle$

For $\left(t = 1,2,\ldots,\frac{I-5}{6}\right)$ [7].

Example 3.6. Arf numerical semigroup with multiplicity six,

Frobenius number 29 (conductor 30)

$$G_{13} = \langle 6,31,32,33,34,35\rangle = \{0,6,12,18,24,30,\ldots\}$$

for RF-matrices $PF(G_{13}) = \{25,26,27,28,29\}$,

$f = 25 \in PF(G_{13})$, we get

$$RF(25) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 4 & -1 & 1 & 0 & 0 & 0 \\ 4 & 0 & -1 & 1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 1 & 0 \\ 4 & 0 & 0 & 0 & -1 & 1 \\ 13 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Frobenius number 29 (conductor 30)

$$G_{14} = \langle 6,20,22,31,33,35 \rangle = \{0,6,12,18,20,22,24,26,28,30,\dots\}$$

for RF-matrices $PF(G_{14}) = \{14,16,25,27,29\}$,

$f = 14PF(G_{14})$, we obtain

$$RF(14) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 & 0 \\ 6 & 0 & -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & 1 & 0 \\ 2 & 0 & 0 & 0 & -1 & 1 \\ 3 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Frobenius number 29 (conductor 30)

$$G_{15} = \langle 6,14,16,31,33,35 \rangle = \{0,6,12,14,16,18,20,22,24,26,28,30,\dots\}$$

for RF-matrices $PF(G_{15}) = \{8,10,25,27,29\}$,

for $f = 8 \in PF(G_{15})$,

$$RF(8) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 4 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Frobenius number 29 (conductor 30)

$$G_{16} = \langle 6,8,10,31,33,35 \rangle = \{0,6,8,10,12,14,16,18,20,22,24,26,28,30,\dots\}$$

for RF-matrices $PF(G_{16}) = \{2,4,25,27,29\}$,

if $f = 2 \in PF(G_{16})$,

$$RF(2) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

4. Conclusions

We mentioned about numerical semigroups and Arf numerical semigroups in this paper. We stated the necessary and sufficient condition for a numerical semigroup to be an Arf numerical semigroup. We Showed how to write the *RF*-matrices of any Arf numerical semigroup. The *RF*-matrices that we investigated in this paper can be studied in all other subjects of numerical semigroups.

Acknowledgements

The author would like to thank Professor Pedro García-Sánchez for valuable and helpful suggestions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] C. Arf, Une interprétation algébrique de la suite des ordres de multiplicité d'une branche algébrique, *Proceedings of the London Mathematical Society* **s2-50**(1 (1945)), 256 – 287, DOI: 10.1112/plms/s2-50.4.256.

[2] A. Assi and P. A. García-Sánchez, *Numerical Semigroups and Applications*, 1st edition, RSME Springer Series, Springer Cham, xiv + 106 pages (2016), DOI: 10.1007/978-3-319-41330-3.

[3] V. Barucci, D. E. Dobbs and M. Fontana, Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains, *Memoirs of the American Mathematical Society* **125**(598), x + 78 pages (1997), DOI: 10.1090/memo/0598.

[4] A. Campillo, J. I. Farran and C. Munuera, On the parameters of algebraic-geometry codes related to Arf semigroups, *IEEE Transactions on Information Theory* **46**(7) (2000), 2634 – 2638, DOI: 10.1109/18.887872.

[5] P. Du Val, The Jacobian algorithm and the multiplicity sequence of an algebraic branch, *Revue de la faculte des sciences de l'universite d'Istanbul, Serie A* **VII** (1942), 107 – 112.

[6] GAP Group, *GAP — Groups, Algorithms, Programming – a System for Computational Discrete Algebra*, Version 4.7.8, The GAP Group (2014), URL: http://www.gap-system.org, accessed December, 2022.

[7] P. A. García-Sánchez, B. A. Heredia, H. İ. Karakaş and J. C. Rosales, Parametrizing Arf numerical semigroups, *Journal of Algebra and Its Applications* **16**(11) (2017), 1750209, DOI: 10.1142/S0219498817502097.

[8] J. Herzog and K.-I. Watanabe, Almost symmetric numerical semigroups, *Semigroup Forum* **98** (2019), 589 – 630, DOI: 10.1007/s00233-019-10007-2.

[9] J. Lipman, Stable ideals and Arf rings, *American Journal of Mathematics* **93**(3) (1971), 649 – 685, DOI: 10.2307/2373463.

[10] J. C. Rosales and M. B. Branco, The Frobenius problem for numerical semigroups with multiplicity four, *Semigroup Forum* **83** (2011), 468 – 478, DOI: 10.1007/s00233-011-9328-4.

[11] J. C. Rosales and P. A. García-Sánchez, *Numerical Semigroups*, 1st edition, Developments in Mathematics series, Vol. **20**, Springer New York, NY, ix + 181 pages, (2009), DOI: 10.1007/978-1-4419-0160-6.

[12] J. C. Rosales, P. A. García-Sánchez, J. I. García-García and M. B. Branco, Arf numerical semigroups, *Journal of Algebra* **276**(1) (2004), 3 – 12, DOI: 10.1016/j.jalgebra.2004.03.007.

[13] J. J. Sylvester, Mathematical questions with their solutions, *Educational Times* **41** (1884), 171 – 178.