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Spectral Properties for Pseudodifferential Operators via Weighted Modulation Spaces

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Abstract. In this paper we deal with this question: considering spectral representation of a positive trace-class integral operator, if its orthonormal eigenvectors are in modulation space M_m^p? This question actually provide a new framework for studying the connection between operator theory and modulation spaces. Here we use some Schatten class properties of pseudodifferential operators to give a positive answer to this question. Also we investigate convergence conditions for eigenvectors of such operators.

1. **Introduction**

Nowaday modulation spaces have found their place as the exact framework for time-frequency analysis. This class of function spaces which are based on measuring both time and frequency concentration of a function, are one of the more interesting areas of modern analysis, which attract many researchers. On the other hand no one can omit the extensive and essential role of operator theory in several areas of mathematics and applications. Concerning a connection between these two important sub-areas of modern analysis, could provide a wide vision in different branches of mathematics. This connection was first obtained just few years ago. In February 2004, in a workshop on *Wavelets, Frames, and Operator Theory* which was taken part in Oberwolfach, Feichtinger gave out a problem on modulation spaces. After that in [19], authors tried to set a connection between operator theory and modulation spaces by giving a reformulation of the problem in operator theory terms.

Although they stated several interesting and new problems inspired by Feichtinger original question, as well as an operator theory reformulation of this question, but Feichtinger's original problem is interesting by itself. Investigating this problem also may helps to generalize the characterization in [19] depends on

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the spirit of given answer. Actually Feichtinger asked that, considering spectral representation, if orthonormal eigenvectors of an positive trace-class integral operator satisfies in M^1? M^1 plays a crucial role for the rigorous investigation of modulation spaces. In this paper we consider weighted modulation spaces M^p_m. In time-frequency analysis weight functions are used to quantify growth and decay conditions in many problems and contexts, e.g. (i) in the theory of Gabor frames to measure the quality of time-frequency concentration; (ii) in the definition of modulation spaces; (iii) in the definition of symbol classes for pseudodifferential operators; [17, 7].

Another fundamental tool in the theory of modulation spaces, is their Wilson basis approach that we use it extensively in our discussion of integral operators. On the other hand, thanks to deep work of Labate we can invokes some results about absolutely summing operators to prove statements about the summability of the singular values of integral operators with kernel in modulation spaces,[21]which will be known as Schatten class properties. Both these subjects, Wilson bases for modulation spaces and Schatten class properties for pseudodifferential operator help us to yield a particular answer for Feichtinger question.

Now we are going to outline this paper: in section 2 we present weight functions and some famous facts in operator theory. In section 3 we will attempt some backgrounds on modulation spaces. Finally section 4 contains description of integral operators with kernels in modulation spaces, their singular values approximating and Feichtinger problem. Finally we present our findings and results.

1.1. *Notations*

Let $\mathscr{H}$ be an infinite-dimensional separable Hilbert space. Then $\|\cdot\|$ denotes its norm and $\langle\cdot\rangle$ is its inner product. For vectors x and y in $\mathscr{H}$, $(x \otimes y)$ is rank one operator $(x \otimes y)z = \langle z, y\rangle x$ which its operator norm is $\|x\|\|y\|$. We denote by $x \cdot y$ the scalar product and abbreviate $\|x\|^2 = x \cdot x$ by x^2. The Schwartz class is denoted by $\mathscr{S}$ and its dual, the space of tempered distributions, by $\mathscr{S}'$.

If $A \in B(\mathscr{H})$ is a compact operator then we show its eigenvalues with λ_j and its singular values with $s_j(A)$. Sequence $\{s_k(A)\}_{k=1}^{\infty}$ is nonincreasing. For a Banach space B and a weight function m we denote by B_m the weighted Banach space $\{f \in B : fm \in B\}$ with norm $\|f\|_{B_m} = \|fm\|_B$ for $f \in B$.

2. **Preliminaries**

Generally a *weight* v or m is a non-negative continuous function on $\mathbb{R}^{2d}$. Here we mention some famous class of weight functions which we will use later.

(i) v is called *submultiplicative*, if $v(z+z') \leq v(z)v(z')$ for all $z, z' \in \mathbb{R}^{2d}$;

(ii) m is called *v-moderate* with respect to a weight v, if $m(z+z') \leq Cv(z)m(z')$ for some positive constant C and all $z, z' \in \mathbb{R}^{2d}$;

In the present paper we restrict ourselves to the weight functions with at most polynomial growth i.e., $m(z) \le (1+|z|)^N$, for some N and we call m a *polynomial weight*. An important class of weights, v_s for $s \ge 0$, is given by

$$v_s(z) = (1+|z|^2)^{s/2} = (1+x^2+\omega^2)^{s/2}, \quad z=(x,\omega) \in \mathbb{R}^{2d}. \tag{2.1}$$

The weights v_s appear naturally in different branches of mathematics and physics, e.g. Banach convolution algebras, function spaces, harmonic analysis, pseudo-differential operators, classes of matrix algebras and quantum mechanics. The weights v_s have many properties that make them of interest. We recall some of these properties in the following lemma.

Lemma 2.1. *For every polynomial weight v_s we have*

(i) *v_s is submultiplicative for all $s \ge 0$;*
(ii) *If $0 \le t \le s$, then v_s and v_t^{-1} are v-moderate;*

We referred to [16] for complete proof of the assertions of the above lemma. Its formulation in the case of locally compact groups can be found in [7]. Through this paper we consider $s > 0$.

Now we are going to recall some terms in operator theory. Let A be a compact operator in $B(\mathscr{H})$. Then by spectral theory $s_j(A) = \lambda_j(A^*A)^{1/2}$. If A is a self-adjoint operator, then $s_j(A) = |\lambda_j(A)|$ and if A is positive (the case we are concerned with, here), then $s_j(A) = \lambda_j(A)$.

We say that A is in Schatten class I_p if $\|A\|_{I_p} = \Big(\sum_{j=1}^{\infty}(s_j(A))^p\Big)^{1/p} < \infty$. If $p=1$ then A is a trace-class operator, and for $p=2$ we deal with well known Hilbert-Schmidt operators. Let A be any positive trace-class operator in $B(\mathscr{H})$. Spectral representation theorem yields following decomposition of A:

$$A = \sum_j \lambda_j (g_j \otimes g_j) = \sum_j h_j \otimes h_j, \tag{2.2}$$

where $\{g_j\}$ are orthonormal eigenvectors of A corresponding to eigenvalues λ_j, and $h_j = \lambda_j^{1/2} g_j$. Since the eigenvalues of A coincide with its singular values, the trace of A coincides with its trace-class norm, i.e.,

$$\sum_j \|h_j\|^2 = \sum_j \lambda_j = \mathrm{trace}(A) = \|A\|_{I_1} < \infty. \tag{2.3}$$

3. **Modulation spaces and Wilson bases**

During the early 1980's Feichtinger introduced the class of *modulation spaces* in [9]. Later he gave in joint work with Gröchenig a description of modulation spaces as coorbit spaces of the Heisenberg group [11]. Recently modulation spaces have turned out to be the correct mathematical framework for a rigorous treatment of many problems in time-frequency analysis. We refer the reader to Feichtinger's survey article [10] for a thorough treatment and an extensive list of references.

Fundamental object in modulation spaces structure is *short time Fourier transform* (STFT) which arises from unitary operators T_x(translation) and M_ω(modulation) on $L^2(\mathbb{R})$:

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega t} f(t), \tag{3.1}$$

as time and frequency shifts. *Short time Fourier transform* (STFT) of function f with respect to function g, which is called window function or briefly window, is defined to be $V_g f(x,\omega)$ such that

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i\omega t}dt, \quad x,\omega,t \in \mathbb{R}^d,$$

whenever the integral or the inner product exists, e.g. for $(f,g) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ or $(f,g) \in \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}'(\mathbb{R}^d)$. The STFT is a measure of the time-frequency concentration (phase space content) of f at the point $(x,\omega) \in \mathbb{R}^{2d}$ in the time-frequency space. Modulation spaces are defined in terms of a function space norm applied to STFT, to measure and control the time-frequency concentration of function f with respect to window g.

In general, a weighted modulation space is the space of functions with finite STFT in weighted L^p norm.

Definition 3.1. For a non-zero window $g \in \mathscr{S}(\mathbb{R}^d)$, a v-moderate weight function m on $\mathbb{R}^{2d}$ and $1 \le p < \infty$, the modulation space $M_m^p(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d)$ such that $V_g f \in L_m^p(\mathbb{R}^{2d})$. The norm on $M_m^p(\mathbb{R}^d)$ is

$$\|f\|_{M_m^p} = \|V_g f\|_{L_m^p} \tag{3.2}$$

with obvious changes if $p = \infty$.

Note that this norm is independent of the choice of window function g.

In order to find the basic properties of this family of function spaces we need to understand the inversion formula for the STFT. Consider the operator

$$V_\gamma^* F = \iint_{\mathbb{R}^{2d}} F(x,\omega) M_\omega T_x \gamma \, dx\, d\omega,$$

for a non-zero window γ and a function F on $\mathbb{R}^{2d}$. If $F = V_g f$ then the inversion formula

$$f = \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^{2d}} V_g f(x,\omega) M_\omega T_x \gamma \, d\omega\, dx,$$

holds in M_m^p, i.e., $I_{M_m^p} = \langle \gamma, g\rangle^{-1} V_\gamma^* V_g$. Let v be a submultiplicative weight on $\mathbb{R}^{2d}$ and m a v-moderate weight. Then we define $M_v^1(\mathbb{R}^d)$ as above. If v is the trivial weight $v \equiv 1$, then $M_v^1(\mathbb{R}^d)$ is the *Feichtinger algebra* $S_0(\mathbb{R}^d)$, [8]. The search for the smallest Banach space that is invariant under translations and modulations led Feichtinger to the discovery of M^1, then he improved it for construction of all modulation spaces. So M^1 is in special interest. The window class M_v^1 is the smallest space which possess all desired functional properties of M_m^p.

When we use polynomial weight v_s instead of m, we show the corresponding modulation spaces with M_s^p.

Historically, extending the theory of Gabor frames from L^2 to modulation spaces, provided a very strong and flexible framework through modulation spaces. Recall that, given a non-zero window function $g \in L^2(\mathbb{R}^d)$ and lattice parameters $\alpha, \beta > 0$, the set of time-frequency shifts

$$\mathscr{G}(g, \alpha, \beta) = \{T_{\alpha k} M_{\beta n} g : k, n \in \mathbb{Z}^d\}, \tag{3.3}$$

is called a Gabor system.

If $\mathscr{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$, it is called a Gabor frame. Following important fact leads to a characterization of modulation spaces by means of Gabor frames.

Proposition 3.1. *Let v be a moderate weight on $\mathbb{R}^{2d}$. If $\mathscr{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ with $g \in M_v^1(\mathbb{R}^d)$, then $\mathscr{G}(g, \alpha, \beta)$ is a Banach frame for $M^p(\mathbb{R}^d)$, $1 \le p \le \infty$. Consequently, each f in $M^p(\mathbb{R}^d)$ has a discrete representation with respect to a dual window $\gamma \in M_v^1(\mathbb{R}^d)$, i.e.,*

$$f = \sum_{k,n \in \mathbb{Z}^d} \langle f, T_{\alpha k} M_{\beta n} \gamma \rangle T_{\alpha k} M_{\beta n} g, \tag{3.4}$$

that converges unconditional for some $1 \le p < \infty$ and with weak*-converge in $M^\infty_{1/v}(\mathbb{R}^d)$. Furthermore, we have that

$$A\|f\|_{M_m^p} \le \left(\sum_{k,n \in \mathbb{Z}^d} |\langle f, T_{\alpha k} M_{\beta n} g \rangle|^p m(\alpha k, \beta n)^p \right)^{1/p} \le B\|f\|_{M_m^p} \tag{3.5}$$

for all $f \in M_m^p(\mathbb{R}^d)$ with v-moderate weight m, [16].

Now, consider Gabor system $\mathscr{G}(g, \frac{1}{2}\mathbb{Z} \times \mathbb{Z})$ be a Gabor system of redundancy 2 in $L^2(\mathbb{R})$. Then the associated *Wilson system*, $\mathscr{W}(g)$, consists of the symmetric linear combinations of time and frequency shifts of function g:

$$\psi_{kn} = c_n T_{k/2}(M_n + (-1)^{k+n} M_{-n}) g, \quad k, n \in \mathbb{Z}, n \ge 0 \tag{3.6}$$

where $c_0 = 1/2$, $c_n = 1/\sqrt{2}$ for $n \ge 0$.

Theorem 3.2. *Let g be in $L^2(\mathbb{R})$ such that $g = g^*$ and $\|g\|_2 = 1$. If $\mathscr{G}(g, \frac{1}{2}\mathbb{Z} \times \mathbb{Z})$ is a tight Gabor frame for $L^2(\mathbb{R})$, then $\mathscr{W}(g)$ is an orthonormal Wilson basis of $L^2(\mathbb{R})$* [5].

After stating this important theorem for Wilson bases in $L^2(\mathbb{R})$ Feichtinger, Gröchenig and Walnut showed that $\mathscr{W}(g)$ is an unconditional basis for $M_m^p(\mathbb{R})$. This result that make sense the relation between modulation spaces and Wilson bases is given in the next theorem and is one of the main tools that we will use in the sequel. The complete proof can be found in [13].

Theorem 3.3. *Assume that $\mathscr{W}(g)$ is an orthonormal basis for $L^2(\mathbb{R})$ with $g \in M_v^1(\mathbb{R})$ for a moderate weight function v of polynomial weights. Then*

$$\frac{1}{C}\|f\|_{M_m^p} \leq \left(\sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}} | \langle f, \psi_{kn}\rangle |^p \, m(\tfrac{k}{2}, n)^p\right)^{1/p} \leq C\|f\|_{M_m^p}, \tag{3.7}$$

for a constant $C \geq 1$. Furthermore, the orthogonal expansion

$$f = \sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}}\langle f, \psi_{kn}\rangle\psi_{kn}, \tag{3.8}$$

converges unconditionally in the $M_m^p(\mathbb{R})$-norm if $p < \infty$, and weak in $M_{1/v}^1(\mathbb{R})$ otherwise.*

Recall that an unconditional basis for a Banach space B is a set $\{e_i : e \in I\}$ of vectors in B such that

(i) the finite linear combinations of the e_i are dense in B, and
(ii) there exists a constant $C \geq 1$, such that

$$\left\|\sum_{i\in F} c_i\lambda_i e_i\right\| \leq C(\sup|\lambda_i|)\left\|\sum_{i\in F} c_i e_i\right\|, \tag{3.9}$$

holds for any finite subset $F \subseteq I$ and any sequence $\{\lambda_i\}$ in $\mathbb{C}$.

The crucial property of unconditional basis is the existence of series expansion for every f in B

$$f = \sum_{i\in I} c_i e_i, \tag{3.10}$$

where the coefficients c_i are uniquely determined and every rearrangement of the series converges to f, i.e., the series converges unconditionally.

Construction of Wilson basis which is considered in $\mathbb{R}$, can be easily generalized to $\mathbb{R}^d$ by tensor product. Following theorem provided it.

Theorem 3.4. *Let m be a v-moderate weight and $\mathscr{W}(g) = \{\psi_{kn} : k, n \in \mathbb{Z}, n \geq 0\}$. Then $\{\Psi_{rs} = \sqcap_{i=1}^{d}\psi_{r_i s_i}(x_i)\}$ is an unconditional basis for $M_m^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ and weak-$*$-convergent in $M_{1/v}^{\infty}(\mathbb{R}^d)$ for $p = \infty$ [13].*

4. Integral operators and pseudodifferential operators with symbols in weighted modulation spaces

Despite the many benefits of Gabor frames in applications, it seems that for establishing the theory of modulation spaces the existence of an unconditional basis for them provides the correct framework. Wilson basis is a useful tool in various proofs of basic properties of modulation spaces, e.g. in the isomorphism theorem for modulation spaces, the kernel theorem for $M_v^1(\mathbb{R}^d)$, in the approach of Feichtinger and Kozek to quantization, in pseudodifferential Operators,. . . . Pseudodifferential operators appear in many areas of mathematics, physics, and engineering. In fact a pseudodifferential operator A_σ is given by a distributional symbol $\sigma \in M^{\infty}_{1/v}(\mathbb{R}^{2d})$ and the operator A_σ is a map from $M^1_v(\mathbb{R}^d)$ to $M^{\infty}_{1/v}(\mathbb{R}^d)$. So one can study the properties of A_σ through the properties of its symbol σ. We actually need that it also has a kernel k which is provided by Feichtinger's kernel theorem. Other version of this theorem can be found in [12] and [14]. Two standard ways to introduce pseudodifferential operators is through the Weyl- or Kohn-Nirenberg correspondence. The Weyl correspondence $L_\sigma f(x) = \iint \sigma(\frac{x+y}{2}, \xi)e^{2\pi i(x-y)\xi} f(y)dyd\xi$ associates to the symbol σ the operator L_σ and the Kohn-Nirenberg correspondence gives $K_\tau f(x) = \iint \tau(x,\xi)e^{2\pi i(x-y)\xi} f(y)dyd\xi$ for a symbol τ. But one can see that both of them are in integral operator form $A_k f(x) = \int k(x,y)f(y)dy$ with kernel k, [18]. So what we need in our discussion is to investigate desired properties for integral operator A_k, and then all results could be described for pseudodifferential operators K_τ and L_σ as well.

If A is a bounded operator between $M^1_v(\mathbb{R}^d)$ and $M^{\infty}_{1/v}(\mathbb{R}^d)$, then we can associate to A a matrix $(a_{(k,m),(l,n)})$ with respect to the Wilson basis by

$$a_{(k,m),(l,n)} = \langle A\psi_{l,n}, \psi_{k,m}\rangle \quad \text{for } k,l,m,n \in \mathbb{Z}^d,\ m,n \geq 0. \tag{4.1}$$

This fact with Wilson description of modulation spaces are keys in the proof of the next theorem for $M^1_v(\mathbb{R}^d)$ in [16].

Theorem 4.1. *Every distribution $k \in M^{\infty}_{1/v}(\mathbb{R}^{2d})$ defines a bounded operator $A_k : M^1_v(\mathbb{R}^d) \to M^{\infty}_{1/v}(\mathbb{R}^d)$ by*

$$\langle A_k f, g\rangle = \langle k, g \otimes \overline{f}\rangle, \quad f, g \in M^1_v(\mathbb{R}^d) \tag{4.2}$$

and vise-versa [16].

So investigation properties of integral operator A_k could be studied from its kernel. Thanks to work of Labate in [21] we can go from Schatten norm of A_k to M^p norm of k.

Let us mention here that in our discussion of integral operators, the existence of a matrix representation with respect to a multivariate Wilson basis is the crucial tool in our proofs. Let kernel k be in $M^p(\mathbb{R}^{2d})$ and consider the Wilson basis ψ_{mn} for $M^p(\mathbb{R}^d)$. Then by the tensor products, $\Psi_{mn}(x,y) = \psi_{mn}(x)\overline{\psi_{mn}(y)}$ yield a basis for $M^p(\mathbb{R}^{2d})$, so by (3.8), $k = \sum_{m,n\in\mathbb{Z}^{2d}} \langle k, \Psi_{mn}\rangle \Psi_{mn}$ and $A_k = \sum_{m,n\in\mathbb{Z}^{2d}} \langle k, \Psi_{mn}\rangle A_{\Psi_{mn}}$ such that

$$\begin{aligned} A_{\Psi_{mn}}(f) = \int \Psi_{mn}(x,y)f(y)dy &= \int \psi_{mn}(x)\overline{\psi_{mn}(y)}f(y)dy \\ &= \psi_{mn}(x)\langle f, \psi_{mn}(y)\rangle. \end{aligned} \tag{4.3}$$

The last equality arises from duality and shows that $A_{\Psi_{mn}}$ is a rank-one operator.

Now we are going to recall some definitions and results for p-summing operators.

Definition 4.1. Let B be a Banach space, $T \in \mathscr{L}(B)$, and $1 \leq p < \infty$. Then T is absolutely p-summing if there is a constant $c \geq 0$ such that for all sequences $\{(f_i)\}_{i=1}^m$ in B

$$\left(\sum \|Tf_i\|_B^p\right)^{1/p} \leq c \sup_{g \in L(B*)} \left(\sum |\langle g, f_i\rangle|^p\right)^{1/p}.$$

Let $\pi_p(T) = \inf c$, for which above inequality holds.

The collection $\Pi_p(X)$, of p-summing operators on B is a Banach space with norm $\pi_p(T)$.

Remark 4.2. If B is the Hilbert space $\mathscr{H}$, then $\Pi_p(\mathscr{H})$ coincides with the class $\mathscr{I}_2$ of Hilbert-Schmidt operators on $\mathscr{H}$ for $p \in [1, \infty]$, [21].

Next important proposition shows that if the kernel k of an integral operator belongs to a modulation space $M^p(\mathbb{R}^d)$, then the integral operator A_k is p-summing operator for some p.

Proposition 4.3. *Let $1 \leq p < \infty$ and p' be the conjugate exponent of p.*

(i) If $1 \leq p \leq 2$ and $k \in M^p(\mathbb{R}^d)$ then A_k is a compact (for $p = 1$, A_k is weakly compact and completely continuous) operator which maps $M^p(\mathbb{R}^d)$ into itself. Singular values of A_k are 2-summable and

$$\|A_k\|_{\mathscr{I}_2} = \left(\sum s_j(A_k)^2\right)^{1/2} \leq C\|k\|_{M^p}. \tag{4.4}$$

(ii) If $2 \leq p < \infty$ and $k \in M^{p'}(\mathbb{R}^d)$ then A_k is a weakly compact and completely continuous operator which maps $M^{p'}(\mathbb{R}^d)$ into itself. Singular values of A_k are r-summable such that $r = \max\{2, p\}$ and

$$\|A_k\|_{\mathscr{I}_r} = \left(\sum s_j(A_k)^r\right)^{1/r} \leq C\|k\|_{M^{p'}}. \tag{4.5}$$

We refer interested reader to [21] for an extensive proof. This proposition is our main tool to prove the following theorem which enable us to give some results on operators in the Schatten-von Neumann classes $\mathscr{I}_p$. This theorem is based on the extension of symbols of a pseudodifferential operator in terms of Wilson basis. We refer to [22] for reader convenience.

Theorem 4.4. *For integral kernel $k \in M_s^p(\mathbb{R}^d)$ and corresponding integral operator $A_k f(x) = \int k(x,y)f(y)dy$ and $s > 0$, the following statements hold for singular values $\{s_N(A_k)\}$:*

(i) *If $1 \leq p \leq 2$ and $k \in M_s^p(\mathbb{R}^d)$ then $s_N(A_k) = \mathscr{O}(N^{-s/2d-1/2})$*

(ii) *If $2 \leq p < \infty$ and $k \in M_s^{p'}(\mathbb{R}^d)$ then $s_N(A_k) = \mathscr{O}(N^{-s/p'd-1/p'})$, where p' is conjugate exponent of p.*

Now we mention a trace class property of integral operator A_k and then we go through Feichtinger problem.

Proposition 4.5. *If $k \in M^1(\mathbb{R}^{2d})$, then the corresponding integral operator A_k is trace-class.*

Proof. Let $\mathscr{W}(g) = \{\psi_n\}_{n\in\mathbb{N}}$ be a Wilson basis for $L^2(\mathbb{R}^d)$ such that $g \in M^1(\mathbb{R}^d)$. So $\mathscr{W}(g)$ is an unconditional basis for $M^1(\mathbb{R}^d)$. By tensor product $\{\Psi_{mn}(x,y)\}_{m,n\in\mathbb{N}} = \{\psi_m(x)\overline{\psi_n(y)}\}$ is an orthonormal basis for $L^2(\mathbb{R}^{2d})$ and an unconditional basis for $M^1(\mathbb{R}^{2d})$. Therefore by (3.8)

$$k = \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle \Psi_{mn}, \tag{4.6}$$

which series converge in M^1-norm and

$$\|k\|_{M_1} = \left| \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle \right| < \infty. \tag{4.7}$$

Definition of integral operator implies that

$$\begin{aligned} A_k f(x) &= \int_{\mathbb{R}} \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle \Psi_{mn}(x,y) f(y)dy \\ &= \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle \int_{\mathbb{R}} \psi_m(x)\overline{\psi_n(y)} f(y)dy \\ &= \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle \langle f, \psi_n\rangle \psi_m(x) \\ &= \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle (\psi_m \otimes \psi_n)(f)(x) \end{aligned} \tag{4.8}$$

Since series is absolute convergence, interchanging is valid. Therefore

$$A = \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle (\psi_m \otimes \psi_n). \tag{4.9}$$

In the last equation each operator $(\psi_m \otimes \psi_n)$ is trace-class and the scalars $\langle k, \Psi_{mn}\rangle$ are summable, so A_k is in I_1. □

Corollary 4.6. *If $k \in M^p_m(\mathbb{R}^{2d})$, for a v-moderate weight m, then the corresponding integral operator A_k is trace-class.*

Proof. Orthonormal Wilson basis $\mathscr{W}(g)$ is unconditional basis for $k \in M^p_m(\mathbb{R}^{2d})$. □

From unitary operators which switch between different representations of pseudodifferential operators and integral operators, Proposition 4.5 can be stated for pseudodifferential operators with symbol σ or τ in M^1.

Now we are going to state Feightinger problem, but first recall that h_j's are orthonormal eigenvectors as in (2.2).

Feichtinger problem. Let A_k be a positive integral operator whose kernel k lies in M^1. Considering spectral representation of A_k given in (2.2), must it be true that

$$\sum_{n=1}^{\infty} \|h_j\|_{M^1}^2 < \infty?$$

We prove the following result.

Recall that here $\mathscr{H} = L^2(\mathbb{R}^d) = M^2(\mathbb{R}^d)$ and $M_m^p \subseteq L^2$ for $1 \le p \le 2$. Also $\mathscr{W}(g)$ is Wilson orthonormal basis for L^2.

Theorem 4.7. *Let A_k be an integral operator which its kernel k lies in M_m^p, for $1 \le p \le 2$. If h_j are as above then $\{h_j\}$ are in M_m^p, for some v-moderate weight m .*

Proof. From Proposition 4.5 we know

$$A_k = \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle (\psi_m \otimes \psi_n).$$

From (2.2), for each eigenvalue λ_j we have

$$\lambda_j h_j = A_k h_j = \sum_{m,n\in\mathbb{N}} \langle k, \Psi_{mn}\rangle \langle h_j, \psi_m\rangle \psi_n.$$

Since $\|h_j\| = \lambda_j^{1/2} > 0$, $\|\psi_m\| = 1$ and $\|\psi_n\|_{M_m^p} < C$ for some constant C, then

$$\begin{aligned} |\lambda_j| \|h_j\|_{M_m^p} = \|A_k h_j\|_{M_m^p} &\le \sum_{m,n\in\mathbb{N}} |\langle j, \Psi_{mn}\rangle| |\langle h_j, \psi_m\rangle| \|\psi_n\|_{M_m^p} \\ &\le \sum_{m,n\in\mathbb{N}} |\langle k, \Psi_{mn}\rangle| \|h_j\| \|\psi_m\| \|\psi_n\|_{M_m^p} \\ &\le C\lambda_j^{1/2} \sum_{m,n\in\mathbb{N}} |\langle k, \Psi_{mn}\rangle| < \infty. \end{aligned}$$

Thus $\|h_j\|_{M_m^p} < \infty$ for each j and in particular we have $h_j \in M_m^p$ for every j. □

Proposition 4.8. *Let A_k be a finite rank positive self adjoint operator with kernel $k \in M_s^p(\mathbb{R}^{2d})$, for $1 \le p \le 2$, then $\sum_j \|h_j\|_{M_s^p}^2 \le \sum_j j^{\frac{s+d}{2d}}$.*

Proof. Above argument shows that

$$\sum_j \|h_j\|_{M_m^p}^2 \le C \sum_j \frac{1}{\lambda_j} \le C \sum_j \frac{1}{s_j(A_k)},$$

and so the desired result follows from Theorem 4.4(i). □

Corollary 4.9. *If A_k is a finite rank positive self adjoint operator with kernel $k \in M_s^p(\mathbb{R}^{2d})$, for $1 \le p \le 2$, then $\sum_j \|h_j\|_{M_s^p}^2 < \infty$.*

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