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Research Article

Solvability of a Class of Generalized System of Variational Inclusion Problems Involving ⊕ Operation

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Abstract. In this paper, a new type of operator known as (α,ρ)-XOR-NODSM operator and its associated resolvent operator is introduced. Further, some important properties of the resolvent operator associated with the (α,ρ)-XOR-NODSM operator, supported by a well constructed example, have been given. As an application, we have considered a generalized system of variational inclusion problems involving XOR operator in the setting of real ordered positive Hilbert space. Using the resolvent operator technique, we have proved the existence of solution for the system considered. Furthermore, the approximation solvability of the generalized system of variational inclusion problems involving the XOR operator has been studied. The results presented in this paper can be treated as the refinement and generalization of many known results present in the literature in this direction.

Keywords. Variational inclusion, ⊕ operation, Resolvent operator, Algorithm, Convergence

Mathematics Subject Classification (2020). 47H05, 47H10, 47J25, 49J40

1. Introduction

Variational inequalities have been the subject of considerable research owing to its profound contributions in a variety of problems arising in the fields of optimization, economics, transportation, elasticity and applied sciences. The classical variational inequality problem was introduced and studied by Stampacchia [18] in early 1960's. Because of its wide applications, the classical variational inequality problem has been studied and generalized in different directions. Among these generalizations, variational inclusion is of much interest and importance and has been extensively studied in the recent years (see, e.g., Ding and Feng [4], Ding and Lou [5], Fang and Huang [6], Fang *et al*. [7], He *et al*. [8], Malik *et al*. [14], Shan *et al*. [17] and the references therein).

One of the most important and challenging aspect in the theory of variational inequality is the development of an efficient and implementable algorithm for solving variational inequalities and its generalizations. Among several methods proposed for solving variational inclusion problems, resolvent operator technique has been widely used. It is well known that monotonicity of the underlying operators plays a crucial role in the solution of variational inequalities and variational inclusions. In the recent past, several researchers have explored and improved resolvent operator technique to discuss the approximation solvability of several classes of variational inclusions (see, for instance Ding and Feng [4], Ding and Lou [5], Fang and Huang [6], Fang *et al*. [7], Huang and Fang [9], Kazmi *et al*. [10, 11], Malik *et al*. [15], and Zeng *et al*. [19]).

In recent years, the fixed point theory and its applications have been extensively studied in real ordered Banach spaces. Therefore, it is very important and natural to study the generalized nonlinear ordered variational inequalities (inclusions). In 2008, Li [12] introduced the generalized nonlinear ordered variational inequalities and proposed an algorithm to approximate the solution for a class of generalized nonlinear ordered variational inequalities in real ordered Banach spaces. Since then several researchers have used XOR operation and its allied forms to solve some classes of variational inequality and variational inclusion problems in real ordered Hilbert and Banach spaces (see, e.g., Ahmad *et al*. [1–3], Li [13], and Sarfaraz *et al*. [16]).

With inspiration and motivation from recent investigations in this area, we have defined a new type of operator known as (α,ρ)-XOR-NODSM operator and the associated resolvent operator and discussed some of the important properties of the resolvent operator associated with the (α,ρ)-XOR-NODSM operator supported by a well constructed example. As an application, we have considered a generalized system of variational inclusion problems involving XOR operator in the setting of real ordered positive Hilbert space. Using the resolvent operator technique, we proved the existence of solution for the system considered. Furthermore, we have discussed the approximation solvability of the generalized system of variational inclusion problems involving the XOR operator. The results proved in this paper unify and generalize many known results present in the literature in this direction.

2. Preliminaries

Let C be a cone with partial ordering "$\leq$". An ordered Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ is called positive if $0 \leq x$ and $0 \leq y$, then $0 \leq \langle x, y\rangle$ holds. Throughout the paper, $\mathcal{H}_p$ is assumed to be a real ordered positive Hilbert space. We denote by $2^{\mathcal{H}_p}$ (respectively, $C^\star(\mathcal{H}_p)$), the family of nonempty (respectively, compact) subsets of $\mathcal{H}_p$, d is the metric induced by the norm and $\mathcal{D}(\cdot,\cdot)$ is the Hausdorff metric on $C^\star(\mathcal{H}_p)$.

We recall some known concepts and results which are needed to prove the main results of this paper.

Definition 2.1 ([2]). A nonempty closed convex subset C of $\mathcal{H}_p$ is said to be a cone if:

(i) for any $x \in C$ and any $\lambda > 0$, $\lambda x \in C$;

(ii) for $x \in C$ and $-x \in C$, then $x = 0$.

Definition 2.2 ([2]). Let C be the cone, then:

(i) C is called a normal cone if there exists a constant $\lambda_N > 0$ such that $0 \le x \le y$ implies $\|x\| \le \lambda_N \|y\|$, for all $x, y \in \mathcal{H}_p$;

(ii) for any $x, y \in \mathcal{H}_p$, $x \le y$ if and only if $y - x \in C$;

(iii) x and y are said to be comparative to each other if either $x \le y$ or $y \le x$ holds and is denoted by $x \propto y$.

Definition 2.3 ([2]). For any $x, y \in \mathcal{H}_p$, $lub\{x, y\}$ denotes the least upper bound and $glb\{x, y\}$ denotes the greatest lower bound of the set $\{x, y\}$. Suppose $lub\{x, y\}$ and $glb\{x, y\}$ exist, then some binary operations are given below:

(i) $x \vee y = lub\{x, y\}$;

(ii) $x \wedge y = glb\{x, y\}$;

(iii) $x \oplus y = (x - y) \vee (y - x)$;

(iv) $x \odot y = (x - y) \wedge (y - x)$.

The operations $\vee$, $\wedge$, $\oplus$ and $\odot$ are called OR, AND, XOR and XNOR operations, respectively.

Lemma 2.4 ([2]). *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist such that $(x - y) \propto (y - x)$ and $0 \le (x - y) \vee (y - x)$.*

Lemma 2.5 ([13]). *For any natural number n, $x \propto y_n$ and $y_n \to y^\star$ as $n \to \infty$, then $x \propto y^\star$.*

Proposition 2.6 ([13]). *Let $\oplus$ and $\odot$ be an XOR and XNOR operations, respectively. Then, the following relations hold for all $x, y, u, v, w \in \mathcal{H}_p$ and $\alpha, \beta, \lambda \in \mathbb{R}$:*

(i) $x \odot x = 0$, $x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$;

(ii) $x \propto 0$, *then* $-x \oplus 0 \le x \le x \oplus 0$;

(iii) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;

(iv) $0 \le x \oplus y$, *if* $x \propto y$;

(v) *if* $x \propto y$, *then* $x \oplus y = 0$ *if and only if* $x = y$;

(vi) $(x + y) \odot (u + v) \ge (x \odot u) + (y \odot v)$;

(vii) $(x + y) \odot (u + v) \ge (x \odot v) + (y \odot u)$;

(viii) *if x, y and w are comparative to each other, then* $(x \oplus y) \le (x \oplus w) + (w \oplus y)$;

(ix) $\alpha x \oplus \beta x = |\alpha - \beta| x = (\alpha \oplus \beta) x$, *if* $x \propto 0$.

Proposition 2.7 ([2]). *Let C be a normal cone in $\mathcal{H}_p$ with constant λ_N, then for each $x,y \in \mathcal{H}_p$, the following relations hold:*

(i) $\|0 \oplus 0\| = \|0\| = 0$;

(ii) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$;

(iii) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_N \|x \oplus y\|$;

(iv) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|$.

Definition 2.8 ([2]). Let $F : \mathcal{H}_p \to \mathcal{H}_p$ be a single-valued mapping, then

(i) F is said to be comparison mapping, if for each $x, y \in \mathcal{H}_p$, $x \propto y$ then $F(x) \propto F(y)$, $x \propto F(x)$ and $y \propto F(y)$;

(ii) F is said to be strongly comparison mapping, if F is a comparison mapping and $F(x) \propto F(y)$ if and only if $x \propto y$, for all $x, y \in \mathcal{H}_p$.

Definition 2.9 ([2]). A single-valued mapping $F : \mathcal{H}_p \to \mathcal{H}_p$ is said to be β-ordered compression mapping if F is a comparison mapping and

$$F(x) \oplus F(y) \leq \beta(x \oplus y), \quad \text{for } 0 < \beta < 1.$$

Definition 2.10 ([2]). Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. Then:

(i) M is said to be a comparison mapping if for any $v_x \in M(x)$, $x \propto v_x$, and if $x \propto y$, then for $v_x \in M(x)$ and $v_y \in M(y)$, $v_x \propto v_y$, for all $x, y \in \mathcal{H}_p$;

(ii) A comparison mapping M is said to be α-non-ordinary difference mapping if there exists a constant $\theta > 0$ such that:

$$(v_x \oplus v_y) \oplus \alpha(x \oplus y) = 0 \text{ holds, for all } x, y \in \mathcal{H}_p, v_x \in M(x) \text{ and } v_y \in M(y);$$

(iii) A comparison mapping M is said to be θ-ordered rectangular if there exists a constant $\theta > 0$ such that:

$$\langle v_x \odot v_y, -(x \oplus y)\rangle \geq \theta \|x \oplus y\|^2 \text{ holds, for all } x, y \in \mathcal{H}_p,\ v_x \in M(x) \text{ and } v_y \in M(y).$$

Definition 2.11 ([1]). Let $A, B : \mathcal{H}_p \to \mathcal{H}_p$ and $H : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be single-valued mappings. Then $\forall\ x, y \in \mathcal{H}_p$, then H is said to be:

(i) t_1-ordered compression mapping in the first argument, if

$$H(x,\cdot) \oplus H(y,\cdot) \leq t_1(x \oplus y), \quad 0 < t_1 < 1;$$

(ii) t_2-ordered compression mapping in the second argument, if

$$H(\cdot,x) \oplus H(\cdot,y) \leq t_2(x \oplus y), \quad 0 < t_2 < 1;$$

(iii) k_1-ordered compression mapping with respect to A, if

$$H(A(x),\cdot) \oplus H(A(y),\cdot) \leq k_1(x \oplus y), \quad 0 < k_1 < 1;$$

(iv) k_2-ordered compression mapping with respect to B, if

$$H(\cdot,B(x)) \oplus H(\cdot,B(y)) \leq k_2(x \oplus y), \quad 0 < k_2 < 1.$$

Definition 2.12 ([1]). Let $A,B:\mathcal{H}_p\to\mathcal{H}_p$ and $H:\mathcal{H}_p\times\mathcal{H}_p\to\mathcal{H}_p$ be single-valued mappings. Then

(i) H is said to be mixed comparison mapping with respect to A and B, if for each $x,y\in\mathcal{H}_p$, $x\propto y$, then $H(A(x),B(x))\propto H(A(y),B(y))$, $x\propto H(A(x),B(x))$ and $y\propto H(A(y),B(y))$;

(ii) H is said to be mixed comparison mapping with respect to A and B, if for each $x,y\in\mathcal{H}_p$, $x\propto y$, then $H(A(x),B(x))\propto H(A(y),B(y))$, if and only if $x\propto y$.

Definition 2.13 ([1]). A set-valued mapping $M:\mathcal{H}_p\to 2^{\mathcal{H}_p}$ is said to be ρ-XOR-ordered strongly monotone compression mapping if $x\propto y$, then there exists a constant $\rho>0$ such that:

$$\rho(v_x\oplus v_y)\ge x\oplus y,\quad \forall\ x,y\in\mathcal{H}_p,\ v_x\in M(x), v_y\in M(y).$$

Definition 2.14 ([1]). A set-valued mapping $T:\mathcal{H}_p\to C^{\star}(\mathcal{H}_p)$ is said to be $\mathcal{D}$-Lipschitz continuous if for all $x,y\in\mathcal{H}_p$, $x\propto y$, there exists a constant $\lambda_T>0$ such that:

$$\mathcal{D}(T(x),T(y))\le\lambda_T\|x\oplus y\|.$$

Definition 2.15 ([1]). A single-valued mapping $F:\mathcal{H}_p\to\mathcal{H}_p$ is said to be Lipschitz-type continuous if there exists a constant $\lambda_F>0$ such that:

$$\|F(x)\oplus F(y)\|\le\lambda_F\|x\oplus y\|,\quad \forall\ x,y\in\mathcal{H}_p.$$

Definition 2.16. Let $A,B:\mathcal{H}_p\to\mathcal{H}_p$, and $H:\mathcal{H}_p\times\mathcal{H}_p\to\mathcal{H}_p$ be single-valued mappings such that $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B. Then, a set-valued comparison mapping $M:\mathcal{H}_p\times\mathcal{H}_p\to 2^{\mathcal{H}_p}$ is said to be (α,ρ)-XOR-NODSM if M is an α-non-ordinay difference mapping and ρ-XOR-ordered strongly monotone compression mapping and $[H(A,B)\oplus\rho M(\cdot,\zeta)](\mathcal{H}_p)=\mathcal{H}_p$, for some fixed $\zeta\in\mathcal{H}_p$ and $\rho>0$.

Definition 2.17. Let $A,B:\mathcal{H}_p\to\mathcal{H}_p$, and $H:\mathcal{H}_p\times\mathcal{H}_p\to\mathcal{H}_p$ be single-valued mappings such that $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B and $M:\mathcal{H}_p\times\mathcal{H}_p\to 2^{\mathcal{H}_p}$ be (α,ρ)-XOR-NODSM mapping. Then the generalized resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}:\mathcal{H}_p\to\mathcal{H}_p$ is defined for fixed $\zeta\in\mathcal{H}_p$ as:

$$\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(\omega)=[H(A,B)\oplus\rho M(\cdot,\zeta)]^{-1}(\omega),\quad \forall\ \omega\in\mathcal{H}_p. \tag{2.1}$$

Now, we discuss some properties of the generalized resolvent operator.

Proposition 2.18. *Let $A,B:\mathcal{H}_p\to\mathcal{H}_p, H:\mathcal{H}_p\times\mathcal{H}_p\to\mathcal{H}_p$ be single-valued mappings such that $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B. Let $M:\mathcal{H}_p\times\mathcal{H}_p\to 2^{\mathcal{H}_p}$ is the set-valued θ-ordered rectangular mapping with $\rho\theta>|k_1-k_2|$. Then, the generalized resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}:\mathcal{H}_p\to\mathcal{H}_p$ is single-valued.*

Proof. For any given $u\in\mathcal{H}_p$ and $\rho>0$, let $x,y\in[H(A,B)\oplus\rho M(\cdot,\zeta)]^{-1}(u)$. Then,

$$v_x=\frac{1}{\rho}[u\oplus H(A(x),B(x))]\in M(x,\zeta)\quad\text{and}\quad v_y=\frac{1}{\rho}[u\oplus H(A(y),B(y))]\in M(y,\zeta).$$

In view of (i) and (ii) of Proposition 2.6, we have

$$\begin{aligned}
v_x \odot v_y &= \frac{1}{\rho}[u \oplus H(A(x),B(x))] \odot \frac{1}{\rho}[u \oplus H(A(y),B(y))] \\
&= \frac{1}{\rho}\{[u \oplus H(A(x),B(x))] \odot [u \oplus H(A(y),B(y))]\} \\
&= -\frac{1}{\rho}\{[u \oplus H(A(x),B(x))] \oplus [u \oplus H(A(y),B(y))]\} \\
&= -\frac{1}{\rho}\{(u \oplus u) \oplus [H(A(x),B(x)) \oplus H(A(y),B(y))]\} \\
&= -\frac{1}{\rho}\{0 \oplus [H(A(x),B(x)) \oplus H(A(y),B(y))]\} \\
&\le -\frac{1}{\rho}[H(A(x),B(x)) \oplus H(A(y),B(y))] \\
&\le -\frac{1}{\rho}\{[H(A(x),B(x)) \oplus H(A(x),B(y))] \oplus [H(A(x),B(y)) \oplus H(A(y),B(y))]\}. \qquad (2.2)
\end{aligned}$$

Using the fact that M is θ-ordered rectangular mapping, $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B and using (2.2), we have

$$\begin{aligned}
\theta\|x \oplus y\|^2 &\le \langle v_x \odot v_y, -(x \oplus y)\rangle \\
&\le \left\langle -\frac{1}{\rho}\{[H(A(x),B(x)) \oplus H(A(x),B(y))] \oplus [H(A(x),B(y)) \oplus H(A(y),B(y))]\}, -(x \oplus y)\right\rangle \\
&\le \frac{1}{\rho}\{\langle H(A(x),B(x)) \oplus H(A(x),B(y)), x \oplus y\rangle \oplus \langle H(A(x),B(y)) \oplus H(A(y),B(y)), x \oplus y\rangle\} \\
&\le \frac{1}{\rho}\langle k_1(x \oplus y), x \oplus y\rangle \oplus \langle k_2(x \oplus y), x \oplus y\rangle \\
&\le \frac{|k_1 - k_2|}{\rho}\|x \oplus y\|^2.
\end{aligned}$$

i.e.,

$$\left(\theta - \frac{|k_1 - k_2|}{\rho}\right)\|x \oplus y\|^2 \le 0, \quad \text{for } \theta > \frac{|k_1 - k_2|}{\rho},$$

which shows that $\|x \oplus y\| = 0$, which implies $x \oplus y = 0$.
Therefore, $x = y$, that is the resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$ is single-valued for $\rho\theta > |k_1 - k_2|$. □

Proposition 2.19. *Let $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an (α,ρ)-XOR-NODSM set-valued mapping with respect to $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$ such that $H(\cdot,\cdot)$ is mixed strongly comparison mapping with respect to A and B. Then, the generalized resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$ is a comparison mapping.*

Proof. Since M is (α,ρ)-XOR-NODSM set-valued mapping with respect to $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$, thus M is α-non-ordinary difference as well as ρ-XOR-ordered strongly monotone compression mapping with respect to $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$.

For any $x,y \in \mathcal{H}_p$, let $x \propto y$,

$$v_x^* = \frac{1}{\rho}\left[x \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right] \in M\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x),\zeta\right) \tag{2.3}$$

and

$$v_y^* = \frac{1}{\rho}\left[y \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right] \in M\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y),\zeta\right). \tag{2.4}$$

Since M is ρ-XOR-ordered strongly monotone compression mapping with respect to $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$, therefore using (2.3) and (2.4), we have

$$\begin{aligned}
(x \oplus y) &\le \rho(v_x^* \oplus v_y^*)\\
&\le \frac{\rho}{\rho}\left\{\left[x \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right] \oplus \left[y \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\right\}\\
&\le (x \oplus y) \oplus \left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &\le H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\\
0 &\le \left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) - H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\\
&\quad \vee \left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right) - H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right].
\end{aligned}$$

It follows that either

$$0 \le \left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) - H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]$$

or

$$0 \le \left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right) - H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right]$$

Thus, in both cases, we have

$$H(A,B)(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)) \propto H(A,B)(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)).$$

Since $H(\cdot,\cdot)$ is mixed strongly comparison mapping with respect to A,B and $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$, thus we have, $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \propto \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)$, i.e., the resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$ is a comparison mapping. □

Proposition 2.20. *Let the mappings A,B,H,M be same as defined in Proposition* 2.18*, then the generalized resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$ is $\frac{1}{\rho\theta-(k_1+k_2)}$-Lipschitz-type continuous for $\rho\theta > (k_1+k_2)$, i.e.,*

$$\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\| \le \frac{1}{\rho\theta-(k_1+k_2)}\|x \oplus y\|, \ \forall\ x,y \in \mathcal{H}_p.$$

Proof. Let $x,y \in \mathcal{H}_p$, $x \propto y$, and

$$v_x^* = \frac{1}{\rho}\left[x \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right] \in M\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x),\zeta\right)$$

and

$$v_y^* = \frac{1}{\rho}\left[y \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right] \in M\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y),\zeta\right).$$

Now,

$$\begin{aligned}v_x^*\oplus v_y^* &= \frac{1}{\rho}\left\{\left[x\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right]\oplus\left[y\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\right\}\\ &= \frac{1}{\rho}\left\{(x\oplus y)\oplus\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\right\}.\end{aligned} \tag{2.5}$$

Since $M(\cdot,\zeta)$ is θ-ordered rectangular mapping and using (2.5), for any $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\in M(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x),\zeta)$ and $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\in M(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y),\zeta)$, we have

$$\begin{aligned}&\theta\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|^2\\ &\le\left\langle v_x^*\odot v_x^*,-(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y))\right\rangle\\ &\le\left\langle v_x^*\oplus v_x^*,\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\rangle\\ &=\frac{1}{\rho}\left\langle(x\oplus y)\oplus\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right.\right.\\ &\qquad\left.\left.\oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right],\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\rangle\\ &\le\frac{1}{\rho}\left\{\left\|(x\oplus y)\oplus\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right.\right.\right.\\ &\qquad\left.\left.\left.\oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\right\|\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\right\}\\ &\le\frac{1}{\rho}\left\{\left\|(x\oplus y)-\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right.\right.\right.\\ &\qquad\left.\left.\left.\oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\right\|\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\right\}\\ &\le\frac{1}{\rho}\left\{\left[\|x\oplus y\|+\left\|H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right.\right.\right.\\ &\qquad\left.\left.\left.\oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right\|\right]\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\right\}\\ &\le\frac{1}{\rho}\left\{\|x\oplus y\|\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|+\left\|H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right.\right.\\ &\qquad\left.\left.\oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right\|\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\right\}.\end{aligned} \tag{2.6}$$

Since $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B, we have

$$\begin{aligned}&\left\|H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right\|\\ &=\left\|\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right]\right.\\ &\quad\left.\oplus\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\right\|\\ &\le\left\|\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right]\right.\\ &\quad\left.-\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\right\|\\ &\le\left\|H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right\|\end{aligned}$$

$$+\left\|H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right\|$$
$$\le (k_1+k_2)\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|.$$

Thus from (2.6), we obtain

$$\theta\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|^2$$
$$\le \frac{1}{\rho}\|x\oplus y\|\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|+\frac{k_1+k_2}{\rho}\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|^2.$$

This implies,

$$\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\le\frac{1}{\rho\theta-(k_1+k_2)}\|x\oplus y\|,\quad\forall\ x,y\in\mathcal{H}_p,$$

for $\rho\theta>(k_1+k_2)$.

This completes the proof. □

Example 2.1. Let $\mathcal{H}_p=[0,\infty)\times[0,\infty)$ with the usual inner product and norm, and let $C=[0,1]\times[0,1]$ be a normal cone. Let $A,B:\mathcal{H}_p\to\mathcal{H}_p$ and $H:\mathcal{H}_p\times\mathcal{H}_p\to\mathcal{H}_p$ be defined by

$$A(x)=\left(\frac{x_1}{9}+3,\frac{x_2}{9}+6\right),\quad B(x)=\left(\frac{x_1}{3}+1,\frac{x_2}{3}+2\right)$$

and

$$H(A(x),B(x))=\frac{A(x)}{3}\oplus B(x),\quad\forall\ x=(x_1,x_2)\in\mathcal{H}_p.$$

For $x=(x_1,x_2)$, $y=(y_1,y_2)\in\mathcal{H}_p$, $x\propto y$, we have

$$\begin{aligned}
H(A(x),u)\oplus H(A(y),u)&=\left(\frac{A(x)}{3}\oplus u\right)\oplus\left(\frac{A(y)}{3}\oplus u\right)\\
&=\frac{1}{3}(A(x)\oplus A(y))\\
&=\frac{1}{3}[(A(x)-A(y))\vee(A(y)-A(x))]\\
&=\frac{1}{3}\left[\left\{\left(\frac{x_1}{9}+3,\frac{x_2}{9}+6\right)-\left(\frac{y_1}{9}+3,\frac{y_2}{9}+6\right)\right\}\right.\\
&\qquad\left.\vee\left\{\left(\frac{y_1}{9}+3,\frac{y_2}{9}+6\right)-\left(\frac{x_1}{9}+3,\frac{x_2}{9}+6\right)\right\}\right]\\
&=\frac{1}{3}\left[\left(\frac{x_1-y_1}{9},\frac{x_2-y_2}{9}\right)\vee\left(\frac{y_1-x_1}{9},\frac{y_2-x_2}{9}\right)\right]\\
&=\frac{1}{27}[(x-y)\vee(y-x)]\\
&=\frac{1}{27}(x\oplus y)\\
&\le\frac{1}{24}(x\oplus y).
\end{aligned}$$

Hence, H is $\frac{1}{24}$-ordered compression mapping with respect to A. Similarly, we can show that H is $\frac{1}{2}$-ordered compression mapping with respect to B.

Suppose that the set-valued mapping $M:\mathcal{H}_p\to 2^{\mathcal{H}_p}$ be defined by

$$M(x)=\{(3x_1,3x_2)\},\quad\forall\ x=(x_1,x_2)\in\mathcal{H}_p.$$

It can be easily verified that M is a comparison mapping, 1-XOR-ordered strongly monotone compression mapping and 3-non-ordinary difference mapping. Further, it is clear that for $\rho = 1$, $[H(A,B) + \rho M](\mathcal{H}_p) = \mathcal{H}_p$. Hence, M is an $(3,1)$-XOR-NODSM strongly monotone compression mapping.

Let $v_x = (3x_1, 3x_2) \in M(x)$ and $v_y = (3y_1, 3y_2) \in M(y)$, then

$$\begin{aligned}\langle v_x \odot v_y, -(x \oplus y)\rangle &= \langle v_x \oplus v_y, x \oplus y\rangle \\ &= \langle 3x \oplus 3y, x \oplus y\rangle \\ &= 3\langle x \oplus y, x \oplus y\rangle \\ &= 3\|x \oplus y\|^2 \\ &\geq 2\|x \oplus y\|^2,\end{aligned}$$

i.e.,

$$\langle v_x \odot v_y, -(x \oplus y)\rangle \geq 2\|x \oplus y\|^2, \quad \forall\ x, y \in \mathcal{H}_p.$$

Thus, M is 2-ordered rectangular comparison mapping.

The resolvent operator defined by (2.1) is given by

$$\mathcal{R}_{\rho,M}^{H(A,B)}(x) = \left(\frac{27x_1}{73}, \frac{27x_2}{73}\right), \quad \forall\ x = (x_1, x_2) \in \mathcal{H}_p.$$

It is easy to verify that the resolvent operator defined above is comparison and single-valued mapping.

Further,

$$\begin{aligned}\left\|\mathcal{R}_{\rho,M}^{H(A,B)}(x) \oplus \mathcal{R}_{\rho,M}^{H(A,B)}(y)\right\| &= \left\|\frac{27x}{73} \oplus \frac{27y}{73}\right\| \\ &= \frac{27}{73}\|x \oplus y\| \\ &\leq \frac{24}{35}\|x \oplus y\|.\end{aligned}$$

i.e.,

$$\left\|\mathcal{R}_{\rho,M}^{H(A,B)}(x) \oplus \mathcal{R}_{\rho,M}^{H(A,B)}(y)\right\| \leq \frac{24}{35}\|x \oplus y\|, \quad \forall\ x, y \in \mathcal{H}_p.$$

This shows that the resolvent operator is $\mathcal{R}_{\rho,M}^{H(A,B)}$ is $\frac{24}{35}$-Lipschitz-type-continuous.

3. Generalized System of Set-Valued Variational Inclusion Problems and Associated Fixed Point Formulation

Let $A, B, g_i, p_i, G_i : \mathcal{H}_p \to \mathcal{H}_p, N_i, H : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p, F_i : \mathcal{H}_p \times \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$, for $i = 1,2$ be single-valued mappings and $S, T : \mathcal{H}_p \to C^*(\mathcal{H}_p)$, $M_i : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be set-valued mappings. Then, for any fixed $\zeta \in \mathcal{H}_p$, we consider the following *generalized system of set-valued variational inclusion problems* (in short, GSSVIP):

Find $u, v \in \mathcal{H}_p, x \in S(u), y \in T(u)$ such that

$$\left.\begin{aligned}&0 \in N_1((g_1 - p_1)(x), G_1(y)) \oplus F_1(u, x, y) \oplus M_1(u, \zeta), \\ &0 \in N_2((g_2 - p_2)(y), G_2(x)) \oplus F_2(v, x, y) \oplus M_2(v, \zeta).\end{aligned}\right\} \tag{3.1}$$

Here, we remark that the problem considered in [1] can be deduced from GSSVIP (3.1) by taking $N_1 \equiv N_2 \equiv N$, $(g_1 - p_1) \equiv (g_2 - p_2) \equiv I$, $G_1 \equiv G_2 \equiv I$, $F_1 \equiv F_2 \equiv 0$ and $M_1(u,\zeta) = M_2(v,\zeta) = M(u)$ and the problem considered in [3] can be obtained by taking $N_1 \equiv N_2 \equiv 0$, $F_1 \equiv F_2 \equiv F$ and $M_1(u,\zeta) = M_2(v,\zeta) = M(u)$. Furthermore, under appropriate selections of different mappings and the underlying space $\mathcal{H}_p$ in GSSVIP (3.1), one can get many new and known classes of variational inequalities and variational inclusions (see, e.g., Ahmad *et al*. [1–3], Li [13] and the related references cited therein).

Lemma 3.1. *Let $u, v \in \mathcal{H}_p, x \in S(u)$ and $y \in T(u)$, then (u,v,x,y) is a solution of GSSVIP* (3.1) *involving $\oplus$ operation if and only if it satisfies:*

$$u = \mathcal{R}^{H(A,B)}_{\rho_1,M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x),G_1(y)) \oplus F_1(u,x,y)] \oplus H(A,B)(u)\} \tag{3.2}$$

and

$$v = \mathcal{R}^{H(A,B)}_{\rho_2,M_2(\cdot,\zeta)}\{\rho_2[N_2((g_2-p_2)(y),G_2(x)) \oplus F_2(v,x,y)] \oplus H(A,B)(v)\}, \tag{3.3}$$

where $\mathcal{R}^{H(A,B)}_{\rho_i,M_i(\cdot,\zeta)} = [H(A,B) \oplus \rho_i M_i(\cdot,\zeta)]^{-1}$, for $i = 1,2$ and $\rho_i > 0$.

Proof. Using the definition of the generalized resolvent operator, we have by (3.2)

$$\begin{aligned}
& u = \mathcal{R}^{H(A,B)}_{\rho_1,M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x),G_1(y)) \oplus F_1(u,x,y)] \oplus H(A,B)(u)\} \\
\Longleftrightarrow\ & u = [H(A,B) \oplus \rho_1 M_1(\cdot,\zeta)]^{-1}\{\rho_1[N_1((g_1-p_1)(x),G_1(y)) \oplus F_1(u,x,y)] \oplus H(A,B)(u)\} \\
\Longleftrightarrow\ & H(A,B)(u) \oplus \rho_1 M_1(u,\zeta) \ni \rho_1[N_1((g_1-p_1)(x),G_1(y)) \oplus F_1(u,x,y)] \oplus H(A,B)(u) \\
\Longleftrightarrow\ & 0 \in N_1((g_1-p_1)(x),G_1(y)) \oplus F_1(u,x,y) \oplus M_1(u,\zeta).
\end{aligned}$$

Similarly, using (3.3), we can prove that

$$0 \in \{N_2((g_2-p_2)(y),G_2(x)) \oplus F_2(v,x,y)\} \oplus M_2(v,\zeta). \qquad \square$$

4. Iterative Algorithm, Existence Result and Convergence Analysis

Lemma 3.1 along with Nadler's Theorem allows us to suggest the following iterative algorithm for finding the approximate solution of GSSVIP (3.1).

Iterative Algorithm 4.1. *For any arbitrary $u_0, v_0 \in \mathcal{H}_p$, choose $x_0 \in S(u_0)$, $y_0 \in T(u_0)$, let*

$$u_1 = (1-\alpha)u_0 + \alpha\mathcal{R}^{H(A,B)}_{\rho_1,M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x_0),G_1(y_0)) \oplus F_1(u_0,x_0,y_0)] \oplus H(A,B)(u_0)\}$$

and

$$v_1 = (1-\alpha)v_0 + \alpha\mathcal{R}^{H(A,B)}_{\rho_2,M_2(\cdot,\zeta)}\{\rho_2[N_1((g_2-p_2)(y_0),G_2(x_0)) \oplus F_2(v_0,x_0,y_0)] \oplus H(A,B)(v_0)\}.$$

Since $x_0 \in S(u_0)$, $y_0 \in T(u_0)$, by Nadler's Theorem, there exists $x_1 \in S(u_1)$, $y_1 \in T(u_1)$ and using Proposition 2.7, *we have*

$$\|x_0 \oplus x_1\| \le \|x_0 - x_1\| \le (1+1)\mathcal{D}(S(u_0),S(u_1))$$

and

$$\|y_0 \oplus y_1\| \le \|y_0 - y_1\| \le (1+1)\mathcal{D}(T(u_0),T(u_1)),$$

where $\mathcal{D}(\cdot,\cdot)$ *is the Hausdorff metric on* $C^\star(\mathcal{H}_P)$. *Let*

$$u_2=(1-\alpha)u_1+\alpha\mathcal{R}^{H(A,B)}_{\rho_1,M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x_1),G_1(y_1))\oplus F_1(u_1,x_1,y_1)]\oplus H(A,B)(u_1)\}$$

and

$$v_2=(1-\alpha)v_1+\alpha\mathcal{R}^{H(A,B)}_{\rho_2,M_2(\cdot,\zeta)}\{\rho_2[N_2((g_2-p_2)(y_1),G_2(x_1))\oplus F_2(v_1,x_1,y_1)]\oplus H(A,B)(v_1)\}.$$

Again by Nadler's Theorem, there exists $x_2\in S(u_2)$, $y_2\in T(u_2)$ *such that*

$$\|x_1\oplus x_2\|\le\|x_1-x_2\|\le(1+2^{-1})\mathcal{D}(S(u_1),S(u_2))$$

and

$$\|y_1\oplus y_2\|\le\|y_1-y_2\|\le(1+2^{-1})\mathcal{D}(T(u_1),T(u_2)).$$

Continuing the process inductively, we have the following scheme:

$$u_{n+1}=(1-\alpha)u_n+\alpha\mathcal{R}^{H(A,B)}_{\rho_1,M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x_n),G_1(y_n))\oplus F_1(u_n,x_n,y_n)]\oplus H(A,B)(u_n)\}$$

and

$$v_{n+1}=(1-\alpha)v_n+\alpha\mathcal{R}^{H(A,B)}_{\rho_2,M_2(\cdot,\zeta)}\{\rho_2[N_1\big((g_2-p_2)(y_n),G_2(x_n))\oplus F_2(v_n,x_n,y_n)]\oplus H(A,B)(v_n)\}.$$

Since $x_{n+1}\in S(u_{n+1})$, $y_{n+1}\in T(u_{n+1})$ *such that*

$$\|x_n\oplus x_{n+1}\|\le\|x_n-x_{n+1}\|\le(1+(1+n)^{-1})\mathcal{D}(S(u_n),S(u_{n+1}))$$

and

$$\|y_n\oplus y_{n+1}\|\le\|y_n-y_{n+1}\|\le(1+(1+n)^{-1})\mathcal{D}(T(u_n),T(u_{n+1})),$$

where $\alpha\in[0,1]$, $n=0,1,2,\dots$.

Next, we prove the following theorem which ensures the existence of solution of GSSVIP (3.1) and convergence of sequences generated by the Iterative Algorithm 4.1.

Theorem 4.2. *Let* $C\subset\mathcal{H}_p$ *be a normal cone with constant* λ_N. *For* $i=1,2$, *let* $A,B,g_i,p_i:\mathcal{H}_p\to\mathcal{H}_p$ *and* $H,N_i:\mathcal{H}_p\times\mathcal{H}_p\to\mathcal{H}_p$ *be single-valued mappings such that* $H(\cdot,\cdot)$ *is* k_1*-ordered compression mapping with respect to* A *and* k_2*-ordered compression mapping with respect to* B; N_i *be* τ_i*-Lipschitz-type continuous with respect to* (g_i-p_i) *in first argument and* σ_i*-Lipschitz-type continuous with respect to* G_i *in second argument, respectively. Let* $M_i:\mathcal{H}_p\times\mathcal{H}_p\to 2^{\mathcal{H}_p}$ *and* $S,T:\mathcal{H}_p\to C^\star(\mathcal{H}_p)$ *be set-valued mappings such that* M_i *is* (α_i,ρ_i)*-XOR-NODSM and* θ_i*-ordered rectangular mapping, respectively;* S *is* γ_1*-*$\mathcal{D}$*-Lipschitz continuous and* T *is* γ_2*-*$\mathcal{D}$*-Lipschitz continuous. Further, let* $F_i:\mathcal{H}_p\times\mathcal{H}_p\times\mathcal{H}_p\to\mathcal{H}_p$ *be* (l_i,m_i,n_i)*-Lipschitz-type continuous in first, second and third arguments, respectively. If* $u_{n+1}\propto u_n$, $v_{n+1}\propto v_n$, *for* $n=0,1,2,\dots$ *and following conditions are satisfied:* $\varphi<1$ *and* $\vartheta<1$, *where*

$$\left.\begin{aligned}&\varphi=\{\lambda_N(1-\alpha)+\alpha\lambda_N\Theta(k_1+k_2)+\alpha\lambda_N|\rho_1|\Theta[\gamma_1(\tau_1+m_1)+\gamma_2(\sigma_1+n_1)+l_1]\},\\&\vartheta=\{\lambda_N(1-\alpha)+\alpha\lambda_N\Theta'+\alpha\lambda_N\Theta'(k_1+k_2)\},\\&\Theta=\tfrac{1}{\rho_1\theta_1-(k_1+k_2)},\ \Theta'=\tfrac{1}{\rho_2\theta_2-(k_1+k_2)}.\end{aligned}\right\}\tag{4.1}$$

Then GSSVIP (3.1) *has a solution* (u,v,x,y), *where* $u,v\in\mathcal{H}_p$, $x\in S(u)$, $y\in T(u)$. *Also, the Iterative sequences* $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, $\{y_n\}$ *generated by the Iterative Algorithm* 4.1 *converge strongly to* u,v,x,y, *respectively.*

Proof. By Algorithm 4.1 and Proposition 2.6, we have

$$
\begin{aligned}
0 &\le u_{n+1} \oplus u_n \\
&= \Big[(1-\alpha)u_n + \alpha \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x_n), G_1(y_n)) \oplus F_1(u_n, x_n, y_n)] \oplus H(A,B)(u_n)\}\Big] \\
&\quad \oplus \Big[(1-\alpha)u_{n-1} + \alpha \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x_{n-1}), G_1(y_{n-1})) \oplus F_1(u_{n-1}, x_{n-1}, y_{n-1})] \\
&\quad \oplus H(A,B)(u_{n-1})\}\Big] \\
&= (1-\alpha)(u_n \oplus u_{n-1}) + \alpha \Big[\mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x_n), G_1(y_n)) \oplus F_1(u_n, x_n, y_n)] \\
&\quad \oplus H(A,B)(u_n)\} \oplus \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x_{n-1}), G_1(y_{n-1})) \oplus F_1(u_{n-1}, x_{n-1}, y_{n-1})] \\
&\quad \oplus H(A,B)(u_{n-1})\}\Big].
\end{aligned}
$$

Now, using Proposition 2.7 and Lipschitz-type continuity of the generalized resolvent operator, we have

$$
\begin{aligned}
&\|u_{n+1} \oplus u_n\| \\
&\le \lambda_N \Big\|(1-\alpha)(u_n \oplus u_{n-1}) + \alpha \Big[\mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x_n), G_1(y_n)) \oplus F_1(u_n, x_n, y_n)] \\
&\quad \oplus H(A,B)(u_n)\} \oplus \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x_{n-1}), G_1(y_{n-1})) \oplus F_1(u_{n-1}, x_{n-1}, y_{n-1})] \\
&\quad \oplus H(A,B)(u_{n-1})\}\Big]\Big\| \\
&\le \lambda_N(1-\alpha)\|u_n \oplus u_{n-1}\| + \alpha\lambda_N \Big\|\mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x_n), G_1(y_n)) \oplus F_1(u_n, x_n, y_n)] \\
&\quad \oplus H(A,B)(u_n)\} \oplus \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\{\rho_1[N_1((g_1-p_1)(x_{n-1}), G_1(y_{n-1})) \oplus F_1(u_{n-1}, x_{n-1}, y_{n-1})] \\
&\quad \oplus H(A,B)(u_{n-1})\}\Big\| \\
&\le \lambda_N(1-\alpha)\|u_n \oplus u_{n-1}\| + \alpha\lambda_N\Theta|\rho_1|\|N_1((g_1-p_1)(x_n), G_1(y_n) \\
&\quad \oplus N_1((g_1-p_1)(x_{n-1}), G_1(y_{n-1})\| + \alpha\lambda_N\Theta|\rho_1|\|F_1(u_n, x_n, y_n) \oplus F_1(u_{n-1}, x_{n-1}, y_{n-1})\| \\
&\quad + \alpha\lambda_N\Theta\|H(A,B)(u_n) \oplus H(A,B)(u_{n-1})\|.
\end{aligned} \tag{4.2}
$$

Since, XOR operator is associative, $N_1 : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ is τ_1-Lipschitz-type continuous with respect to $(g_1 - p_1)$ in first argument and σ_1-Lipschitz-type continuous with respect to G_1 in second argument, respectively, and S, T are γ_1, γ_2-$\mathcal{D}$-Lipschitz-type continuous,respectively, therefore in view of Algorithm 4.1, we have

$$
\begin{aligned}
&\|N_1((g_1-p_1)(x_n), G_1(y_n) \oplus N_1((g_1-p_1)(x_{n-1}), G_1(y_{n-1})\| \\
&\quad \le \|N_1((g_1-p_1)(x_n), G_1(y_n) \oplus N_1((g_1-p_1)(x_{n-1}), G_1(y_n)\| \\
&\quad\quad + \|N_1((g_1-p_1)(x_{n-1}), G_1(y_n) \oplus N_1((g_1-p_1)(x_{n-1}), G_1(y_{n-1})\| \\
&\quad \le \tau_1\|x_n \oplus x_{n-1}\| + \sigma_1\|y_n \oplus y_{n-1}\| \\
&\quad \le \tau_1(1+n^{-1})\mathcal{D}(S(u_n), S(u_{n-1})) + \sigma_1(1+n^{-1})\mathcal{D}(T(u_n), T(u_{n-1})) \\
&\quad \le \tau_1\gamma_1(1+n^{-1})\|u_n - u_{n-1}\| + \sigma_1\gamma_2(1+n^{-1})\|u_n - u_{n-1}\| \\
&\quad = [(\tau_1\gamma_1 + \sigma_1\gamma_2)(1+n^{-1})]\|u_n - u_{n-1}\|.
\end{aligned} \tag{4.3}
$$

Since $F_1 : \mathcal{H}_p \times \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ is (l_1, m_1, n_1)-Lipschitz-type continuous in first, second and third arguments, respectively, and using Algorithm 4.1, we have

$$\begin{aligned}
&\|F_1(u_n,x_n,y_n) \oplus F_1(u_{n-1},x_{n-1},y_{n-1})\| \\
&\quad \le \|F_1(u_n,x_n,y_n) \oplus F_1(u_{n-1},x_n,y_n)\| + \|F_1(u_{n-1},x_n,y_n) \oplus F_1(u_{n-1},x_{n-1},y_n)\| \\
&\qquad + \|F_1(u_{n-1},x_{n-1},y_n) \oplus F_1(u_{n-1},x_{n-1},y_{n-1})\| \\
&\quad \le l_1\|u_n \oplus u_{n-1}\| + m_1\|x_n \oplus x_{n-1}\| + n_1\|y_n \oplus y_{n-1}\| \\
&\quad \le l_1\|u_n - u_{n-1}\| + m_1\gamma_1(1+n^{-1})\|u_n - u_{n-1}\| + n_1\gamma_2(1+n^{-1})\|u_n - u_{n-1}\| \\
&\quad = [l_1 + (m_1\gamma_1 + n_1\gamma_2)(1+n^{-1})]\|u_n - u_{n-1}\|. \qquad (4.4)
\end{aligned}$$

Since $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B, we have

$$\begin{aligned}
\|H(A,B)(u_n) \oplus H(A,B)(u_{n-1})\| &= \|H(A(u_n),B(u_n)) \oplus H(A(u_{n-1}),B(u_{n-1}))\| \\
&\le \|H(A(u_n),B(u_n)) \oplus H(A(u_{n-1}),B(u_n))\| \\
&\quad + \|H(A(u_{n-1}),B(u_n)) \oplus H(A(u_{n-1}),B(u_{n-1}))\| \\
&\le k_1\|u_n \oplus u_{n-1}\| + k_2\|u_n \oplus u_{n-1}\| \\
&\le (k_1+k_2)\|u_n - u_{n-1}\|. \qquad (4.5)
\end{aligned}$$

Using (4.3)-(4.5) in (4.2), we have

$$\begin{aligned}
\|u_{n+1} \oplus u_n\| &\le \{\lambda_N(1-\alpha) + \alpha\lambda_N\Theta|\rho_1|[(\tau_1\gamma_1 + \sigma_1\gamma_2)(1+n^{-1})] \\
&\quad + \alpha\lambda_N\Theta|\rho_1|[l_1 + (m_1\gamma_1 + n_1\gamma_2)(1+n^{-1})] + \alpha\lambda_N\Theta(k_1+k_2)\}\|u_n - u_{n-1}\| \\
&= \{\lambda_N(1-\alpha) + \alpha\lambda_N\Theta(k_1+k_2) \\
&\quad + \alpha\lambda_N|\rho_1|\Theta[(\gamma_1(\tau_1+m_1) + \gamma_2(\sigma_1+n_1))(1+n^{-1}) + l_1]\}\|u_n - u_{n-1}\|.
\end{aligned}$$

Since $u_{n+1} \propto u_n$, $n = 0,1,2,\dots$, we have

$$\|u_{n+1} - u_n\| \le \varphi_n\|u_n - u_{n-1}\|,$$

where

$$\varphi_n = \{\lambda_N(1-\alpha) + \alpha\lambda_N\Theta(k_1+k_2) + \alpha\lambda_N|\rho_1|\Theta[(\gamma_1(\tau_1+m_1) + \gamma_2(\sigma_1+n_1))(1+n^{-1}) + l_1]\}.$$

Let

$$\varphi = \{\lambda_N(1-\alpha) + \alpha\lambda_N\Theta(k_1+k_2) + \alpha\lambda_N|\rho_1|\Theta[\gamma_1(\tau_1+m_1) + \gamma_2(\sigma_1+n_1) + l_1]\}.$$

We know that $\varphi_n \to \varphi$ as $n \to \infty$. It follows from condition (4.1) that $0 < \varphi < 1$, and consequently $\{u_n\}$ is a Cauchy sequence in $\mathcal{H}_p$ and since $\mathcal{H}_p$ is complete, there exists $u \in \mathcal{H}_p$ such that $u_n \to u$ as $n \to \infty$. Proceeding the same way, we arrive at

$$\begin{aligned}
\|v_{n+1} \oplus v_n\| &\le \alpha\lambda_N|\rho_2|\Theta'\{[\gamma_1(\sigma_2+m_2) + \gamma_2(\tau_2+n_2)](1+n^{-1})\}\|u_n - u_{n-1}\| \\
&\quad + \{\lambda_N(1-\alpha) + \alpha\lambda_N\Theta' + \alpha\lambda_N\Theta'(k_1+k_2)\}\|v_n - v_{n-1}\|.
\end{aligned}$$

As $v_{n+1} \propto v_n$, $n = 0,1,2,\dots$, we have

$$\|v_{n+1} - v_n\| \le \alpha\lambda_N|\rho_2|\Theta'\{[\gamma_1(\sigma_2+m_2) + \gamma_2(\tau_2+n_2)](1+n^{-1})\}\|u_n - u_{n-1}\| + \vartheta\|v_n - v_{n-1}\|,$$

where

$$\vartheta = \{\lambda_N(1-\alpha)+\alpha\lambda_N\Theta'+\alpha\lambda_N\Theta'(k_1+k_2)\}.$$

Using the fact that $\{u_n\}$ is a Cauchy sequence and $0<\vartheta<1$, by condition (4.1), it follows that $\{v_n\}$ is a Cauchy sequence in $\mathcal{H}_p$ and since $\mathcal{H}_p$ is complete, there exists $v\in\mathcal{H}_p$ such that $v_n\to v$, as $n\to\infty$.

By Algorithm 4.1 and the $\mathcal{D}$-Lipschitz-type continuity of S and T, we have

$$\left.\begin{aligned}\|x_{n+1}\oplus x_n\| &\le \|x_{n+1}-x_n\| \le (1+(1+n)^{-1})\gamma_1\|u_{n+1}-u_n\|,\\ \|y_{n+1}\oplus y_n\| &\le \|y_{n+1}-y_n\| \le (1+(1+n)^{-1})\gamma_2\|v_{n+1}-v_n\|.\end{aligned}\right\} \tag{4.6}$$

Sequences $\{u_n\}$ and $\{v_n\}$ being Cauchy in $\mathcal{H}_p$, (4.6) implies that $\{x_n\}$ and $\{y_n\}$ are also Cauchy sequence in $\mathcal{H}_p$. Thus, there exist x, y in $\mathcal{H}_p$ such that $x_n\to x$, $y_n\to y$ as $n\to\infty$.
Now, we show that $x\in S(u)$ and $y\in T(u)$. Since $x_n\in S(u_n)$, we have

$$\begin{aligned} d(x,S(u)) &\le \|x-x_n\|+d(x_n,S(u))\\ &\le \|x-x_n\|+\mathcal{D}\big(S(u_n),S(u)\big)\\ &\le \|x-x_n\|+\gamma_1\|u_n-u\|\to 0 \ \text{ as } \ n\to\infty.\end{aligned}$$

Since $S(u)$ is closed, it follows that $x\in S(u)$. Similarly, we can show that $y\in T(u)$. Thus, in view of Lemma 3.1, we conclude that (u,v,x,y) is a solution of GSSVIP (3.1). This completes the proof. □

5. Conclusion

The problems considered in this paper are more general than previously studied problems in ordered spaces. The problem considered in [1] can be deduced from our problem by taking $N_1\equiv N_2\equiv N$, $(g_1-p_1)\equiv(g_2-p_2)\equiv I$, $G_1\equiv G_2\equiv I$, $F_1\equiv F_2\equiv 0$ and $M_1(u,\zeta)=M_2(v,\zeta)=M(u)$ and that considered in [3] can be obtained by taking $N_1\equiv N_2\equiv 0$, $F_1\equiv F_2\equiv F$ and $M_1(u,\zeta)=M_2(v,\zeta)=M(u)$. It is pertinent to mention that the solution of variational inclusions involving $\oplus$ operator is of recent origin and can be exploited to solve various classes of known and new variational inclusions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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