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Research Article

λ-Δ^m-Statistical Convergence on Intuitionistic Fuzzy Normed Spaces

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Abstract. The basic purpose of our work is to define λ-statistical convergence for the generalized difference sequences (i.e. λ-Δ^m-statistical convergence) on *Intuitionistic Fuzzy Normed space* (IFN space). We have proven topological results about this generalized method of sequence convergence. Also, we have given the λ-Δ^m-statistical Cauchy sequences along with its Cauchy criteria of convergence on these spaces.

Keywords. λ-statistical convergence, Difference sequences, Intuitionistic fuzzy normed space

Mathematics Subject Classification (2020). 40A35, 26E50, 40G15, 46S40

1. Introduction

Fuzzy sets are very useful in various branches of natural sciences as well as in engineering where exact solution of the problem is not necessary to exist. The effectiveness of fuzzy sets is seen in complex physical systems where the precise mathematical structure, fast solution or initial estimate fails to exists. Zadeh [22] brought this novel theory and later on numerous admirable research outcomes came across in the literature based on the conception of fuzzy sets ([2, 3, 10, 11, 16]). Park [18] has given the framework of intuitionistic fuzzy metric spaces and afterward Saadati and Park [19] explored a useful concept as an *Intuitionistic Fuzzy Normed space* (IFN space) in the modified form by working on certain conditions like separation condition using t-norm and continuous t-norm. IFN space with its analytic properties and various generalizations becomes a key area of research for providing mathematical structure for real-life situations.

Definition 1.1 ([19]). An IFN space is 5-tuple $(X,\varphi,\vartheta,\otimes,\odot)$ with a vector space X, a continuous t-norm $\otimes$ on $[0,1]$, a continuous t-conorm $\odot$ on $[0,1]$, and two fuzzy sets φ and ϑ on $X\times(0,\infty)$ in which for $x,y\in X$ and $s,t>0$, we have

(i) $\varphi(x,t)+\vartheta(x,t)\le 1$,

(ii) $\varphi(x,t)>0$ and $\vartheta(x,t)<1$,

(iii) $\varphi(x,t)=1$ and $\vartheta(x,t)=0$ iff $x=0$,

(iv) $\varphi(\mu x,t)=\varphi\left(x,\frac{t}{|\mu|}\right)$ and $\vartheta(\mu x,t)=\vartheta\left(x,\frac{t}{|\mu|}\right)$ for $\mu\neq 0$,

(v) $\varphi(x,s)\otimes\varphi(y,t)\le\varphi(x+y,s+t)$ and $\vartheta(x,s)\odot\vartheta(y,t)\ge\vartheta(x+y,s+t)$,

(vi) $\varphi(x,\circ):(0,\infty)\to[0,1]$ and $\vartheta(x,\circ):(0,\infty)\to[0,1]$ are continuous,

(vii) $\lim\limits_{t\to\infty}\varphi(x,t)=1$ and $\lim\limits_{t\to 0}\varphi(x,t)=0$,

(viii) $\lim\limits_{t\to\infty}\vartheta(x,t)=0$ and $\lim\limits_{t\to 0}\vartheta(x,t)=1$.

Here (φ,ϑ) is termed an intuitionistic fuzzy norm.

Example 1.1 ([19]). Let $(X,\|\cdot\|)$ be a normed space. Define

(i) $\mu_1\otimes\mu_2=\mu_1\mu_2$ and $\mu_1\odot\mu_2=\min\{\mu_1+\mu_2,1\}$ for all $\mu_1,\mu_2\in[0,1]$,

(ii) $\varphi(x,t)=\frac{t}{t+\|x\|}$, $\vartheta(y,t)=\frac{\|x\|}{t+\|x\|}$ for $x\in X$ and $t>0$.

Then $(X,\varphi,\vartheta,\otimes,\odot)$ is an IFN space.

The convergence of sequences is also defined by Saadati and Park in [19] on IFN space as:

Definition 1.2 ([19]). Let $(X,\varphi,\vartheta,\otimes,\odot)$ be any IFN space. Then, sequence $x=\{x_k\}$ can be recognized as convergent to $x_0\in X$ corresponding to norm (φ,ϑ) provided with any $t>0$ and every $\epsilon>0$ there exists $k_0\in N$ satisfying $\varphi(x_k-x_0,t)>1-\epsilon$ and $\vartheta(x_k-x_0,t)<\epsilon$ for all $k\ge k_0$. Symbolically, $(\varphi,\vartheta)-\lim x=x_0$.

Karakus *et al.* [13] extended the concept of sequence convergence statistically on IFN space. The term statistical convergence [9] rely on natural density's perception. The expression of natural density for any set A, where $A\subset N$, has given by $\delta(A)=\lim\limits_{n\to\infty}\frac{1}{n}|\{a\le n: a\in A\}|$ where $|\cdot|$ indicates order of the enclosed set. Any sequence $x=\{x_k\}$ converges statistically to some x_0, provided that the set $A(\epsilon)=\{a\le n:|x_k-x_0|>\epsilon\}$ has zero natural density.

Definition 1.3 ([13]). Let $(X,\varphi,\vartheta,\otimes,\odot)$ be any IFN space. Then, sequence $x=\{x_k\}$ can be recognized as statistically convergent to $x_0\in X$ corresponding to norm (φ,ϑ) provided with any $t>0$ and every $\epsilon>0$ satisfying

$$\delta(\{k\in N:\varphi(x_k-x_0,t)\le 1-\epsilon \text{ or } \vartheta(x_k-x_0,t)\ge\epsilon\})=0.$$

Symbolically, $S^{(\varphi,\vartheta)}-\lim x=x_0$.

One of the generalized type of sequence convergence is λ-statistical convergence, which has been presented by Mursaleen [17] using a non-decreasing sequence $\lambda = \{\lambda_n\}$ which tends to ∞ with $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$. Also, generalized de La Vallée-Poussin mean has been described as

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k, \quad \text{where } I_n = [n - \lambda_n + 1, n].$$

Throughout the article, we use I_n for $[n - \lambda_n + 1, n]$.

Definition 1.4 ([17]). A sequence $x = \{x_k\}$ can be recognized as λ-statistically convergent to x_0 provided with every $\epsilon > 0$ satisfying

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - x_0| \geq 1 - \epsilon\}| = 0$$

or

$$\delta_\lambda(\{k \in I_n : |x_k - x_0| \geq 1 - \epsilon\}) = 0.$$

Symbolically, $S_\lambda - \lim_{n\to\infty} x_k = x_0$.

Kizmaz [14] has discovered the difference sequence spaces conception by considering the set $Z(\Delta) = \{x = \{x_k\} : \{\Delta x_k\} \in Z\}$ for $Z = l_\infty$ (spaces of all the bounded sequences) c_0 (spaces of all the convergent sequences) and c_0 (spaces of all the null sequences), where $\Delta x = \{\Delta x_k\} = \{x_k - x_{k+1}\}$, and $x = \{x_k\}$ is a real sequence for all $k \in N$. In particular, $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are also recognized as Banach spaces, due to the norm endowed by $\|x\|_\Delta = |x_1| + \sup_k |\Delta x_k|$. Moreover, the generalized difference sequence spaces were defined by Et and Çolak [6] by $Z(\Delta^m) = \{x = \{x_k\} : \{\Delta^m x_k\} \in Z\}$, where m be any fixed positive integer, for $Z = l_\infty$, c, c_0 and $\Delta^m x = \{\Delta^m x_k\} = \{\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\}$ so that $\Delta^m x_k = \sum_{r=0}^{m} (-1)^r \binom{m}{r} x_{k+r}$. The Δ^m-statistical convergence concept was studied and considered by Et and Nuray [7] with the help of statistical convergence.

Definition 1.5 ([7]). A sequence $x = \{x_k\}$ can be recognized as Δ^m-statistically convergent to x_0 provided with every $\epsilon > 0$, we have

$$\delta\left(\{k \leq n : |\Delta^m x_k - x_0| \geq \epsilon\}\right) = 0.$$

Symbolically, $St - \lim \Delta^m x_k = x_0$.

A lot of work is done by various researchers related to convergence of difference sequences in extended way with different structures, some connected results can be seen in [1, 4, 5, 8, 12, 15, 20, 21].

2. Main Results

In this section, we are introducing the idea of λ-statistical convergence for generalized difference sequences on IFN space, i.e., λ-Δ^m-statistical convergence on IFN space and further establish some important results on this convergence structure.

Definition 2.1. Let $(X,\varphi,\vartheta,\otimes,\odot)$ be any IFN space. Then, sequence $x=\{x_k\}$ can be recognized as λ-Δ^m-statistically convergent to $x_0\in X$ corresponding to norm (φ,ϑ) provided with any $t>0$ and every $\epsilon>0$ satisfies

$$\delta_\lambda(\{k\in I_n:\varphi(\Delta^m x_k-x_0,t)\le 1-\epsilon \text{ or } \vartheta(\Delta^m x_k-x_0,t)\ge\epsilon\})=0.$$

Symbolically, $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m x=x_0$.

Definition 2.2. Let $(X,\varphi,\vartheta,\otimes,\odot)$ be any IFN space. Then, sequence $x=\{x_k\}$ can be recognized as λ-Δ^m-statistically Cauchy corresponding to norm (φ,ϑ) provided with any $t>0$ and every $\epsilon>0$ there exists $k_0\in N$ satisfying

$$\delta_\lambda(\{k\in I_n:\varphi(\Delta^m x_k-\Delta^m x_s,t)\le 1-\epsilon \text{ or } \vartheta(\Delta^m x_k-\Delta^m x_s,t)\ge\epsilon\})=0, \quad \text{for all } \ k\ge k_0.$$

The next results can be easily obtained using Definition 2.1 and Definition 2.2.

Theorem 2.1. *Let $x=\{x_k\}$ be any sequence in an IFN space $(X,\varphi,\vartheta,\otimes,\odot)$. If $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m x=x_0$, then limit ξ is unique.*

Proof. Assume that, $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m x=x_0$ and $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m x=x_1$ and $\xi_1\ne\xi_2$.
For $t>0$ and $\epsilon>0$, take $\kappa>0$ with $(1-\kappa)\otimes(1-\kappa)>1-\epsilon$ and $\kappa\odot\kappa<\epsilon$. Define

$$A_{1,\varphi}(\kappa,t)=\left\{k\in I_n:\varphi\left(\Delta^m x_k-x_0,\frac{t}{2}\right)\le 1-\kappa\right\},$$
$$A_{2,\varphi}(\kappa,t)=\left\{k\in I_n:\Delta^m\varphi\left(x_k-x_1,\frac{t}{2}\right)\le 1-\kappa\right\},$$
$$A_{1,\vartheta}(\kappa,t)=\left\{k\in I_n:\vartheta\left(\Delta^m x_k-x_0,\frac{t}{2}\right)\ge\kappa\right\},$$
$$A_{2,\vartheta}(\kappa,t)=\left\{k\in I_n:\vartheta\left(\Delta^m x_k-x_0,\frac{t}{2}\right)\ge\kappa\right\}.$$

Since $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m x=x_0$, then due to Definition 2.1 we get

$$\delta_\lambda(A_{1,\varphi}(\kappa,t))=\delta_\lambda(A_{1,\vartheta}(\kappa,t))=0.$$

Further $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m x=x_1$, due to Definition 2.1 we get

$$\delta_\lambda(A_{2,\varphi}(\kappa,t))=\delta_\lambda(A_{2,\vartheta}(\kappa,t))=0.$$

Consider $A_{\varphi,\vartheta}(\kappa,t)=(A_{1,\varphi}(\kappa,t)\cup A_{2,\varphi}(\kappa,t))\cap(A_{1,\vartheta}(\kappa,t)\cup A_{2,\vartheta}(\kappa,t))$.
Clearly,

$$\delta_\lambda(A_{\varphi,\vartheta}(\kappa,t))=0\Leftrightarrow\delta_\lambda(I_n-A_{\varphi,\vartheta}(\kappa,t))=1.$$

If $k\in I_n-A_{\varphi,\vartheta}(\kappa,t)$ then either

$$k\in I_n-(A_{1,\varphi}(\kappa,t)\cup A_{2,\varphi}(\kappa,t)) \ \text{ or } \ k\in I_n-(A_{1,\vartheta}(\kappa,t)\cup A_{2,\vartheta}(\kappa,t)).$$

If $k\in I_n-(A_{1,\varphi}(\kappa,t)\cup A_{2,\varphi}(\kappa,t))$, then

$$\varphi(x_0-x_1,t)\ge\varphi(\Delta^m x_k-x_0,t/2)\otimes\varphi(\Delta^m x_k-x_1,t/2)>(1-\kappa)\otimes(1-\kappa)>1-\epsilon.$$

As $\epsilon>0$, so we get $\varphi(x_0-x_1,t)=1$ for all $t>0$, then $x_0=x_1$.

On other hand if $k \in I_n - (A_{1,\vartheta}(\kappa,t) \cup A_{2,\vartheta}(\kappa,t))$, then

$$\vartheta(x_0 - x_1, t) \leq \vartheta(\Delta^m x_k - x_0, t/2) \odot \vartheta(\Delta^m x_k - x_1, t/2) < \kappa \odot \kappa < \epsilon.$$

As $\epsilon > 0$, so we get $\varphi(x_0 - x_1, t) = 0$ for all $t > 0$, then $x_0 = x_1$. Hence, limit is unique. □

Theorem 2.2. *Let $(X, \varphi, \vartheta, \otimes, \odot)$ be any IFN space. If $(\varphi,\vartheta)_\lambda - \lim \Delta^m x = x_0$, then $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$. But counter part does not hold.*

Proof. Assume $(\varphi,\vartheta)_\lambda - \lim \Delta^m x = x_0$. For given $\epsilon > 0$ and $t > 0$ we get $k_0 \in N$ satisfying

$$\varphi(\Delta^m x_k - x_0, t) > 1 - \epsilon \text{ and } \vartheta(\Delta^m x_k - x_0, t) < \epsilon$$

for all $k \geq k_0$. This provides the set

$$\{k \in I_n : \varphi(\Delta^m x_k - x_0, t) \leq 1 - \epsilon \text{ or } \vartheta(\Delta^m x_k - x_0, t) \geq \epsilon\},$$

with finite members. As per rule λ density of every finite set becomes zero. Thus,

$$\delta_\lambda\{k \in I_n : \varphi(\Delta^m x_k - x_0, t) \leq 1 - \epsilon \text{ or } \vartheta(\Delta^m x_k - x_0, t) \geq \epsilon\} = 0,$$

i.e.,

$$S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0.$$ □

However, counter part of the above mentioned result fails to exist, this can be explained using next example:

Example 2.1. Let $(R, |\cdot|)$ be any normed space. Define

(i) $\mu_1 \otimes \mu_2 = \mu_1\mu_2$ and $\mu_1 \odot \mu_2 = \min\{\mu_1 + \mu_2, 1\}$ for all $\mu_1, \mu_2 \in [0,1]$,

(ii) $\varphi(x,t) = \frac{t}{t+|x|}$, $\vartheta(x,t) = \frac{|x|}{t+|x|}$ for any $t > 0$ and every $x \in R$.

Then $(R, \varphi, \vartheta, \otimes, \odot)$ is an IFN space.

Consider a sequence $x = \{x_k\}$ such that

$$\Delta^m x_k = \begin{cases} 1, & n - \sqrt{\lambda_n} + 1 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

For $t > 0$ and $\epsilon > 0$, we have

$$\begin{aligned} A(\epsilon,t) &= \{k \in I_n : \varphi(\Delta^m x_k - x_0, t) \leq 1 - \epsilon \text{ or } \vartheta(\Delta^m x_k - x_0, t) \geq \epsilon\} \qquad (x_0 = 0) \\ &= \left\{k \in I_n : \frac{t}{t + |\Delta^m x_k|} \leq 1 - \epsilon \text{ or } \frac{|\Delta^m x_k|}{t + |\Delta^m x_k|} \geq \epsilon\right\} \\ &= \left\{k \in I_n : |\Delta^m x_k| \geq \frac{\epsilon t}{1-\epsilon} > 0\right\} \\ &= \{k \in I_n : |\Delta^m x_k| = 1\} \\ &= \{k \in I_n : k \in [n - \sqrt{\lambda_n} + 1 \leq k \leq n]\}. \end{aligned}$$

Now,

$$\begin{aligned} &\frac{1}{\lambda_n}|A(\epsilon,t)| \leq \frac{\sqrt{\lambda_n}}{\lambda_n} \to 0 \text{ as } n \to \infty \\ \Rightarrow \quad &\lim_{n\to\infty} \frac{1}{\lambda_n}|A(\epsilon,t| = 0. \end{aligned}$$

Thus, $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = 0$, i.e. $x = \{x_k\}$ is λ-Δ^m-statistical convergent on $(R,\varphi,\vartheta,\otimes,\odot)$.
Moreover, using above defined sequence, we get

$$\varphi(\Lambda x_k,t) = \begin{cases} \frac{t}{t+1}, & n-\sqrt{\lambda_n}+1 \le k \le n, \\ 1, & \text{otherwise,} \end{cases}$$

i.e.. $\varphi(\Delta^m x_k,t) \le 1, \quad \forall\ k$

and

$$\vartheta(\Delta^m x_k,t) = \begin{cases} \frac{1}{t+1}, & n-\sqrt{\lambda_n}+1 \le k \le n, \\ 0, & \text{otherwise,} \end{cases}$$

i.e. $\vartheta(\Delta^m x_k,t) \ge 0, \quad \forall\ k.$

This implies $(\varphi,\vartheta)_\Lambda - \lim \Delta^m x \ne 0$.

Next, we will discuss some algebraic features of λ-Δ^m-statistical sequences in IFN space as follows:

Theorem 2.3. *Let $(X,\varphi,\vartheta,\otimes,\odot)$ be an IFN space. Let $x = \{x_k\}$ and $y = \{y_k\}$ be sequences from X. Then*

(i) *If $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$ then $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m ax = ax_0$, $a \in R$,*

(ii) *If $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$ and $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m y = y_0$ then $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m (x+y) = x_0 + y_0$.*

Proof. (i) Assume $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$. Then, for the fixed $\epsilon > 0$ and any $t > 0$, we can take

$$A(\epsilon,t) = \{k \in I_n : \varphi(\Delta^m x_k - x_0,t) \le 1-\epsilon \text{ or } \vartheta(\Delta^m x_k - x_0,t) \ge \epsilon\}.$$

Which provides

$$\delta_\lambda(A(\epsilon,t)) = 0 \ \text{ so that } \ \delta_\lambda([A(\epsilon,t)]^c) = 1.$$

Let $k \in [A(\epsilon,t)]^c$ and $a \ne 0$, then

$$\begin{aligned} \varphi(\Delta^m(ax_k) - ax_0,t) &= \varphi(a(\Delta^m x_k - x_0),t) \\ &= \varphi\left(\Delta^m x_k - x_0, \frac{t}{|a|}\right) \\ &\ge \varphi(\Delta^m x_k - x_0,t) \otimes \varphi\left(0, \frac{t}{|a|} - t\right) \\ &= \varphi(\Delta^m x_k - x_0,t) \otimes 1 \\ &> 1-\epsilon \end{aligned}$$

and

$$\begin{aligned} \vartheta(\Delta^m(ax_k) - ax_0,t) &= \vartheta(a(\Delta^m x_k - x_0),t) \\ &= \vartheta\left(\Delta^m x_k - x_0, \frac{t}{|a|}\right) \\ &\le \vartheta(\Delta^m x_k - x_0,t) \odot \vartheta\left(0, \frac{t}{|a|} - t\right) \\ &\le \vartheta(\Delta^m x_k - x_0,t) \odot 0 \\ &< \epsilon. \end{aligned}$$

Therefore, $\delta_\lambda([A(\epsilon,t)]^c)=1$, i.e.

$$\delta_\lambda\{k\in I_n:\varphi(\Delta^m(ax_k)-ax_0,t)>1-\epsilon \text{ and } \vartheta(\Delta^m(ax_k)-kx_0,t)<\epsilon\}=1.$$

Hence, $S_\lambda^{(\varphi,\vartheta)}-\lim ax=ax_0$, $a\neq 0$.

When $a=0$, we get

$$\varphi(0\Delta^m x_k,t)>1-\epsilon \text{ and } \vartheta(0\Delta^m x_k,t)<\epsilon.$$

Hence, $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m ax=ax_0$, $a\in R$.

(ii) As $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m x=x_0$ and $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m y=y_0$. Then, for $t>0$ and $\epsilon>0$, take $\kappa>0$ with $(1-\kappa)\otimes(1-\kappa)>1-\epsilon$ and $\kappa\odot\kappa<\epsilon$.

Define sets for the given sequences $x=\{x_k\}$ and $y=\{y_k\}$ sets

$$A_x(\kappa,t)=\{k\in I_n:\varphi(\Delta^m x_k-x_0,t/2)\le 1-\kappa \text{ or } \vartheta(\Delta^m x_k-x_0,t/2)\ge\kappa\}$$

and

$$A_y(\kappa,t)=\{k\in I_n:\varphi(\Delta^m y_k-y_0,t/2)\le 1-\kappa \text{ or } \vartheta(\Delta^m y_k-y_0,t/2)\ge\kappa\}.$$

We have, $\delta_\lambda(A_x(\kappa,t))=\delta_\lambda(A_y(\kappa,t))=0$.

Consider $A(\kappa,t)=A_x(\kappa,t)\cap A_y(\kappa,t)$, then $\delta_\lambda(A(\kappa,t))=0$ i.e. $\delta([A(\kappa,t)]^c)=1$.

For all $k\in[A(\kappa,t)]^c$,

$$\begin{aligned}\varphi(\Delta^m(x_k+y_k)-(x_0+y_0),t)&=\varphi(\Delta^m x_k-x_0+\Delta^m y_{kr}-y_0,t)\\&\ge\varphi(\Delta^m x_k-x_0,t/2)\otimes\varphi(\Delta^m y_k-y_0,t/2)\\&\ge(1-\kappa)\otimes(1-\kappa)\\&>1-\epsilon\end{aligned}$$

and

$$\begin{aligned}\vartheta(\Delta^m(x_k+y_k)-(x_0+y_0),t)&=\vartheta(\Delta^m x_k-x_0+\Lambda y_k-y_0,t)\\&\le\vartheta(\Delta^m x_k-x_0,t/2)\odot\vartheta(\Delta^m y_k-y_0,t/2)\\&\le\kappa\odot\kappa\\&<\epsilon\end{aligned}$$

$$\Rightarrow\qquad S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m(x+y)=x_0+y_0.$$ □

Theorem 2.4. *Let $(X,\varphi,\vartheta,\otimes,\odot)$ be an IFN space. Then sequence $x=\{x_k\}$ from X is $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m x=x_0$ if and only if there exist set $J=\{j_1<j_2<j_3<\dots\}\subseteq I_n$ with $\delta_\lambda(J)=1$ and $(\varphi,\vartheta)_\lambda-\lim\Delta^m x_{j_n}=x_0$.*

Proof. Necessary part: Consider $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m x=x_0$. For $t>0$ and $\kappa\in N$, we consider

$$A(\kappa,t)=\{k\in I_n:\varphi(\Delta^m x_k-x_0,t)>1-1/\kappa \text{ and } \vartheta(\Delta^m x_k-x_0,t)<1/\kappa\},$$

and

$$K(\kappa,t)=\{k\in I_n:\varphi(\Delta^m x_k-x_0,t)\le 1-1/\kappa \text{ or } \vartheta(\Delta^m x_k-x_0,t)\ge 1/\kappa\}.$$

Since $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$, then $\delta_\lambda(K(\kappa,t)) = 0$. Moreover, $A(\kappa,t) \supset A(\kappa+1,t)$, and

$$\delta_\lambda(A(\kappa,t)) = 1. \tag{2.1}$$

Next, for any $k \in A(\kappa,t)$, we have $(\varphi,\vartheta)_\lambda - \lim \Delta^m x = x_0$.
We are going to prove this part by contradiction. Assume that for any $k \in A(\kappa,t)$ we found $\mu > 0$ and $k_0 \in N$ satisfying

$$\varphi(\Delta^m x_k - x_0, t) \le 1-\mu \text{ or } \vartheta(\Delta^m x_k - x_0, t) \ge \mu, \quad \text{for all } \ k \ge k_0.$$

This implies that

$$\varphi(\Delta^m x_k - x_0, t) > 1-\mu \text{ and } \vartheta(\Delta^m x_k - x_0, t) < \mu, \quad \text{for all } k < k_0.$$

Therefore,

$$\delta_\lambda\{k \in I_n : \varphi(\Delta^m x_k - x_0, t) > 1-\mu \text{ and } \vartheta(\Delta^m x_k - x_0, t) < \mu\} = 0.$$

As $\alpha > \frac{1}{\kappa}$, we have $\delta(A(\kappa,t)) = 0$, which leads a contradiction to (2.1). Thus, we get set $A(\kappa,t)$ with $\delta(A(\kappa,t)) = 1$ and $x = \{x_k\}$ is λ-Δ^m-statistical convergent to x_0.

Sufficient part: Suppose there exist a subset $J = \{j_1 < j_2 < j_3 < \dots\} \subseteq N$ so that $\delta_\lambda(J) = 1$ and $(\varphi,\vartheta)_\lambda - \lim \Delta^m y_{j_n} = x_0$, i.e. $\exists\ N_0 \in N$ for every $\epsilon > 0$ and any $t > 0$ satisfying

$$\varphi(\Delta^m x_k - x_0, t) > 1-\epsilon \text{ and } \vartheta(\Delta^m x_k - x_0, t) < \epsilon, \quad k \ge N_0.$$

Take

$$K(\epsilon,t) = \{k \in I_n : \varphi(\Delta^m x_k - x_0, t) \le 1-\epsilon \text{ or } \vartheta(\Delta^m x_k - x_0, t) \ge \epsilon\}.$$

Then,

$$K(\epsilon,t) \subseteq I_n - \{j_{N_0+1}, j_{N_0+2}, \dots\}.$$

Due to $\delta_\lambda(J) = 1$ we get $\delta_\lambda(K(\epsilon,t)) \le 0$. Therefore, $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$. □

Theorem 2.5. *Let $(X,\varphi,\vartheta,\otimes,\odot)$. Then $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$ if and only if there is a sequence $y = \{y_k\}$ with $(\varphi,\vartheta)_\lambda - \lim \Delta^m y = x_0$ and $\delta_\lambda(\{k \in I_n : \Delta^m x = \Delta^m y\}) = 1$.*

Proof. Necessary part: Consider $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$. By Theorem 2.4, we get a set $J \subseteq I_n$ with $\delta_\lambda(J) = 1$ and $(\varphi,\vartheta)_\lambda - \lim \Delta^m x_{j_n} = x_0$.
Consider a sequence $y = \{y_k\}$ such that

$$\Delta^m y_k = \begin{cases} \Delta^m x_k, & k \in J, \\ x_0, & \text{otherwise}. \end{cases}$$

Then $y = \{y_k\}$ serve our purpose.

Sufficient part: Consider $x = \{x_k\}$ and $y = \{y_k\}$ be the sequences from X with $(\varphi,\vartheta)_\lambda - \lim \Delta^m y = x_0$ and $\delta_\lambda(\{k \in I_n : \Delta^m x = \Delta^m y\}) = 1$. Then for any $t > 0$ and every $\epsilon > 0$, we have

$$\{k \in I_n : \varphi(\Delta^m y_k - x_0, t) \le 1-\epsilon \text{ or } \vartheta(\Delta^m y_k - x_0, t) \ge \epsilon\} \subseteq A \cup B,$$

where

$$A = \{k \in I_n : \varphi(\Delta^m x_k - x_0, t) \le 1-\epsilon \text{ or } \vartheta(\Delta^m x_k - x_0, t) \ge \epsilon\},$$

$$B = \{k \in I_n : \Delta^m y_k \neq \Delta^m x_k\}.$$

Since $(\varphi,\vartheta)_\lambda - \lim \Delta^m x = x_0$ then above defined set A has at most finitely many elements. Also $\delta_\lambda(B) = 0$ as $\delta_\lambda(B^c) = 1$ where $B^c = \{k \in I_n : \Delta^m y_k = \Delta^m x_k\}$. Therefore

$$\delta_\lambda(\{k \in I_n : \varphi(\Delta^m x_k - x_0, t) \leq 1-\epsilon \text{ or } \vartheta(\Delta^m x_k - x_0, t) \geq \epsilon\}) = 0.$$

Hence $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$. □

Theorem 2.6. *Let $x = \{x_k\}$ be a sequence from an IFN space $(X,\varphi,\vartheta,\otimes,\odot)$. Then $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$ if and only if there exist two sequences $y = \{y_k\}$ and $z = \{z_k\}$ from X with $\Delta^m x_k = \Delta^m y_k + \Delta^m z_k$ for all $k \in I_n$ where $(\varphi,\vartheta)_\lambda - \lim \Delta^m y = x_0$ and $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m z = x_0$.*

Proof. Necessary part: Let $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$. By Theorem 2.4 we get a set $J = \{k_q : q = 1,2,3,\dots\} \subseteq N$ with $\delta_\lambda(J) = 1$ and $(\varphi,\vartheta)_\lambda - \lim_{k_q\to\infty} \Delta^m y_{k_q} = x_0$.

Consider the sequences $y = \{y_k\}$ and $z = \{z_k\}$

$$\Delta^m y_k = \begin{cases} \Delta^m z_k, & k \in J, \\ x_0, & \text{otherwise} \end{cases}$$

and

$$\Delta^m x_k = \begin{cases} 0, & k \in J, \\ \Delta^m y_{jk} - x_0, & \text{otherwise} \end{cases}$$

which gives the required result.

Sufficient part: If two such sequences $y = \{y_k\}$ and $z = \{z_k\}$ exists in X with the required properties, then the result follows using Theorem 2.2 and Theorem 2.3. □

Theorem 2.7. *Let $(X,\varphi,\vartheta,\otimes,\odot)$ be an IFN space with norm (φ,ϑ). Then $S^{(\varphi,\vartheta)}(\Delta^m) \subseteq S_\lambda^{(\varphi,\vartheta)}(\Delta^m)$ iff $\lim_{k\to\infty} \inf \frac{\lambda_n}{n} > 0$.*

Proof. For given $\epsilon > 0$ and $t > 0$ we have

$$\{k \leq n : \varphi(\Delta^m x_k - x_0; t) \leq 1-\epsilon \text{ or } \vartheta(\Delta^m x_k - x_0; t) \geq \epsilon\}$$
$$\supseteq \{k \in I_n : \varphi(\Delta^m x_k - x_0; t) \leq 1-\epsilon \text{ or } \vartheta(\Delta^m x_k - x_0; t) \geq \epsilon\}.$$

This provides

$$\frac{1}{n}|\{k \leq n : \varphi(\Delta^m x_k - x_0; t) \leq 1-\epsilon \text{ or } \vartheta(\Delta^m x_k - x_0; t) \geq \epsilon\}|$$
$$\geq \frac{1}{\lambda_n}|\{k \in I_n : \varphi(\Delta^m x_k - x_0; t) \leq 1-\epsilon \text{ or } \vartheta(\Delta^m x_k - x_0; t) \geq \epsilon\}|$$
$$\geq \frac{\lambda_n}{n}.\frac{1}{\lambda_n}|\{k \in I_n : \varphi(\Delta^m x_k - x_0; t) \leq 1-\epsilon \text{ or } \vartheta(\Delta^m x_k - x_0; t) \geq \epsilon\}|.$$

Taking limit as $n \to \infty$ we get $S^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$ (as $\lim_{k\to\infty} \inf \frac{\lambda_n}{n} > 0$).

Hence $S_\lambda^{(\varphi,\vartheta)} - \lim \Delta^m x = x_0$.

Conversely, suppose that $\lim_{k\to\infty} \inf \frac{\lambda_n}{n} = 0$. We can take a sub-sequence $\{n_j\}$ such that $\frac{\lambda_{n_j}}{n_j} < \frac{1}{j}$.

Consider a sequence $x=\{x_k\}$ such that

$$\Delta^m y_k=\begin{cases}1, & k\in I_{n_j},\\ 0, & \text{otherwise.}\end{cases}$$

Then take $t>0$ and $\epsilon\in(0,1)$ such that $1\notin B(0,\epsilon,t)$. Also, to each $n\in N$ we are able to get $n_j\in N$ such that $n_j\le n$ for $j>0$.

$$\frac{1}{n}|\{k\le n:\varphi(\Delta^m x_k;t)\le 1-\epsilon \text{ or } \vartheta(\Delta^m x_k;t)\ge\epsilon\}|<\frac{1}{j}.$$

Then $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m x=0$. For $k\notin I_{n_j}$ we get

$$\lim_{j\to\infty}\frac{1}{\lambda_{n_j}}|\{k\in I_{n_j}:\varphi(\Delta^m x_k;t)\le 1-\epsilon \text{ or } \vartheta(\Delta^m x_k;t)\ge\epsilon\}|=1,$$

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{k\in I_n:\varphi(\Delta^m x_k-1;t)\le 1-\epsilon \text{ or } \vartheta(\Delta^m x_k-1;t)\ge\epsilon\}|=1.$$

This implies that $x\notin S_\lambda^{(\varphi,\vartheta)}(\Delta^m)$. □

Next we establish the result related to Cauchy criterion for λ-Δ^m-statistical convergent sequences in IFN space.

Theorem 2.8. *A sequence $x=\{x_k\}$ from an IFN space $(X,\varphi,\vartheta,\otimes,\odot)$ is λ-Δ^m-statistical convergent corresponding to (φ,ϑ) if and only if it is λ-Δ^m-statistical Cauchy corresponding to (φ,ϑ).*

Proof. Necessary part: Consider $S_\lambda^{(\varphi,\vartheta)}-\lim\Delta^m x=x_0$. Then, for any $t>0$ and every $\epsilon>0$, we take $\kappa>0$ with $(1-\kappa)\otimes(1-\kappa)>1-\epsilon$ and $\kappa\odot\kappa<\epsilon$.
Define $A(\kappa,t)=\{k\in I_n:\varphi(\Delta^m x_k-x_0,t/2)\le 1-\kappa \text{ or } \vartheta(\Delta^m x_k-x_0,t/2)\ge\kappa\}$.
Therefore, $\delta_\lambda(A(\kappa,t))=0$ and $\delta_\lambda([A(\kappa,t)]^c)=1$.
Consider $B(\epsilon,t)=\{k\in I_n:\varphi(\Delta^m x_k-\Lambda x_s,t)\le 1-\epsilon \text{ or } \vartheta(\Delta^m x_k-\Delta^m x_s,t)\ge\epsilon\}$.
Here, for the result we show that $B(\epsilon,t)\subset A(\kappa,t)$, as $k\in B(\epsilon,t)-A(\kappa,t)\Rightarrow\varphi(\Delta^m x_k-x_0,t/2)\le 1-\kappa$ or $\vartheta(\Delta^m x_k-x_0,t/2)\ge\kappa$

$$\begin{aligned}1-\epsilon&\ge\varphi(\Delta^m x_k-\Delta^m x_s,t)\\&\ge\varphi(\Delta^m x_k-x_0,t/2)\otimes\varphi(\Delta^m x_s-x_0,t/2)\\&>(1-\kappa)\otimes(1-\kappa)\\&>1-\epsilon\end{aligned}$$

and

$$\begin{aligned}\epsilon&\le\vartheta(\Delta^m x_k-\Delta^m x_s,t)\\&\le\vartheta(\Delta^m x_k-x_0,t/2)\odot\vartheta(\Delta^m x_s-x_0,t/2)\\&<\kappa\odot\kappa\\&<\epsilon,\end{aligned}$$

which is not possible. This implies that $B(\epsilon,t)\subset A(\kappa,t)$ and $\delta_\lambda(B(\epsilon,t))=0$, i.e. λ-Δ^m-statistical Cauchy corresponding to (φ,ϑ).

Sufficient part: Let $x=\{x_k\}$ be λ-Δ^m-statistical Cauchy corresponding to (φ,ϑ) but not λ-Δ^m-statistical convergent corresponding to (φ,ϑ). Then, for any $t>0$ and $\epsilon>0$, we have $\delta_\lambda(C(\epsilon,t))=0$ where

$$C(\epsilon,t)=\{k\in I_n:\varphi(\Delta^m x_k-\Delta^m x_{k_0},t)\le 1-\epsilon \text{ or } \vartheta(\Delta^m x_k-\Delta^m x_{k_0},t)\ge\epsilon\}.$$

Take $\kappa>0$ such that $(1-\kappa)\otimes(1-\kappa)>1-\epsilon$ and $\kappa\odot\kappa<\epsilon$. Let $D(\kappa,t)=\{k\in I_n:\varphi(\Delta^m x_k-x_0,t/2)>1-\kappa$ or $\vartheta(\Delta^m x_k-x_0,t/2)<\kappa\}$.
Now for $k\in D(\epsilon,t)$ we get

$$\begin{aligned}\varphi(\Delta^m x_k-\Delta^m x_{k_0},t)&\ge\varphi(\Delta^m x_k-x_0,t/2)\otimes\varphi(\Delta^m x_{k_0}-\xi,t/2)\\&>(1-\kappa)\otimes(1-\kappa)\\&>1-\epsilon\end{aligned}$$

and

$$\begin{aligned}\vartheta(\Delta^m x_k-\Delta^m x_{k_0},t)&\le\vartheta(\Delta^m x_k-\xi,t/2)\odot\vartheta(\Delta^m x_{k_0}-\xi,t/2)\\&<\kappa\odot\kappa\\&<\epsilon.\end{aligned}$$

Since $x=\{x_k\}$ is not λ-Δ^m-statistical convergent sequence corresponding to (φ,ϑ). Therefore, $\delta_\lambda([C(\epsilon,t)]^c)=0$, i.e. $\delta_\lambda(C(\epsilon,t))=1$, which leads to contradiction for $x=\{x_k\}$, assumed to be λ-Δ^m-statistical Cauchy. Thus, $x=\{x_k\}$ converges λ-Δ^m-statistically corresponding to (φ,ϑ). □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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