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Review Article

A Survey on Branciari Metric Spaces

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Abstract. The motive of this review article is to collect most of the results on v-generalized metric space and its upto date various generalizations. We try to update the literature for continuous development on the results of v-generalized metric and its generalizations.

Keywords. Metric, v-generalized metric, 2-generalized metric, 3-generalized metric, Rectangular b-metric, cone rectangular metric, Rectangular S-metric, Partial rectangular metric, Rectangular M-metric, Complex valued rectangular metric, Fixed point, Compatible topology

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1. Introduction

The idea of *Metric space* was introduced by M. Fréchet [66] in 1906. Later, Hausdorff [75] formulated in the abstract manner and named it *Metric*. Metric space generalized the notion of Euclidean distance function to abstract spaces. Several authors generalized the notion of metric by modifying or reducing some of the metric axioms and as a result 2-metric [69], D-metric [54], G-metric [119], S-metric [147], b-metric [45, 46], cone metric [76], partial metric [112], parametric metric [78], F-metric [83], fuzzy metric [92] have been introduced and studied ([176]-[31], and so on). A lot of metric results have been extended to the generalized metric spaces. Among those generalized metric spaces, one remarkable space is v-generalized metric space introduced by Branciari [37] in 2000 where the triangle inequality is replaced by a v-generalized inequality.

Definition 1.1 ([37]). Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping and $\upsilon \in \mathbb{N}$. d is said to be a υ-generalized metric if it satisfies the following conditions:

(a) $d(x,y) = 0 \iff x = y$;

(b) $d(x,y) = d(y,x)$;

(c) $d(x,y) \le d(x,u_1) + d(u_1,u_2) + d(u_2,u_3) + \cdots + d(u_\upsilon, y)$,

where $x, y, u_1, u_2, \cdots, u_\upsilon \in X$ and all of $x, y, u_1, u_2, \cdots, u_\upsilon \in X$ are distinct points in X. Then, the pair (X,d) is called a υ-generalized metric space.

It is clear that every 1-generalized metric space is metric space. A 2-generalized metric space is called generalized metric space or rectangular metric space [37]. A 2-generalized metric space is always understood to be a topological space with the topology induced by its convergence and its sequential sense is of Franklin [67]. The sense of compactness and continuity are similar to metric space. Many authors established some fixed point theorems in υ-generalized metric spaces in similar technique to metric spaces. But there were some difficulties in proving those theorems. To overcome such problems, there are lot of efficient remarks have been developed by researchers.

The triangle inequality of Metric axioms has strong impact to the following statements:

(i) the distance function is continuous in both variables;

(ii) each open ball is an open set;

(iii) the induced topology is Hausdorff;

(iv) the limit of a sequence is unique, if exists;

(v) every convergent sequence is Cauchy.

υ-generalized metric spaces not necessarily satisfy all the above statements (i)-(v). For which they often fails to be metrizable. So, the study of the topological structure of those spaces is one of the main task of research. For the metrizable spaces, the topological study and fixed point results are easily followed from the theory of metric spaces. But it does not mean that all results of those spaces are redundant. That's why researchers are still interested for those metrizable spaces also. The fixed point theory for non-metrizable spaces also have wide applications in many directions. For example, non-Hausdorff spaces are useful for the Tarskian approach to programming language semantics [112].

Being an extension of metric space, the metrization problem in υ-generalized metric spaces are very interesting topics for researchers. It is worth mentioning the work in υ-generalized metric spaces (see [67, 160, 162, 167, 169–171, 171]).

In 1922, Banach [27] in his PhD thesis stated the existence of the fixed point in the abstract setting of metric spaces. Later which became familiar to us as the 'Banach contraction principle'. Banach contraction principle is a classic method to non-linear analysis and heavily researched fixed point theorem. So far, there are so many mode of modification of the celebrated 'Banach contraction principle'. This modification were done in two ways: one is by changing contraction condition in a general manner and another one by reforming it into new generalized metric spaces.

Following this trend, researchers involves themselves on generalization of metric fixed point theorems to υ-generalized metric spaces. Day-by-day, more generalization of υ-generalized metric spaces, such as cone rectangular metric spaces [25], partial rectangular metric spaces [152], rectangular b-metric spaces [72], extended rectangular b-metric spaces [121], rectangular S-metric spaces [4], etc. have been introduced.

Based on the research development from beginning to present scenario on the structure of v-generalized metric spaces and fixed point results related to such spaces are collected in this article which will be helpful for researchers in future for further development and study.

2. v-Generalized Metric Spaces and Its Topological Structure

There are a large group of researchers, interested in v-generalized metric spaces and its all kind of hybrid spaces. Suzuki's group is one among them. Suzuki and his co-authors exercised on the problem of topology on v-generalized metric space and in 2014, Suzuki [163] justified by proper examples that all the v-generalized metric spaces does not necessarily have a topology compatible with metric topology.

Definition 2.1 ([163]). Let (X,d) be a v-generalized metric space. Then a net $\{x_n\}$ is said to converge to x iff $\lim_{n\to\infty} d(x_n,x)=0$.

Definition 2.2 ([163]). Let X be a topological space with topology τ and d be a v-generalized metric space on X. Then τ is compatible with d if for any net $\{x_n\}$ and $x\in X$, $\lim_{n\to\infty} d(x_n,x)=0$ iff $\{x_n\}$ converges to x in (X,τ).

By the following example very smoothly Suzuki [163] justified the ambiguity for compatible topology of v-generalized metric spaces.

Example 2.3 ([163]). Let $X=\{(0,0)\}\cup((0,1]\times[0,1])$ and define d on $X\times X$ by

$$d(x,x)=0;\ d((0,0),(s,0))=d((s,0),(0,0))=s,\quad \text{if } s\in(0,1],$$
$$d((s,0),(p,q))=d((p,q),(s,0))=|S-p|+q,\quad \text{if } s,p,q\in(0,1];\ d(x,y)=3,\ \text{otherwise}.$$

Then d is a 2-generalized metric but not a metric. Moreover, X do not have a topology which is compatible with d.

If τ be the topology induced by a subbase $\{B(x,r):x\in X, r>0\}$, then also τ is not compatible with d where $B(x,r)=\{y\in X:d(x,y)<r\}$.

After that, he gave an example which shows that there exist a v-generalized metric space which is not a p-generalized metric space for $p<v$.

Example 2.4 ([163]). Take $X=\mathbb{N}$ and $v\in\mathbb{N}$ such that $v\geq 2$. Define a function d by

$$d(x,x)=0;\ d(x,1)=\begin{cases} d(1,x)=v+1, & \text{if } x\in\mathbb{N}\setminus\{1,2\},\\ d(x,y)=1, & \text{otherwise.}\end{cases}$$

Then (X,d) is not a p-generalized metric space for $p<v$, $p\in\mathbb{N}$ but (X,d) is a q-generalized metric space for $q\geq v$, $q\in\mathbb{N}$.

Since (X,d) does not necessarily have a compatible topology, so does not necessarily have uniformity which is compatible with d. Hence following definitions, given by Suzuki [163] are important to work further in v-generalized metric spaces.

Definition 2.5 ([163])**.** Let (X,d) be a v-generalized metric space.

(i) A sequence $\{x_n\}$ is said to be Cauchy iff $\lim_{n\to\infty}\sup_{n>m} d(x_n,x_m)=0$.

(ii) X is said to be complete iff every Cauchy sequence converges to some point in X.

(iii) X is Hausdorff iff every convergent sequence have a unique limit point.

Lastly, Suzuki [163] finished this article by proving the CJM type fixed point (for details, see [44, 60, 79, 103, 113, 164, 172]) in v-generalized metric spaces.

Theorem 2.6 ([163])**.** *Let (X,d) be a complete v-generalized metric space and T be a self mapping on X, for any $x,y\in X$ which satisfies the following conditions:*

(i) *for every $\epsilon>0$, $\exists\ \delta>0$ such that $d(x,y)<\epsilon+\delta$ implies $d(Tx,Ty)\le\epsilon$,*

(ii) *$x\neq y$ implies $d(Tx,Ty)<d(x,y)$.*

Then T has a unique fixed point $x\in X$. Moreover, $\lim_{n\to\infty} d(T^n y,x)=0$, for any $y\in X$.

Theorem 2.7 ([163])**.** *Let (X,d) be a complete v-generalized metric space and T be a self mapping on X. Assume that $\exists$ two functions $\psi,\phi:[0,\infty)\to[0,\infty)$ such that*

(i) *$\psi(d(Tx,Ty))\le\psi(d(x,y))-\phi(d(x,y))$,*

(ii) *ψ is non-decreasing,*

(iii) *$\inf\phi([s,t])>0$, $\forall\ s,t\in(0,\infty)$, $s<t$,*

for all $x,y\in X$. Then T is a CJM contraction.

Suzuki and his group left a remarkable contribution on the research on v-generalized metric spaces. Suzuki developed a lot of results ([162, 167, 169–171]) on the topology on v-generalized metric spaces.

First, we recall some definitions and then useful results of Suzuki and his co-authors ([7, 165, 174]).

Definition 2.8 ([174])**.** A v-generalized metric space (X,d) is said to be

(i) compact if for any sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{f(n)}\}$ of $\{x_n\}$ converging to some $z\in X$.

(ii) compact in the strong sense if for any sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{f(n)}\}$ of $\{x_n\}$ converging to some $z\in X$ in the strong sense.

Definition 2.9. Let (X,d) be a v-generalized metric space and $\{x_n\}$ be a sequence in X. Let $\kappa\in\mathbb{N}$.

(i) [165] $\{x_n\}$ is said to be $(\sum,\neq)$-Cauchy if x_n are all different and $\sum_{j=1}^{\infty} d(x_j,x_{j+1})<\infty$ holds.

(ii) [165] X is $(\sum,\neq)$-complete if every $(\sum,\neq)$-Cauchy sequence converges.

(iii) [174] $\{x_n\}$ is said to converge exclusively to x if $\lim_{n\to\infty} d(x_n,x)=0$ and $\liminf_{n\to\infty} d(x_n,y)>0$ hold for any $y\in X\setminus\{x\}$.

(iv) [174] $\{x_n\}$ is said to converge to x in the strong sense if $\{x_n\}$ is Cauchy and converges to x.

(v) [7] $\{x_n\}$ is said to be κ-Cauchy if $\lim_{n\to\infty}\sup\{d(x_n,x_{n+1+j\kappa}):j=0,1,\cdots\}=0$ hold.

(vi) [7] $\{x_n\}$ is said to converge only to x if $\lim_{n\to\infty} d(x_n,x)=0$ and $\limsup_{n\to\infty} d(x_n,y)>0$ hold for any $y\in X\setminus\{x\}$.

(vii) [7] X is κ-complete if every κ-Cauchy sequence converges.

(viii) [7] A mapping T on X is said to be sequentially continuous iff $\{Tx_n\}$ converges to Tx whenever $\{x_n\}$ converges to x.

(ix) [7] A mapping T on X is said to be sequentially lower semi-continuous iff $f(x)\le \liminf_n f(x_n)$ whenever $\{x_n\}$ converges to x.

Proposition 2.10 ([7]). *Let (X,d) be a v-generalized metric space and $\lambda,\kappa\in\mathbb{N}$ be such that κ divides λ. Then,*

(i) *every κ-Cauchy sequence is λ-Cauchy.*

(ii) *if X is λ-complete then it is κ-complete.*

Proposition 2.11 ([7]). *Let (X,d) be a v-generalized metric space and $\{x_n\}$ be a v-Cauchy sequence such that all x_n are different.*

(i) *If v is odd, then $\{x_n\}$ is Cauchy.*

(ii) *If v is even, then $\{x_n\}$ is 2-Cauchy.*

Lemma 2.12 ([7]). *Let (X,d) be a v-generalized metric space and $\{x_n\}$ be a sequence such that all x_n are different.*

(i) *If $\sum_{j=1}^{\infty} d(x_j,x_{j+1})<\infty$, then $\{x_n\}$ is v-Cauchy.*

(ii) *If $\lim_{j\to\infty} d(x_j,x_{j+1})=0$, and $\{x_n\}$ converges to some $z\in X$, then $\{x_n\}$ converges only to $z\in X$.*

Lemma 2.13 ([7]). *Let (X,d) be a v-generalized metric space satisfying either of the following:*

(i) *v is odd and X is complete.*

(ii) *v is even and X is 2-complete.*

Let $\{x_n\}$ be a sequence such that all x_n are different, $\lim_{j\to\infty} d(x_j,x_{j+1})<\infty$. Then, $\exists\ z\in X$ such that $\{x_n\}$ converges only to z.

Lemma 2.14 ([165]). *Let (X,d) be a v-generalized metric space and $\{x_n\}$ be a κ-Cauchy sequence converging to some $z\in X$ such that all x_n are different. Then $\{x_n\}$ is Cauchy.*

Lemma 2.15 ([165]). *Let (X,d) be a v-generalized metric space and $\{x_n\}$ be a $(\sum,\neq)$-Cauchy sequence in X. Then the following hold:*

(i) *If v is odd, then $\{x_n\}$ is Cauchy.*

(ii) *$\{x_n\}$ is 2-Cauchy.*

(iii) *If $\{x_n\}$ converges, then $\{x_n\}$ is Cauchy, that is, $\{x_n\}$ converges in the strong sense.*

Lemma 2.16 ([165]). *Let (X,d) be a v-generalized metric space satisfying either of the following:*

(i) *v is odd and X is complete.*

(ii) *X is 2-complete.*

Then X is $(\sum,\neq)$ complete.

Lemma 2.17 ([165]). *Let (X,d) be a v-generalized metric space and $\{x_n\}$ be a Cauchy sequence in X.*

(i) *If $\{x_n\}$ converges to some $z \in X$ and $\{y_n\}$ be a sequence in X satisfying $\lim d(x_j, y_j) = 0$, then $\{y_n\}$ also converges to z.*

(ii) *If $\{x_n\}$ satisfies $\liminf d(x_j, z) = 0$, for some $z \in X$, then $\{y_n\}$ converges to z.*

Lemma 2.18 ([165]). *Let (X,d) be a $(\sum, \neq)$-complete, v-generalized metric space. Then X is complete.*

Lemma 2.19 ([165]). *Let (X,d) be a 2-complete, v-generalized metric space. Then X is Hausdorff.*

Lemma 2.20 ([165]). *Let (X,d) be a $(\sum, \neq)$-complete, Hausdorff, v-generalized metric space. Then X is 2-complete.*

From these lemmas Suzuki established some important results on completeness.

Proposition 2.21 ([165]). *Let (X,d) be a v-generalized metric space where v is odd. Then followings are equivalent:*

(i) *X is complete.*

(ii) *X is $(\sum, \neq)$-complete.*

Proposition 2.22 ([165]). *Let (X,d) be a v-generalized metric space where v is odd. Then followings are equivalent:*

(i) *X is 2-complete.*

(ii) *X is $(\sum, \neq)$-complete and Hausdorff.*

Proposition 2.23 ([165]). *Let (X,d) be a v-generalized metric space where v is odd. Then followings are equivalent:*

(i) *X is complete.*

(ii) *X is $(\sum, \neq)$-complete.*

(iii) *X is 2-complete.*

Before going to the main results, Suzuki [171] proved some useful lemmas to establish his claims.

Definition 2.24 ([171]). Let (X,d) be a v-generalized metric space andτ be a topology on X. Then τ is said to be strongly compatible with d if the followings are equivalent for any net $\{x_\alpha\}$ and x in X:

(i) $\lim_\alpha d(x, x_\alpha) = 0$ and $\lim_\alpha \sup\{d(x_\alpha, x_\beta) : \beta \geq \alpha\} = 0$.

(ii) $\{x_\alpha\}$ converges to x in τ.

Lemma 2.25 ([171]). *For any $(x,y,z) \in X^3$,*

$$d(x,z) \leq d(x,y) + d(y,z) + 2\eta(x) \quad \text{and} \quad (x,z) \leq d(x,y) + d(y,z) + 2\eta(y)$$

hold where $\eta(x) = \inf\{\delta(x, u_1, \cdots, u_{v+2}) : (x, u_1, \cdots, u_{v+2}) \in X^{(v+3)}\}$, $\forall\ x \in X$ *and*

$$\delta(x, u_1, \cdots, u_{v+2}) = \max\{d(x, u_{\sigma(1)}) + \sum_{j=1}^{v+1} d(u_{\sigma(j)}, u_{\sigma(j+1)}) : \sigma \in S_{v+2}\},$$

S_{v+2} *is the permutation group consisting of all bijective mappings on* $\{1, 2, \cdots, (v+2)\}$.

Lemma 2.26 ([171])**.** *For any* $(x, y) \in X^2$, $\eta(y) \le d(x, y) + 3\eta(x)$ *holds.*

Lemma 2.27 ([171])**.** *Let* $\{x_\alpha\}$ *be a net in* X *satisfying*

(i) $\lim_{\alpha} \sup\{d(x_\alpha, x_\beta) : \beta \ge \alpha\} = 0$,

(ii) *for any* $\alpha \in D$, $\exists\ \beta \ge \alpha$ *such that* $x_\alpha \ne x_\beta$.

Then the following hold:

(i) $\lim_{\alpha} \eta(x_\alpha) = 0$.

(ii) *if* $\lim_{\alpha} d(x, x_\alpha) = 0$ *for some* $x \in X$, *then* $\eta(x) = 0$.

Theorem 2.28 ([171])**.** *Let* (X, d) *be a* v*-generalized metric space satisfying* $Card(X) \ge v + 3$. *Define a function* $\rho : X \times X \to [0, \infty)$ *by*

$$\rho(x, y) = \begin{cases} 0, & \text{if } x = y, \\ d(x, y) + \eta(x) + \eta(y), & \text{if } x \ne y, \end{cases}$$

for $x, y \in X$. *Then the following hold:*

(i) (X, ρ) *is a metric space.*

(ii) *For every* $x \in X$ *and for every net* $\{x_\alpha : \alpha \in D\}$ *in* X, $\lim_{\alpha} \rho(x, x_\alpha) = 0$ *iff* $\lim_{\alpha} d(x, x_\alpha) = 0$ *and* $\lim_{\alpha} \sup\{d(x_\alpha, x_\beta) : \beta \ge \alpha\} = 0$.

Theorem 2.29 ([171])**.** *Let* (X, d) *be a* v*-generalized metric space satisfying* $Card(X) < \infty$. *Define a function* $\rho : X \times X \to [0, \infty)$ *by*

$$\rho(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \ne y, \end{cases}$$

for $x, y \in X$. *Then the same conclusion of Theorem* 2.28 *holds.*

Corollary 2.30 ([171])**.** *Every* v*-generalized metric space has the strongly compatible topology, which is metrizable.*

Later, he discussed on the completeness and compactness of (X, d) and (X, ρ).

Theorem 2.31 ([171])**.** *Let* (X, d) *be a* v*-generalized metric space satisfying* $Card(X) \ge v + 3$ *and* $\{x_n\}$ *be a sequence in* X. *Then*

(i) $\{x_n\}$ *is Cauchy in* (X, d) *iff* $\{x_n\}$ *is Cauchy in* (X, ρ).

(ii) (X, d) *is complete iff* (X, ρ) *is complete.*

(iii) (X, d) *is compact in the strong sense iff* (X, ρ) *is compact.*

Theorem 2.32 ([171])**.** *Let* (X, d) *be a* v*-generalized metric space satisfying* $Card(X) < \infty$ *and* $\{x_n\}$ *be a sequence in* X. *Then*

(i) $\{x_n\}$ *is Cauchy in* (X,d) *iff* $\{x_n\}$ *is Cauchy in* (X,ρ).

(ii) (X,d) *is complete iff* (X,ρ) *is complete.*

(iii) (X,d) *is compact in the strong sense iff* (X,ρ) *is compact.*

Lastly, he applied Theorem 2.28, to generalize [174, Proposition 2.7] and which also shown that d is continuous in the strongly compatible topology.

Proposition 2.33 ([171]). *Let* (X,d) *be a* v*-generalized metric space. Let* $x,y \in X$ *and* $\{x_\alpha\}$, $\{y_\alpha\}$ *be nets in* X *satisfying*

$$\lim_\alpha d(x,x_\alpha) = \lim_\alpha \sup\{d(x_\alpha,x_\beta) : \beta \geq \alpha\} = 0,$$

$$\lim_\alpha d(y,y_\alpha) = \lim_\alpha \sup\{d(y_\alpha,y_\beta) : \beta \geq \alpha\} = 0.$$

Then $\lim_\alpha d(x_\alpha,y_\alpha) = d(x,y)$ *holds.*

Definition 2.34 ([170]). Let (X,d) be a v-generalized metric space and τ be a topology on X. Then τ is said to be sequentially compatible with d if for any sequence $\{x_n\}$ and x in X,

$$\lim_n d(x,x_n) = 0 \iff \{x_n\} \text{ converges to } x \text{ in } \tau.$$

Theorem 2.35 ([170]). *Let* (X,d) *be a* v*-generalized metric space and* τ *be a topology on* X. *Then* (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) *holds:*

(i) τ *is sequentially compatible with* d.

(ii) *For any open subset* U *of* X, $\exists d > 0$ *satisfying* $S(x,\delta) \subset U$, $x \in U$.

(iii) *If a net* $\{x_\alpha\}$ *satisfies* $\lim_\alpha d(x,x_\alpha) = 0$, *for some* $x \in X$, *then* $\{x_\alpha\}$ *converges to* x *in* τ.

Lemma 2.36 ([170]). *Let* $\{x_n\}$ *be a sequence in a* v*-generalized metric space* (X,d). *Then there exist a subsequence* $\{y_n\}$ *of* $\{x_n\}$ *and a subset* Z *of* X *such that either (a) or (b) holds:*

(a) (a1)–(a3) *hold.*

- (a1) $\liminf d(u,y_n) > 0$, *for any* $u \in X \setminus Z$.
- (a2) $\lim d(z,y_n) = 0$, *for any* $z \in Z$.
- (a3) $Card(Z) < \infty$.

(b) (b1)–(b2) *hold.*

- (b1) $\lim d(z,y_n) = 0$, *for any* $z \in Z$.
- (b2) $Card(Z) = \infty$.

Lemma 2.37 ([170]). *Let* (X,d) *be a* v*-generalized metric space and let* $\{y_n\}$ *and* Z *satisfy* (b1)–(b2) *of Lemma* 2.36. *Let* $x,y,z \in X$ *satisfy* $z \in Z$ *and* $d(x,y) < \inf\{d(x,v) : v \in \{y_n : n \in \mathbb{N}\} \cup \{z\}\}$. *Then*

$$\inf\{d(y,v) : v \in \{y_n : n \in \mathbb{N}\} \cup \{z\}\} > 0$$

holds.

To discuss on the strongly sequentially topology on v-generalized metric spaces, Suzuki [170] used some notations.

(X,d) be a v-generalized metric space with $v \geq 2$. Let $F = \{f : X \to [0,\infty)$ be a function$\}$ and define $T(x,\delta) = S(x,\delta) \setminus \{x\}$, for $x \in X$ and $\delta > 0$. Define $S(x,f,\delta) = S(x, \min\{f(x),\delta\}$ and $T(x,f,\delta) = S(x,f,\delta) \setminus \{x\}$ for $x \in X$, $f \in F$ and $\delta > 0$. Define

$$U(x,f,\delta,n) = \begin{cases} S(x,f,\delta), & \text{if } n = 1, \\ S(x,f,\delta) \cup \cup\{U(y,\delta - d(x,y), n-1) : y \in T(x,f,\delta)\}, & \text{if } n > 1, \end{cases}$$

and $U(x,f) = U(x,f,f(x),v)$ for $x \in X$, $f \in F$, $\delta > 0$ and $n \in \mathbb{N}$. Let τ be a topology on X induced by a sub-base $\{U(x,f) : x \in X, f \in F\}$.

Lemma 2.38 ([170]). *Let $x \in X$ and $f, g \in F$. Then, the following hold:*

(i) *If $f \leq g$ holds, then $U(x,f) \subset U(x,g)$ holds.*

(ii) *If the restrictions of f and g to $U(x,f)$ coincide, then $U(x,f) = U(x,g)$ holds.*

Lemma 2.39 ([170]). *Define $V(x,f,n)$ and $V(x,f)$ by*

$$V(x,f,n) = \begin{cases} S(x,f(x)), & \text{if } n = 1, \\ S(x,f(x)) \cup \{V(y,f,n-1) : y \in T(x,f(x))\}, & \text{if } n > 1, \end{cases}$$

and $V(x,f) = V(x,f,v)$ for $x \in X$, $f \in F$, and $n \in \mathbb{N}$. Then, $U(x,f) \subset V(x,f)$ holds.

Lemma 2.40 ([170]). *Let $x \in X$ and $f \in F$. Then the following hold:*

(i) *For any $z \in U(x,f)$, $\exists \epsilon > 0$ satisfying $S(z,\epsilon) \subset U(x,f)$.*

(ii) *For any $z \in U(x,f)$, $\exists g \in F$ satisfying $U(z,g) \subset U(x,f)$.*

Lemma 2.41 ([170]). *Let U be an open subset of (X,τ). Then the following hold:*

(i) *For any $x \in U$, $\exists\ \epsilon > 0$ satisfying $S(x,\epsilon) \subset U$.*

(ii) *For any $x \in U$, $\exists\ f \in F$ satisfying $U(x,f) \subset U$.*

Lemma 2.42 ([170]). *Let U be a subset of X. Then U is open in τ iff (A) holds.*

Lemma 2.43 ([170]). *Let $x \in X$, let $\{y_n\}$ be a sequence in X and let $Z \subset X$ satisfying* (b1)–(b2) *of Lemma* 2.36. *Fix $z \in Z$ and define a function $f : X \to [0,\infty)$ by*

$$f(u) = \inf\{d(u,v) : v \in \{y_n : n \in \mathbb{N}\} \cup \{z\}\}.$$

Then, the following hold:

(i) *If $f(x) > 0$ holds, then $f(y) > 0$ holds for any $y \in S(x,f(x))$.*

(ii) *If $f(x) > 0$ holds, then $f(u) > 0$ holds for any $u \in U(x,f)$ and $\{y_n : n \in \mathbb{N}\} \cap U(x,f) = \phi$.*

Lemma 2.44 ([170]). *Let $x \in X$, let $\{y_n\}$ be a sequence in X and let $Z \subset X$ satisfying* (a1)–(a3) *of Lemma* 2.36. *Define a function $f : X \to [0,\infty)$ by*

$$f(u) = \inf\{d(u,v) : v \in \{y_n : n \in \mathbb{N}\} \cup Z\}/2.$$

Then, the following hold:

(i) *If $\inf\{d(x,y_n) : n \in \mathbb{N}\}$ holds, then $f(x) > 0$ holds.*

(ii) *If $f(x) > 0$ holds, then $f(y) > 0$ holds for any $y \in S(x,f(x))$.*

(iii) *If $\inf\{d(x,y_n) : n \in \mathbb{N}\} > 0$ holds, then $f(u) > 0$ holds for any $u \in U(x,f)$ and $\{y_n : n \in \mathbb{N}\} \cap U(x,f) = \phi$.*

Using the above lemmas, finally we can conclude on compatibility.

Theorem 2.45 ([170]). *τ is sequentially compatible with d.*

Remark 2.46. τ is the strongest topology that is sequentially compatible with d.

Theorem 2.47 ([170]). *Every v-generalized metric space (X,d) has a sequentially compatible topology with d.*

Theorem 2.48 ([170]). *Let (X,d) be a v-generalized metric space. Then, the following are equivalent:*

(i) *X has the compatible topology with d.*

(ii) *τ is the compatible topology with d.*

(iii) *For any $x \in X$ and $\delta > 0$, $\exists$ an open neighborhood U at x in τ satisfying $U \subset S(x,\delta)$.*

Theorem 2.49 ([170]). *Let (X,d) be a v-generalized metric space and let τ be a topology on X which is sequentially compatible with d. Then (X,τ) is T_1.*

Theorem 2.50 ([170]). *Let (X,d) and τ be as in Theorem 2.49. Then* (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) *holds:*

(i) *(X,τ) is T_2.*

(ii) *If $\lim_n d(x_n,x) = 0$ holds for some $x \in X$, then $\lim_n d(x_n,y) > 0$ holds for any $y \in X \setminus \{x\}$.*

(iii) *If $\lim_n d(x_n,x) = \lim_n d(x_n,y) = 0$ holds for some $x,y \in X$, then $x = y$ holds.*

To give the answer of the open question (Problem 5.1) of Suzuki *et al*. [175] on the metrizability of v-generalized metric spaces, Dung and Hung [59] gave an interesting counter example. Since the answer was negative, so they felt interest on this topic and as a result proved a sufficient condition for a v-generalized metric space with $v \geq 4$ having a metric with the same convergence of sequences.

The following example of Dung and Hung [59] shown that there exists a v-generalized metric space (X,d) that has a non-metrizable topology being compatible with d in the sense of Definition 2.2 which ensures of a negative answer to Problem 5.1 [175].

Example 2.51 ([59]). Let $X = \{0,2\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and $d : X \times X \to [0,\infty)$ be a function defined by

$$d(x,y) = d(y,x) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } (x,y) = (0,2) \text{ or } (x,y) = \left(\frac{1}{n},\frac{1}{m}\right),\ n \neq m, \\ y, & \text{if } x \in \{0,2\},\ y \in \left\{\frac{1}{n} : n \in \mathbb{N}\right\}. \end{cases}$$

Then, d is not a 3-generalized metric on X but a v-generalized metric space for $v \geq 2$, $v \neq 3$. In particular, d is not a metric on X. There exists a non-metrizable topology τ on X that is compatible with d. There exists a convergent sequence which is not a Cauchy in (X,d).

Next, they worked on the existence of limit point of a convergent sequence in a v-generalized metric space.

Proposition 2.52 ([59]). *Let (X,d) be a v-generalized metric space and $\{x_n\}$ be a convergent sequence. If $v=3$ then $\{x_n\}$ has a unique limit point and if $v=2$ or $v\geq 4$ and $\lim_{n\to\infty} d(x_n,x_{n+1})=0$ then $\{x_n\}$ has a unique limit point.*

Theorem 2.53 ([59]). *Let (X,d) be a v-generalized metric space and $\rho : X\times X \to [0,\infty)$ be a function defined by*

$$\rho(x,y)=\inf\left\{\sum_{j=0}^{n} d(u_j,u_{j+1}) : n\in\mathbb{N}\cup\{0\},\, u_0=x,\, u_1,\cdots,u_n\in X,\, u_{n+1}=y\right\}$$

for all $x,y\in X$, and every convergent sequence is a Cauchy sequence in (X,d). Then

(i) *ρ is a metric on X.*

(ii) *For every sequence $\{x_n\}$ in X, $\lim_{n\to\infty} d(x_n,x)=0$ iff $\lim_{n\to\infty}\rho(x_n,x)=0$.*

If a v-generalized metric space (X,d) is non-metrizable then there may exists a convergent sequence which is not Cauchy. So the assumption 'every convergent sequence is a Cauchy sequence' in Theorem 2.53 is necessary.

Dung and Hung [59] by the following example justified that the inversion of Theorem 2.53 does not hold.

Example 2.54 ([59]). Let $X=\{0\}\cup\left\{\frac{1}{n}:n\in\mathbb{N}\right\}$ and $d : X\times X\to[0,\infty)$ be a function defined by

$$d(x,y)=d(y,x)=\begin{cases}0, & \text{if } x=y,\\ \frac{1}{n}, & \text{if } (x,y)=\left(0,\frac{1}{n}\right),\\ 1, & \text{otherwise.}\end{cases}$$

Then, d is a v-generalized metric on X for all $v\geq 2$ and (X,d) is metrizable but there exists a convergent sequence which is not Cauchy in (X,d).

Since Theorem 2.53 is a sufficient condition for a v-generalized metric space with $v\geq 4$ having a metric with the same convergence of sequences, so Dung and Hung [59] finished their discussion with an open question that "Can any one establish a condition for a v-generalized metric space with $v\geq 4$ to be metrizable?" With the motive to give an answer to their question, Suzuki [169] gave a necessary and sufficient conditions on the conclusion of their main theorem (Theorem 2.53) and based on this they claimed that they have almost completed the metrization problem on v-generalized metric spaces with respect to Dung and Hung's [59] method.

Notations are used in [169] are of [171] and [59].

Lemma 2.55 ([169]). *Let (X,d) be a v-generalized metric space and $\{x_n\}$ be a Cauchy sequence in X such that for any $n\in\mathbb{N}$, $\exists\ m>n$ satisfying $x_m\neq x_n$. Then, the following hold:*

(i) $\lim_{n\to\infty}\eta(x_n)=0$.

(ii) *If $\lim_{n\to\infty} d(x_n,x)=0$ for some $x\in X$, then $\eta(x)=0$.*

Lemma 2.56 ([169]). *Let (X,d) be a v-generalized metric space. Let $x,y,z\in X$ satisfy $\eta(y)=0$. Then, the following hold:*

(i) *$d(x,y)=\rho(x,y)$.*

(ii) *$d(x,z)\leq d(x,y)+d(y,z)$.*

Lemma 2.57 ([169]). *Let (X,d) be a ν-generalized metric space and $x\in X$. Then the following are equivalent:*

(i) $\eta(x)=0$.

(ii) $\inf\{d(x,y): y\in X\setminus\{x\}\}=0$ *holds and every sequence converging to x is Cauchy.*

(iii) *there exists a Cauchy sequence $\{x_n\}$ in $X\setminus\{x\}$ converging to x.*

Lemma 2.58 ([169]). *Let (X,d) be a ν-generalized metric space and τ be a topology on X. Assume that τ is first-countable. Then the following are equivalent:*

(i) *τ is compatible with d.*

(ii) *τ is sequentially compatible with d.*

Next, Suzuki established some important results on characterizations of X of being metrizable, which are the followings.

Theorem 2.59 ([169]). *Let (X,d) be a ν-generalized metric space. Then the following are equivalent:*

(i) *(X,ρ) is a metric space and the topology induced by (X,ρ) is compatible with d.*

(ii) *If $\{x_n\}$ is $\sum$-Cauchy and $\liminf_n d(x_n,x)=0$ holds, then $\lim_{n\to\infty} d(x_n,x)=0$ holds.*

(iii) *X is $\sum$-precomplete.*

Theorem 2.60 ([169]). *Let (X,d) be a ν-generalized metric space. Then the following are equivalent:*

(i) *(X,ρ) is a complete metric space and the topology induced by (X,ρ) is compatible with d.*

(ii) *(X,d) is $\sum$-complete.*

Theorem 2.61 ([169]). *Let (X,d) be a ν-generalized metric space. Then the following are equivalent:*

(i) *(X,ρ) is a compact metric space and the topology induced by (X,ρ) is compatible with d.*

(ii) *(X,d) is $\sum$-precomplete and compact.*

Theorem 2.62 ([169]). *Let (X,d) be a ν-generalized metric space. Then the following are equivalent:*

(i) *(X,ρ) is a metric space, the topology induced by (X,ρ) is compatible with d and every ρ-convergent net is d-Cauchy.*

(ii) *(X,ρ) is a metric space, the topology induced by (X,ρ) is compatible with d and every ρ-convergent sequence is d-Cauchy.*

(iii) $\inf\{d(x,y): y\in X\setminus\{x\}\}=0 \implies \eta(x)=0$.

(iv) *Every d-convergent sequence is d-Cauchy.*

(v) *d is sequentially jointly continuous in the sense, $\lim d(x_n,y_n)=d(x,y)$ whenever*

$$\lim d(x_n,x)=0 \text{ and } \lim d(y_n,y)=0.$$

(vi) *d is sequentially separately continuous in the sense, $\lim d(x_n,y)=d(x,y)$ whenever*

$$\lim d(x_n,x)=0.$$

Theorem 2.63 ([169])**.** *Let (X,d) be a v-generalized metric space. Then the following are equivalent:*

(i) *(X,ρ) is a compact metric space, the topology induced by (X,ρ) is compatible with d and every ρ-convergent net is d-Cauchy.*

(ii) *(X,d) is compact in strong sense.*

Using The results of Theorem 2.59-2.63, Suzuki [169] makes some other remarks on v-generalized metric spaces.

Lemma 2.64 ([169])**.** *Let (X,d) be a $\sum$-precomplete v-generalized metric space. Then X is Hausdorff.*

Proposition 2.65 ([169])**.** *For a $\sum$-complete v-generalized metric space (X,d), X is Hausdorff and X is κ-complete for any $\kappa\in\mathbb{N}$.*

Proposition 2.66 ([169])**.** *Let (X,d) be a v-generalized metric space. Then X is $\sum$-complete provided either of the following holds:*

(i) *X is $\sum$-precomplete and compact.*

(ii) *X is compact in the strong sense.*

In 2016, Suzuki worked on some another direction of v-generalized metric spaces. He used some results of [1] and by computer, he established some different results [166] on v-generalized metric spaces.

Next, we discussed on some fixed point result on v-generalized metric spaces.

Suzuki *et al.* [173] proved some fixed point theorems in 2015. Due to which, first they derived two lemmas on Cauchy sequence.

Lemma 2.67 ([173])**.** *Let (X,d) be a v-generalized metric space and $\{x_n\}$ 2-Cauchy sequence such that x_n are all different and $\sup_{n\to\infty} d(x_n,x_{n+2})=0$. Then $\{x_n\}$ is Cauchy.*

Lemma 2.68 ([173])**.** *Let (X,d) be a v-generalized metric space and $\{x_n\}$ be a sequence such that x_n are all different, $\sum_{n=1}^{\infty} d(x_n,x_{n+1})<\infty$ and $\sup_{n\to\infty} d(x_n,x_{n+2})=0$. Then $\{x_n\}$ is Cauchy.*

Lemma 2.69 ([173])**.** *Let (X,d) be a v-generalized metric space and T be a mapping on X. Assume that $\sum_{n=1}^{\infty} d(T^n(u),T^{n+1}(u))<\infty$ for some $u\in X$. Assume either v is odd or v is even and $\lim_{n\to\infty} d(T^n(u),T^{n+1}(u))=0$ holds. Then $\{T^n(u)\}$ is Cauchy.*

Their main motive was to establish a generalized version of Banach [27], Kannan [93], and Ciric's [43] contraction principles to v-generalized metric spaces, which are collected in the next theorem.

Theorem 2.70 ([173])**.** *Let (X,d) be a complete v-generalized metric space and T be a self mapping on X, for any $x,y\in X$ which satisfies either of the following:*

[1] T. Suzuki, Edelstein's fixed point theorem in generalized metric spaces – part II, *preprint*.

(i) $d(Tx,Ty) \le rd(x,y)$ *where* $r \in [0,1)$,

(ii) $d(Tx,Ty) \le r[d(x,Tx)+d(y,Ty)]$ *where* $r \in [0,\frac{1}{2})$,

(iii) $d(Tx,Ty) \le r\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$ *where* $r \in [0,1)$.

Then T *has a unique fixed point* z. *Moreover, for any* $x \in X$, $\{T^n x\}$ *converges to* z *in the strong sense.*

In 2018, Suzuki [168] generalized the famous Edelstein's fixed point theorem in compact v-generalized metric spaces. We start with some lemmas.

Lemma 2.71 ([168])**.** *Let* (X,d) *be a* v*-generalized metric space and* $\{x_n\}$ *be a sequence on* X *converging to* z. *Then the following hold:*

(i) *If* $\{x_n\}$ *is Cauchy, then* $\{x_n\}$ *converges exclusively to* z.

(ii) *If* $\{x_n\}$ *converges to* z, $\lim_{n\to\infty} d(x_n,x_{n+1})=0$ *and* $x_n \neq z$ *for any* $n \in \mathbb{N}$, *then* $\{x_n\}$ *is Cauchy, that is,* $\{x_n\}$ *converges to* z *in the strong sense.*

(iii) *If* X *is Hausdorff, then* $\{x_n\}$ *converges exclusively to* z.

(iv) $Card\{n \in \mathbb{N} : x_n = x\} < \infty$ *for any* $x \in X \setminus \{z\}$.

(v) *If* $\lim_{n\to\infty} d(x_n,w)=0$ *for some* $w \in X \setminus \{z\}$, *then* $Card\{n \in \mathbb{N} : x_n = x\} < \infty$ *for any* $x \in X$.

Lemma 2.72 ([168])**.** *Let* (X,d) *be a* v*-generalized metric space and* T *be a self mapping on* X. *Define a sequence* $\{x_n\}$ *in* X *by* $x_1 \in X$ *and* $x_{n+1} = Tx_n$. *If* $\{x_n\}$ *converges to* z *and* $\lim_{n\to\infty} d(x_n,x_{n+1})=0$, *then* $\{x_n\}$ *is Cauchy that is,* $\{x_n\}$ *converges to* z *in the strong sense.*

Lemma 2.73 ([168])**.** *Let* (X,d) *be a* v*-generalized metric space and* $\{x_n\}$ *be a Cauchy sequence in* X. *Let* $A=\{y \in X : \lim_n d(x_n,y)=0\}$. *Then if* $\liminf_n d(x_n,z)=0$ *holds for some* $z \in X$, *then* $z \in A$ *and* $Card(A) \le 1$. *Moreover, if* v *is odd, then for any sequence* $\{x_n\}$, $Card(A) \le \max\{1,\frac{v-1}{2}\}$.

Lemma 2.74 ([168])**.** *Let* (X,d) *be a* v*-generalized metric space and* $v \in \{1,3\}$. *Then* X *is Hausdorff.*

Proposition 2.75 ([168])**.** *Let* (X,d) *be a* v*-generalized metric space. Then the following are equivalent:*

(i) X *is compact in strong sense.*

(ii) X *is compact and* d *is sequentially continuous.*

Lemma 2.76 ([168])**.** *Let* (X,d) *be a* v*-generalized metric space. If* X *is compact, then* X *is complete.*

Then, they proved [160, Theorem 3.2] and [174, Theorem 3.4] in another way ([168, Lemma 26]) and also derived a finer result ([168, Theorems 28-30]) than [160, Theorem 3.2]. In [168, Section 8], they also gave proper examples in support of their claims.

The following are also some fixed point results, which are generalization of Subrahmanyam's and Caristi's fixed point theorem ([157], [39]), on v-generalized metric spaces, derived by Alamri *et al*. [7].

Theorem 2.77 ([7]). *Let (X,d) be as in Lemma 2.13 and T be a sequentially continuous mapping on X satisfying*

$$d(Tx,T^2x) \le cd(x,Tx)$$

for all $x \in X$ where $c \in [0,1)$. Then, for any $x \in X$, $\{T^nX\}$ converges only to a fixed point of T.

Theorem 2.78 ([7]). *Let (X,d) be as in Lemma 2.13 and T be a self mapping on X. Let f be a proper, sequentially lower semi-continuous functions function from X into $(-\infty,\infty]$ satisfying*

$$f(Tx)+d(x,Tx) \le f(x)$$

for all $x \in X$. Then, T has a unique fixed point.

Suzuki [165] also generalized Subrahmanyam's and Caristi's fixed point theorem in $(\sum,\neq)$-complete, v-generalized metric spaces.

Theorem 2.79 ([165]). *Let (X,d) be a $(\sum,\neq)$-complete, v-generalized metric space. Let T be a set-valued mapping on X such that Tx is non-empty subset of X, for any $x \in X$. Moreover, if a sequence $\{y_n\} \subset Tx$ converges to y implies $y \in Tx$ and $\exists\ r \in [0,1)$ satisfying $\delta(Tx,Ty) \le rd(x,y)$, $\forall\ x,y \in X$, where $\delta = \sup_{a\in A} \inf_{b\in B} d(a,b)$, then $\exists\ z \in X$ satisfying $z \in Tz$.*

Theorem 2.80 ([165]). *Let (X,d) be a $(\sum,\neq)$-complete, v-generalized metric space. Let T be a sequentially continuous mapping on X satisfying $d(Tx,T^2x) \le cd(x,Tx)$ for all $x \in X$ where $c \in [0,1)$. Then, for any $x \in X$, $\{T^nX\}$ converges to a fixed point of T in the strong sense.*

Recently, Suzuki [167] in 2020, exercised on several completeness of v-generalized metric spaces. At first, he proved the following lemmas.

Lemma 2.81 ([167]). *Let (X,d) be a v-generalized metric space. Let $\{a_n\}$ and $\{b_n\}$ be sequences in X satisfying $\lim_n \sup\{d(a_n,b_n) : m\ geqn\} = 0$. Define two subsets of X by $A = \{x \in X : Card\{n \in \mathbb{N} : a_n = x\} = \infty\}$ and $B = \{x \in X : Card\{n \in \mathbb{N} : b_n = x\} = \infty\}$. Then the following hold:*

(i) $\lim_n d(a,b_n) = 0$ *holds for all $a \in A$.*

(ii) $\lim_n d(a_n,b) = 0$ *holds for all $b \in B$.*

(iii) *If $A \neq \phi$ and $B \neq \phi$ hold, then $\exists z \in X$ satisfying $A = B = \{z\}$.*

(iv) *If $Card\{a_n : n \in \mathbb{N}\} < \infty$ and $Card\{b_n : n \in \mathbb{N}\} < \infty$ hold, then $\exists\ z \in X$ and $\mu \in \mathbb{N}$ satisfying $a_n = b_n = z$ for all $n \in \mathbb{N}$ with $n \ge \mu$.*

Lemma 2.82 ([167]). *Let (X,d) be a v-generalized metric space and $\{a_n\}$ and $\{b_n\}$ be sequences in X. Assume that there exists $\mu \in \mathbb{N}$ satisfying $\lim_n \sup\{d(a_n,b_m) : m \ge n+\mu\} = 0$. Then $\lim_n \sup\{\max\{d(a_n,b_m),d(a_m,b_n) : m \ge n\}\} = 0$ holds.*

Lemma 2.83 ([167]). *Let (X,d) be a v-generalized metric space and $\{a_n\}$ and $\{b_n\}$ be sequences in X satisfying $\lim_n \sup\{d(a_n,b_m) : m \ge n\} = 0$. Then $\lim_n \sup\{d(a_m,b_n) : m \ge n\} = 0$ holds.*

Lemma 2.84 ([167]). *Let (X,d) be a v-generalized metric space and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{e_n\}$ be sequences in X satisfying $\lim_n \sup\{d(a_n,b_m) : m \ge n\} = 0$, $\lim_n \sup\{d(b_n,c_m) : m \ge n\} =$*

0, $\limsup_n\{d(c_n,e_m) : m \geq n\} = 0$. *Assume* $Card\{b_n : n \in \mathbb{N}\} = Card\{c_n : n \in \mathbb{N}\} = \infty$. *Then* $\limsup_n\{d(a_n,e_m) : m \geq n\} = 0$ *holds.*

Lemma 2.85 ([167]). *Let (X,d) be a Hausdorff, v-generalized metric space and $\{a_n\}$ be a sequences in X converging to a. Then $\{a_n\}$ converges exclusively to a.*

Using Lemmas 2.81-2.85, Suzuki proved that 1-completeness is equivalent to 3-completeness and also discussed in 5-completeness.

Theorem 2.86 ([167]). *In a v-generalized metric space, the followings are equivalent:*

(i) *X is complete.*

(ii) *X is 3-complete.*

Theorem 2.87 ([167]). *Let (X,d) be a v-generalized metric space. Let $\lambda \in \mathbb{N}$ with $\lambda \geq 4$. Assume that X is λ-complete. Then X is $(\lambda-2)$-complete.*

Lemma 2.88 ([167]). *Let (X,d) be a κ-complete, v-generalized metric space where $\kappa \in \mathbb{N} \setminus \{1,3\}$ holds. Then X is Hausdorff.*

Theorem 2.89 ([167]). *Let (X,d) be a v-generalized metric space. Let $\kappa \in \mathbb{N}$ with $\lambda \geq 4$. Assume that X is κ-complete. Then X is $(\kappa+2)$-complete.*

Theorem 2.90 ([167]). *In a v-generalized metric space, the followings are equivalent:*

(i) *X is 5-complete.*

(ii) *X is $(2\kappa+3)$-complete for any $\kappa \in \mathbb{N}$.*

(iii) *X is $(2\kappa+3)$-complete for some $\kappa \in \mathbb{N}$.*

Theorem 2.91 ([167]). *Let (X,d) be a 2-complete, v-generalized metric space. Then X is κ-complete for any $\kappa \in \mathbb{N}$.*

Theorem 2.92 ([167]). *In a v-generalized metric space, the followings are equivalent:*

(i) *X is 2-complete.*

(ii) *X is 2κ-complete for any $\kappa \in \mathbb{N}$.*

(iii) *X is 2κ-complete for some $\kappa \in \mathbb{N}$.*

Theorem 2.93 ([167]). *Let (X,d) be a v-generalized metric space where v is odd. Then the followings are equivalent:*

(i) *X is complete and Hausdorff.*

(ii) *X is κ-complete for any $\kappa \in \mathbb{N} \setminus \{1,3\}$.*

(iii) *X is κ-complete for some $\kappa \in \mathbb{N} \setminus \{1,3\}$.*

Theorem 2.94 ([167]). *Let (X,d) be a v-generalized metric space. Then*

(i) $(\sum) \implies (2) \implies (\sum,\neq) \implies (1)$ *and* $(\sum) \implies (2) \implies (5) \implies (1)$.

(ii) *if v is odd with $v \geq 5$, then* $(\sum) \implies (2) \iff (5) \implies (\sum,\neq) \iff (1)$.

3. 2-Generalized(Rectangular or Generalized) Metric Spaces

2-generalized metric spaces are also known as rectangular or generalized metric spaces was introduced by Branciari [37], is a particular case of v-generalized metric spaces for $v = 2$.

In this section, we discuss on the development and research on generalized metric spaces. We start with the definition given by Branciari.

Definition 3.1 ([37]). Let X be a non-empty set and $d : X \times X \to [0,\infty)$ be a mapping such that for all $x, y \in X$ and for all distinct point $u, v \in X \setminus \{x, y\}$ which satisfies the following conditions:

(a) $d(x, y) = 0 \iff x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, y) \le d(x,u) + d(u,v) + d(v, y)$ (quadrilateral inequality).

Then the pair (X,d) is a generalized metric space.

Remark 3.2. Branciari [37] considered a weaker assumption(quadrilateral inequality) and so every metric space is a generalized metric space but a generalized metric space need not be a metric space.

Azam and Arshad [23] justified this by the following example.

Example 3.3. Let $X = \{1,2,3,4\}$. Define a function d on X as follows:

$$d(1,2) = d(2,1) = 3; \quad d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1;$$

$$d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = 4.$$

Then (X,d) is generalized metric space but not a metric space.

Branciari [37] also defined the notion of convergence of a sequence, Cauchy sequence, open ball in a standard way.

Definition 3.4 ([37]). Let (X,d) be a generalized metric space.

(i) A sequence $\{x_n\} \subseteq X$ is said to be convergent and converges to some $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$ and denoted this by $x_n \to x$.

(ii) A sequence $\{x_n\} \subseteq X$ is said to be a Cauchy sequence if for all $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $d(x_n, x_{n+m}) < \epsilon$, $\forall\ n \ge N$, $m = 1,2,\cdots$.

(iii) (X,d) is called complete if every Cauchy sequence in X is convergent and converges in X.

(iv) For any $x \in X$ and $r > 0$, open ball is defined by the set $B(x,r) = \{y \in X : d(x, y) < r\}$.

Ahmad *et al.* [5] provided a method to construct a generalized metric space from a family of generalized metric spaces.

Example 3.5 ([5]). Let $\{(X_n,d_n) : n \in J \subset \mathbb{N}\}$ be a family of disjoint generalized metric spaces and $X = \cup_{i\in J} X_i$. Define a mapping $d : X \times X \to [0,\infty)$ by

$$d(x, y) = \begin{cases} d_n(x, y), & \text{if } x, y \in X_n, \\ 1, & \text{if } x \in X_n,\ y \in X_m,\ m \neq n. \end{cases}$$

After wards a large number of research work have been done on fixed point theory of generalized metric spaces and most of them are the extension of metric fixed point theories. Now we collect some set of conditions which ensure the existence of fixed point for self mappings in generalized metric spaces.

Theorem 3.6. *Let (X,d) be complete generalized metric space and $f:X\to X$ be a self mapping. Consider the following conditions:*

(i) [36] $d(fx,fy)\le cd(x,y)$ *where* $c\in[0,1)$.

(ii) [47] $d(fx,fy)\le \frac{c}{2}[d(x,fx)+d(y,fy)]$ *where* $c\in[0,1)$.

(iii) [26] $d(fx,fy)\le c\max\{d(x,y),d(x,fx),d(y,fy)\}$ *where* $c\in[0,1)$.

(iv) [5] $d(fx,fy)\le \begin{cases}\lambda d(x,y)+\mu\frac{d(x,fx)d(x,fy)+d(y,fx)d(y,fy)}{d(x,fy)+d(y,fx)}, & \text{if } d(x,fy)+d(y,fx)\neq 0,\\ 0, & \text{if } d(x,fy)+d(y,fx)=0\end{cases}$
for some $\lambda,\mu\in[0,1)$ *with* $\lambda+\mu<1$.

(v) [5] $d(f^nx,f^ny)\le \begin{cases}\lambda d(x,y)+\mu\frac{d(x,f^nx)d(x,f^ny)+d(y,f^nx)d(y,f^ny)}{d(x,f^ny)+d(y,f^nx)}, & \text{if } d(x,f^ny)+d(y,f^nx)\neq 0,\\ 0, & \text{if } d(x,f^ny)+d(y,f^nx)=0\end{cases}$
for some $\lambda,\mu\in[0,1)$ *with* $\lambda+\mu<1$, $n\ge 1$.

(vi) [5] $d(f^nx,f^ny)\le\lambda d(x,y)$ *where* $\lambda\in[0,1)$, $n\ge1$.

(vii) [48] $d(fx,fy)\le\psi(d(x,y))$
where $\psi:\bar{P}\to[0,\infty)$ *is upper semi-continuous from right on* $\bar{P}$ *(the closure of the range d) satisfying* $\phi(t)<t$, $\forall\ t\in\bar{P}\setminus\{0\}$.

(viii) [82] $\theta(d(Tx,Ty))\le\theta(d(x,y))^k$ *where* $k\in(0,1)$ *and* $\theta\in\mathscr{L}$, *a set of functions* $\theta:(0,\infty)\to(1,\infty)$ *satisfying*

 (a) *θ is non-decreasing;*

 (b) *for each sequence* $\{t_n\}\subset(0,\infty)$, $\lim_{n\to\infty}\theta(t_n)=1 \iff \lim_{n\to\infty}t_n=0^+$;

 (c) *there exist* $r\in(0,1)$ *and* $l\in(0,\infty)$ *such that* $\lim_{t\to0^+}\frac{\theta(t)-1}{t^r}=l$.

(ix) [94] $\alpha(x,y)d(fx,fy)\le\psi(d(x,y))$ *where* $\alpha:X\times X\to[0,\infty)$ *and* ψ *is a function satisfying the followings:*

 (a) *ψ is continuous at 0;*

 (b) $\psi(t)<t$ *for any* $t\in\mathbb{R}^+$;

 (c) $\{\psi^n(t)\}\to0$ *as* $n\to\infty$ *for any* $t\in\mathbb{R}^+$;

 (d) $\sum_{n=1}^{\infty}\psi^n(t)$ *converges for any* $t\in\mathbb{R}^+$.

(x) [142] $\int_0^d(fx,fy)\phi(t)\le c\int_0^d(x,y)\phi(t)dt$ *where* $c\in(0,1)$ *and* $\phi:[0,\infty)\to[0,\infty)$ *is a Lebesgue-integrable mapping which is summable on each compact subset of* $[0,\infty)$, *nonnegative, and such that* $\forall\ \epsilon>0$, $\int_0^\epsilon\phi(t)dt>0$.

(xi) [89] *f is α-ψ contractive and suppose*

 (a) *f is α-admissible;*

 (b) $\exists\ x_0\in X$ *such that* $\alpha(x_0,fx_0)\ge1$ *and* $\alpha(x_0,f^2x_0)\ge1$;

 (c) *f is continuous.*

If one of the conditions discussed above is satisfied by f then f has a unique fixed point in X.

There are a lot of fixed point results for α-admissible mapping have been developed by several authors [95, 106, 107, 140].

To investigate the existence and uniqueness of fixed point theorem in generalized metric space, Asadi *et al*. [13] exercise on the concept of Geraghty [73], and Samet *et al*. [143].

Definition 3.7 ([143]). Let (X,d) be a generalized metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. A self mapping T on X is called α-ψ-Geraghty contraction if there exists Geraghty function β such that for all $x,y \in X$,

$$\alpha(x,y)\psi(d(Tx,Ty)) \le \beta(\psi(d(x,y)))\psi(d(x,y)),$$

where $\psi \in \varphi$, denotes the family of non-decreasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$\sum_{n=1}^{\infty} \psi^n(t) < \infty, \quad \text{for each } t > 0 \text{ and } \psi(t) < t.$$

Theorem 3.8 ([143]). *Let (X,d) be a complete generalized metric space, T be a self mapping, and $\alpha : X \times X \to \mathbb{R}$ be another mapping. Suppose that the following conditions are satisfied:*

(i) *T is an α-ψ-Geraghty contraction mapping;*

(ii) *T is triangular α-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_0,T^2x_0) \ge 1$;*

(iv) *T is continuous.*

Then T has a fixed point x^* in X and $\{T^n x_0\}$ converges to x^*.

Next definition is due to Asadi *et al*. [13].

Definition 3.9 ([13]). Let (X,d) be a generalized metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A self mapping T on X is called α-ψ-Geraghty contraction mapping if there exists Geraghty function β such that for all $x,y \in X$,

$$\alpha(x,y)\psi(d(Tx,Ty)) \le \beta(\psi(M(x,y)))\psi(M(x,y)),$$

where $\psi \in \varphi$, $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$.

Theorem 3.10 ([13]). *Let (X,d) be a complete generalized metric space, T be a self mapping, and $\alpha : X \times X \to \mathbb{R}$ be another mapping. Suppose that the following conditions are satisfied:*

(i) *T is an α-ψ-Geraghty contraction mapping;*

(ii) *T is triangular α-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_0,T^2x_0) \ge 1$;*

(iv) *T is continuous.*

Then T has a fixed point x^* in X and $\{T^n x_0\}$ converges to x^*.

After that Aydi *et al*. [20] established fixed point results for mapping involving generalized $(\alpha$-$\psi)$-contractive mappings.

Definition 3.11 ([20]). Let (X,d) be a generalized metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A self mapping T on X is called $(\alpha$-$\psi)$-Geraghty contraction mapping of type-I if for all $x,y \in X$,

$$\alpha(x,y)d(Tx,Ty) \le \psi(M(x,y)),$$

where $\psi \in \varphi$, $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$.

Definition 3.12 ([20])**.** Let (X,d) be a generalized metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A self mapping T on X is called $(\alpha$-$\psi)$-Geraghty contraction mapping of type-II if for all $x,y \in X$,

$$\alpha(x,y)d(Tx,Ty) \le \psi(N(x,y)),$$

where $\psi \in \varphi$, $N(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}\right\}$.

Theorem 3.13 ([20])**.** *Let (X,d) be a complete generalized metric space, T be an α-ψ-Geraghty contraction mapping of type-I on X. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_0,T^2x_0) \ge 1$;*

(iii) *T is continuous.*

Then T has a fixed point in X.

Following theorem is a generalized version of Kannan [93] type fixed point result in generalized metric spaces.

Theorem 3.14 ([2])**.** *Let (X,d) be a complete generalized metric space, and T,S be two self mappings on X such that T is continuous, one-one, and subsequentially convergent. If for $0 \le c < \frac{1}{2}$ and $\forall$ $x,y \in X$, S and T satisfies*

$$d(TSx,TSy) \le c[d(Tx,TSx) + d(Ty,TSy)]$$

then S has a unique fixed point in X. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Ninsri and Sintunavarat [123] proved some fixed point results for partial α-ψ-contractive mappings. They proved some fixed point results endowed with an arbitrary binary relation and also endowed with graph.

Definition 3.15 ([123])**.** Let (X,d) be a complete generalized metric space. A self mapping T on X is said to be partial α-ψ-contractive mapping if $\exists$ $\alpha : X \times X[0,\infty)$ and $\psi \in \varphi$ such that for all $x,y \in X$,

$$\alpha(x,y) \ge 1 \implies d(Tx,Ty) \le \psi(d(x,y)).$$

Theorem 3.16 ([123])**.** *Let (X,d) be a complete generalized metric space, and T be a partial α-ψ-contractive self mapping satisfying the following properties:*

(i) *T is an α-admissible mapping;*

(ii) *$\exists$ $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_0,T^2x_0) \ge 1$;*

(iii) *T is a continuous mapping.*

Then T has a fixed point in X.

[2]S. Moradi, Kannan fixed-point theorem on complete metric spaces and on generalized metric spaces depended an another function, *arXiv:0903.1577v1[math.FA], March 9, 2009*, DOI: 10.48550/arXiv.0903.1577.

Next, we discuss some notations on binary relation used by them.

Let (X,d) be a complete generalized metric space, and $\mathscr{R}$ be a binary relation over X. Denote $\mathscr{S} = \mathscr{R} \cup \mathscr{R}^{-1}$. Then $\mathscr{S}$ is a symmetric relation attached to $\mathscr{R}$.

Definition 3.17 ([123])**.** Let X be a nonempty set, and $\mathscr{R}$ be a binary relation over X. A self mapping T is called:

(i) a comparative mapping if

$$x,y \in X \text{ with } x\mathscr{S}y \implies (Tx)\mathscr{S}(Ty).$$

(ii) a partial α-contractive mapping with respect to $\mathscr{S}$ if $\exists$ a function $\psi \in \varphi$ such that

$$x,y \in X \text{ with } x\mathscr{S}y \implies d(Tx,Ty) \le \psi(d(x,y)).$$

Theorem 3.18 ([123])**.** *Let (X,d) be a complete generalized metric space, $\mathscr{R}$ be a binary relation over X, and T be a partial α-contractive self mapping with respect to $\mathscr{S}$ satisfying the following properties:*

(i) *T is an comparative mapping;*

(ii) *$\exists\ x_0 \in X$ such that $(x_0)\mathscr{S}(Tx_0)$ and $(x_0)\mathscr{S}(T^2x_0)$;*

(iii) *T is a continuous mapping.*

Then T has a fixed point in X.

After this, they established the existence of fixed point theorems on a generalized metric space endowed with graph as in the following:

Let (X,d) be a generalized metric space. A set $\{(x,x) : x \in X\}$ is called a diagonal of the Cartesian product $X \times X$ and is denoted by Δ. G be a graph and the set $V(G)$ be its vertices coincides with X and the set $E(G)$ of its edges contains all loops, that is, $\Delta \subseteq E(G)$. Also assume G has no parallel edges, so we can identify G with the pair $(V(G),E(G))$. Moreover, G can be treated as a weighted graph by assigning to each edge the distance between its vertices.

Definition 3.19 ([123])**.** Let X be a nonempty set endowed with graph G. A self mapping T is called:

(i) preserve edge if

$$x,y \in X \text{ with } (x,y) \in E(G) \implies (Tx,Ty) \in E(G).$$

(ii) partial α-contractive mapping with respect to $E(G)$ if $\exists$ a function $\psi \in \varphi$ such that

$$x,y \in X \text{ with } (x,y) \in E(G) \implies d(Tx,Ty) \le \psi(d(x,y)).$$

Theorem 3.20 ([123])**.** *Let (X,d) be a complete generalized metric space endowed with graph G, and T be a partial α-contractive self mapping with respect to $E(G)$ satisfying the following properties:*

(i) *T is preserve edge;*

(ii) *$\exists\ x_0 \in X$ such that $(x_0,Tx_0) \in E(G)$ and $(x_0,T^2x_0) \in E(G)$;*

(iii) *T is a continuous mapping.*

Then T has a fixed point in X.

Recently, Xue *et al*. [180], extended the concept of Boyd-Wong [35], and Das and Dey [48] proved the following fixed point results.

Theorem 3.21 ([180]). *Let (X,d) be a complete generalized metric space and T be a self mappings on X such that for all $x,y \in X$,*

$$\psi(d(Tx,Ty)) \le \phi(d(x,y)),$$

where ψ and ϕ are defined by Das and Dey [48] *with $\psi(r) > \phi(r)$, and $\liminf_{t\to r^+}\psi(t) > \lim\sup_{\phi}(t)$. Then T has a unique fixed point in X.*

Corollary 3.22 ([180]). *Let (X,d) be a complete generalized metric space and T be a self mappings on X such that for all $x,y \in X$,*

$$d(Tx,Ty) \le \phi(d(x,y)),$$

where ϕ are defined by Das and Dey [48] *with $r > \phi(r)$, and $t > \limsup_{t\to r^+}\phi(t)$, $\forall\ t > 0$. Then T has a unique fixed point in X.*

Theorem 3.23 ([180]). *Let (X,d) be a complete generalized metric space and T be a self mappings on X such that for all $x,y \in X$,*

$$\psi(d(Tx,Ty)) \le \phi(d(x,y)),$$

where ψ and ϕ are defined by Das and Dey [48] *with $\psi(r) > \phi(r)$, and $\psi(t)$ and is $\phi(t)$ are upper semi-continuous and lower semi-continuous functions from the right. Then T has a unique fixed point in X.*

Corollary 3.24 ([180]). *Let (X,d) be a complete generalized metric space and T be a self mappings on X such that for all $x,y \in X$,*

$$d(Tx,Ty) \le \phi(d(x,y)),$$

where ϕ is defined by Das and Dey [48] *with $r > \phi(r)$, and $\phi(t)$ are upper semi-continuous functions from the right. Then T has a unique fixed point in X.*

Generalized metric spaces are not necessarily satisfy all the properties (i)-(v) mentioned in introduction. Initially it was overlooked by some authors and that's why the proofs of the fixed point results does not seem right ([23, 37, 108], [3]).

This fact was first examined by Samet [144], and Sharma *et al*. [150].

Example 3.25 ([144]). Let $A = \{0,2\}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $X = A \cup B$. Define

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y;\ \{x,y\} \in A \text{ or } \{x,y\} \in B, \\ y, & x \in A,\ y \in B, \\ x, & x \in B,\ y \in A. \end{cases}$$

Then d is a generalized metric on X. It is clear that:

(i) the sequence $\{\frac{1}{n}\}$ converges to both 0 and 2 and not a Cauchy sequence;

(ii) there does not any $s > 0$ such that $B_s(0) \cap B_s(2) = \phi$ and hence the respective topology is not Hausdorff;

(iii) there does not exist any $s > 0$ such that $B_s(0) \subseteq B_{\frac{2}{3}}(\frac{1}{3})$;

(iv) d is not a continuous function.

Kirk and Shahzad [102] imposed condition to prove the distance function to be continuous in another form.

Proposition 3.26 ([102])**.** *If (X,d) is a generalized metric space which satisfies for each pair of distinct points $a,b \in X$, there is a number $r_{a,b} < 0$ such that for every $c \in X$,*

$$r_{a,b} \leq d(a,c) + d(c,b)$$

then the distance function is continuous.

Proposition 3.27 ([102])**.** *Let $\{x_n\}$ be a Cauchy sequence in a generalized metric space (X,d) and suppose $\lim_{n\to\infty} d(x_n,x) = 0$. Then $\lim_{n\to\infty} d(y,x_n) = d(y,x)$ for all $y \in X$. In particular, $\{x_n\}$ does not converge to y if $x \neq y$.*

Remark 3.28. The above preposition shows that the quadrilateral inequality implies a weaker but useful form of distance continuity.

Al-Bsoul *et al*. [8] studied the properties of generalized metric space and gave necessary and sufficient conditions for the generalized metric spaces to be a metric space.

Proposition 3.29 ([8])**.** *Let (X,d) be a generalized metric space. Let $x_i \in X$, $0 \leq i \leq n$, $n \in N$, $x_0 = x$, $x_n = y$, $x \neq x_i$. Then, either*

$$\sum_{i=1}^{n} d(x_{i-1},x_i) \geq d(x,y) \ \text{ or } \ \sum_{i=1}^{n} d(x_{i-1},x_i) \geq d(x,x_1) + d(x_1,y).$$

To work on generalized metric spaces afterwards, researchers assumed usually the Hausdorffness of the induced topology on generalized metric spaces.

Next, we collect those type results.

Theorem 3.30. *Let (X,d) be a Hausdorff and complete generalized metric space. Suppose that $T : X \to X$ such that for all $x,y \in X$, T satisfies either of the following conditions:*

(i) [136] $d(Tx,Ty) \leq k d(x,y)$ *where* $0 \leq k < 1$.

(ii) [21] $d(Tx,Ty) \leq \frac{k}{2}(d(x,Tx) + d(y,Ty))$ *where* $0 \leq k < 1$.

(iii) [21] $d(Tx,Ty) \leq \frac{1}{2}(d(x,Tx) + d(y,Ty)) - \phi(d(x,Tx),d(y,Ty))$
where $\phi : [0,\infty) \times [0,\infty) \to [0,\infty)$ *is continuous, and* $\phi(a,b) = 0$ *if and only if* $a = b = 0$.

(iv) [21] $d(Tx,Ty) \leq \frac{1}{2}(d(x,Tx) + d(y,Ty)) - \psi(\frac{1}{2}(d(x,Tx),d(y,Ty)))$ *where* $\psi : [0,\infty) \to [0,\infty)$ *is continuous and* $\psi^{-1}(\{0\}) = \{0\}$.

(v) [63] *Let φ is the family of all continuous mappings $\theta : [0,\infty) \to [0,\infty)$ such that $\theta(t) = 0 \iff t = 0$ which satisfies either of the followings:*

  (a) $\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y))$ *where $\psi,\phi \in \varphi$, ψ is non-decreasing.*

  (b) $\psi(d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y)) + Lm(x,y)$ *where $\psi,\phi \in \varphi$, ψ is non-decreasing,*

$L > 0$ and

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\},$$

$$m(x,y) = \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

(c) $\psi(d(Tx,Ty)) \le \psi(M(x,y)) - \phi(M(x,y))$ *where* $\psi, \phi \in \varphi$, ψ *is non-decreasing and*

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

(d) $d(Tx,Ty) \le k \max\{d(x,y), d(x,Tx), d(y,Ty)\}$ *where* $0 \le k < 1$.

(e) $d(Tx,Ty) \le k \max\{d(x,y), d(x,Tx), d(y,Ty)\} + L \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$ *where* $0 \le k < \frac{1}{3}$ *and* $L > 0$.

(f) $d(Tx,Ty) \le M(x,y) - \phi(M(x,y)) + Lm(x,y)$ *where* $\phi \in \varphi$ *is non-decreasing,* $L > 0$ *and*

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\},$$

$$m(x,y) = \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

(g) $\psi(d(Tx,Ty)) \le \psi(M(x,y)) - \phi(M(x,y))$ *where* $\psi, \phi \in \varphi$, ψ *is non-decreasing and*

$$M(x,y) = \max\left\{d(x,y), d(y,Ty), \frac{1+d(x,Tx)}{1+d(x,y)}\right\}.$$

(h) $d(Tx,Ty) \le k \max\left\{d(x,y), d(y,Ty), \frac{1+d(x,Tx)}{1+d(x,y)}\right\}$ *where* $0 \le k < 1$.

(vi) [63] *Let* Λ *be the set of all functions* $\phi : [0,\infty) \to [0,\infty)$ *which is Lebesgue-integrable and summable on each compact subset of* $[0,\infty)$*, nonnegative, and such that* $\forall\ \epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$*, satisfying either of the followings:*

(a) $\int_0^{d(Tx,Ty)} f(t) \le \int_0^{M(x,y)} f(t)dt - \int_0^{M(x,y)} g(t)dt + Lm(x,y)$ *where* $f, g \in \Lambda$, $L > 0$ *and*

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\},$$

$$m(x,y) = \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

(b) $\int_0^{d(Tx,Ty)} f(t) \le k \int_0^{M(x,y)} f(t)dt + Lm(x,y)$, *where* $f \in \Lambda$, $L > 0$, $0 \le k < 1$*, and* $m(x,y)$, $M(x,y)$ *defined above in case (a).*

(c) $\int_0^{d(Tx,Ty)} f(t) \le \int_0^{M(x,y)} f(t)dt - \int_0^{M(x,y)} g(t)dt$ *where* $f, g \in \Lambda$ *and*

$$M(x,y) = \max\left\{d(x,y), d(y,Ty), \frac{1+d(x,Tx)}{1+d(x,y)}\right\}.$$

(vii) [109] $\psi(d(Tx,Ty)) \le \psi(d(x,y)) - \phi(d(x,y))$ *where* $\psi \in \varphi$*, the set of all continuous and decreasing functions* $\psi : [0,\infty) \to [0,\infty)$ *such that* $\psi(t) = 0 \iff t = 0$ *and* $\phi \in \vartheta$*, the set of all continuous functions* $\phi : [0,\infty) \to [0,\infty)$ *such that* $\phi(t) = 0 \iff t = 0$.

(viii) [109] Λ *is the set of all functions* $\phi : [0,\infty) \to [0,\infty)$ *which is Lebesgue-integrable and summable on each compact subset of* $[0,\infty)$*, non-negative, and such that* $\forall\ \epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$ *satisfying:*

(a) $\int_0^{d(Tx,Ty)} \alpha(t) \le \int_0^{d(x,y)} \alpha(t)dt - \int_0^{d(x,y)} \beta(t)dt$, $\forall\ \alpha, \beta \in \Lambda$ *or*

(b) $\int_0^{d(Tx,Ty)} \alpha(t) \le k \int_0^{d(x,y)} \alpha(t)dt$ *where* $0 \le k$ *and* $\alpha \in \Lambda$.

(ix) [34] *Let* $\mathscr{F}$ *be the set of functions* $\eta : [0,\infty) \to [0,\infty)$ *satisfying the condition* $\eta(t) = 0$ *if and only if* $t = 0$ *and* $\varphi \subset \mathscr{F}$ *the set of functions* $\psi \in \varphi$ *such that* ψ *is continuous and* *nondecreasing;* $\vartheta \subset \mathscr{F}$ *is the set of functions* $\alpha \in \vartheta$ *such that* α *is continuous and* $\Lambda \subset \mathscr{F}$ *denotes the set of functions* $\beta \in \Lambda$ *such that* β *is lower semi-continuous satisfying:*

(a) $\psi(d(Tx,Ty)) \le \psi(d(x,y)) - \phi(d(x,y))$ *where* $\psi \in \varphi$, $\phi \in \Lambda$ *or*

(b) $\psi(d(Tx,Ty)) \le \alpha(d(x,y)) - \beta(d(x,y))$ *where* $\psi \in \varphi$, $\alpha \in \vartheta$, $\beta \in \Lambda$ *and these mapping satisfies the condition*

$$\psi(t) - \alpha(t) + \beta(t) > 0, \quad \forall\ t > 0.$$

Then, T *has a unique fixed point in* X.

Theorem 3.31 ([183]). *Let* (X,d) *be a Hausdorff complete generalized metric space. If* T *is a* $(\Phi\text{-}\phi)$*-weak contraction self mapping, then* T *has a unique fixed point in* X.

Next, we recollect the definition of compatible mappings in generalized metric spaces given by Jungck and Rhodas ([86], [87]).

Definition 3.32 ([86]). Let (X,d) be a generalized metric space, and let $S,F : X \to X$ be two single-valued functions. We say that S and F are compatible if $\lim_{n\to\infty} d(SFx_n, FSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(Fx_n, Sx_n) = 0$.

Definition 3.33 ([87]). Let F,G be two self mappings on a non-empty set X.

(a) A point $\alpha \in X$ is said to be a common fixed point of F and G if $\alpha = F\alpha = G\alpha$.

(b) A point $\alpha \in X$ is said to be a coincidence fixed point of F and G if $F\alpha = G\alpha$ and β is said to be a point of coincidence if $\beta = F\alpha = G\alpha$.

(c) The mappings F and G are said to be weakly compatible if they commute at their point of coincidence that is $FG\alpha = GF\alpha$ whenever $G\alpha = F\alpha$.

Theorem 3.34 ([30]). *Let* ς *is the family of all continuous non-decreasing mappings* $\sigma : [0,\infty) \to [0,\infty)$ *satisfying* $\sigma(t) = 0 \iff t = 0$ *and* φ *is the family of all lower semi-continuous* $\phi : [0,\infty) \to [0,\infty)$ *satisfying* $\phi(t) = 0 \iff t = 0$. *Also let* Λ *be the set of all functions* $\phi : [0,\infty) \to [0,\infty)$ *which is Lebesgue-integrable and summable on each compact subset of* $[0,\infty)$, *nonnegative, and such that* $\forall\ \epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$.
Let (X,d) *be a Hausdorff and complete generalized metric space and* $f,g : X \to X$ *be two self mappings such that* $f(X) \subseteq g(X)$. *Assume that* $(g(X),d)$ *is complete and one of the following conditions hold:*

(a) $\psi(d(fx,fy)) \le \psi(d(gx,gy)) - \phi(d(gx,gy))$ *where* $\psi \in \varsigma$, $\phi \in \varphi$*;*

(b) $\int_0^{d(fx,fy)} \gamma(t) \le \int_0^{d(gx,gy)} \gamma(t)dt - \phi(d(gx,gy))$, $\forall\ \gamma \in \Lambda$ *where* $\psi \in \varsigma$, $\phi \in \varphi$*;*

(c) $\int_0^{d(fx,fy)} \gamma(t) \le \int_0^{d(gx,gy)} \gamma(t)dt$, $\forall\ \gamma \in \Lambda$ *where* $0 < k < 1$*;*

for all $x, y \in X$. *Then* f *and* g *have a unique point of coincidence in* X. *Moreover, if* f *and* g *are weakly compatible, then they have a unique common fixed point.*

Theorem 3.35 ([41]). *Let* φ *be the class of functions* $\phi : [0,\infty) \to [0,\infty)$ *satisfying*

($\phi 1$) ϕ *is a weaker Meir-Keeler function;*

($\phi 2$) $\phi(t) > 0$ *for all* $t > 0$ *and* $\phi(t) = 0 \iff t = 0$*;*

($\phi 3$) $\forall t > 0$, $\{\phi^n(t)\}$ *is decreasing;*

(ϕ4) *if* $\lim_{n\to\infty} t_n = t$, *then* $\lim_{n\to\infty} \phi(t_n) \le t$;
and denote ϑ *be the class of functions* $\theta : [0,\infty) \to [0,\infty)$ *satisfying*
(θ1) θ *is a continuous function;*
(θ2) $\theta(t) > 0$ *for all* $t > 0$ *and* $\theta(t) = 0 \iff t = 0$.
Let (X,d) *be a Hausdorff and complete generalized metric space and* $f : X \to X$ *be a self mapping which satisfies either of the followings:*

(a) $d(fx,fy) \le \phi(d(x,y)) - \theta(d(x,y))$, $\forall x,y \in X$ *and* $\phi \in \varphi$, $\theta \in \vartheta$. *Then* f *has a periodic point* x *in* X.
(b) $d(fx,fy) \le \phi(d(x,y)) - \theta(d(x,y)$, $\forall x,y \in X$ *and* $\phi \in \varphi$ *with* $0 < \phi(t) < t$, $\forall t > 0$ *and* $\theta \in \vartheta$. *Then* f *has a unique fixed point* x *in* X.

Chen [40], and Arshad *et al*. [12] worked on the existence of common fixed points for self mappings on generalized metric spaces.

Definition 3.36 ([40])**.** A function $\phi : [0,\infty) \to [0,\infty)$ is said to be $\mathscr{W}$ function if it satisfies the following conditions

(ϕ1) $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$;

(ϕ2) if $\lim_{n\to\infty} t_n = t$, then $\lim_{n\to\infty} \phi(t_n) < t$.

Lemma 3.37 ([40])**.** *Let* ϕ *be a* $\mathscr{W}$*-function. Then* $\lim_{n\to\infty} \phi^n(t) = 0$, $\forall\ t > 0$.

Theorem 3.38 ([40])**.** *Let* (X,d) *be a Hausdorff and complete generalized metric space and* ϕ *be a* $\mathscr{W}$*-function.*

(a) *Let* $S,T,F,G : X \to X$ *be four single-valued function such that for all* $x,y \in X$,
$$d(Sx,Ty) \le \phi(\max\{d(Fx,Gy), d(fx,Sx), d(Gy,Ty)\}).$$
Assume that $T(X) \subset F(X)$ *and* $S(X) \subset G(X)$ *and the pairs* $\{S,F\}$ *and* $\{T,G\}$ *are compatible. If* F *or* G *is continuous, then* S,T,F*, and* G *have a unique common fixed point in* X.
(b) *Let* $T : X \to X$ *be a single-valued function such that for all* $x,y \in X$,
$$d(Tx,Ty) \le \phi(\max\{d(x,y), d(x,Tx), d(y,Ty)\}).$$
Then T *has a unique fixed point in* X.

Theorem 3.39 ([40])**.** *Let* (X,d) *be a Hausdorff and complete generalized metric space and* ϕ *be a* $\mathscr{S}$ *function, defined by* $\phi : [0,\infty)^3 \to [0,\infty)$ *which satisfies the following conditions*

(ϕ1) ϕ *is strictly increasing and continuous in each coordinate;*

(ϕ2) *for all* $t > 0$, $\phi(t,t,t) < t$, $\phi(t,0,0) < t$, $\phi(0,t,0) < t$, $\phi(0,0,t) < t$.

(a) *Let* $S,T,F,G : X \to X$ *be four single-valued function such that for all* $x,y \in X$,
$$d(Sx,Ty) \le \phi(\max\{d(Fx,Gy), d(fx,Sx), d(Gy,Ty)\}).$$
Assume that $T(X) \subset F(X)$ *and* $S(X) \subset G(X)$ *and the pairs* $\{S,F\}$ *and* $\{T,G\}$ *are compatible. If* F *or* G *is continuous, then* S,T,F*, and* G *have a unique common fixed point in* X.

(b) *Let $T:X \to X$ be a single-valued function such that for all $x,y \in X$,*

$$d(Tx,Ty) \leq \phi(\max\{d(x,y),d(x,Tx),d(y,Ty)\}).$$

Then T has a unique fixed point in X.

Theorem 3.40 ([12]). *Let (X,d) be a Hausdorff generalized metric space and let $f,g:X \to X$ be two self mappings such that $f(X) \subseteq g(X)$. Assume that (gX,d) is complete.*

Let φ be the class of all continuous functions $\theta:[0,\infty) \to [0,\infty)$ which satisfies $\theta(t)=0 \iff t=0$.

Suppose for all $x,y \in X$, f and g satisfies the condition:

(a) $\psi(d(fx,fy)) \leq \psi(M(gx,gy)) - \phi(M(gx,gy))$ *where $\psi,\phi \in \varphi$, ψ is nondecreasing and*

$$M(gx,gy) = \max\{d(gx,gy),d(gx,fx),d(gy,fy)\}.$$

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then they have a unique common fixed point.

(b) $\psi(d(fx,fy)) \leq \psi(M(gx,gy)) - \phi(M(gx,gy))$ *where $\psi,\phi \in \varphi$, ψ is nondecreasing and*

$$M(gx,gy) = \max\left\{d(gx,gy),d(gy,fy),\frac{1+d(gx,fx)}{1+d(gy,fy)}\right\}.$$

Then f and g have a unique common fixed point.

Theorem 3.41 ([12]). *Let (X,d) be a Hausdorff generalized metric space and let $f,g:X \to X$ be two self mappings such that $f(X) \subseteq g(X)$. Assume that (gX,d) is complete.*
Let Λ is the set of all functions $\phi:[0,\infty) \to [0,\infty)$ which is Lebesgue-integrable and summable on each compact subset of $[0,\infty)$, non-negative, and such that $\forall\ \epsilon > 0$, $\int_0^{\epsilon} \phi(t)dt > 0$.
Suppose for all $x,y \in X$, f and g satisfies either of the following conditions:

(a) $\int_0^{d(fx,fy)} \alpha(t)dt \leq \int_0^{M(gx,gy)} \alpha(t)dt - \int_0^{M(gx,gy)} \beta(t)dt\ \forall\ \alpha,\beta \in \Lambda$ *and*

$$M(gx,gy) = \max\{d(gx,gy),d(gx,fx),d(gy,fy)\}.$$

(b) $\int_0^{d(fx,fy)} \alpha(t) \leq k\int_0^{M(gx,gy)} \alpha(t)dt$ *where $0 \leq k$, $\alpha \in \Lambda$, and*

$$M(gx,gy) = \max\{d(gx,gy),d(gx,fx),d(gy,fy)\}.$$

(c) $\int_0^{d(fx,fy)} \alpha(t)dt \leq \int_0^{M(gx,gy)} \alpha(t)dt - \int_0^{M(gx,gy)} \beta(t)dt\ \forall\ \alpha,\beta \in \Lambda$ *and*

$$M(gx,gy) = \max\left\{d(gx,gy),d(gy,fy),\frac{1+d(gx,fx)}{1+d(gy,fy)}\right\}.$$

Then f and g have unique common fixed point.

Corollary 3.42 ([12]). *Let (X,d) be a Hausdorff and complete generalized metric space and let $f,g:X \to X$ be two self mappings such that $f(X) \subseteq g(X)$ satisfying*

$$d(fx,fy) \leq k\max\{d(gx,gy),d(gx,fx),d(gy,fy)\}$$

for all $x,y \in X$ where $0 \leq k < 1$. Then f and g have unique common fixed point.

Corollary 3.43 ([12]). *Let (X,d) be a Hausdorff and complete generalized metric space and let $f,g:X \to X$ be two self mappings such that $f(X) \subseteq g(X)$ satisfying*

$$d(fx,fy) \leq k(d(gx,gy)+d(gx,fx)+d(gy,fy))$$

for all $x,y \in X$ where $0 \leq k < 1$. Then f and g have unique common fixed point.

Corollary 3.44 ([12]). *Let (X,d) be a Hausdorff and complete generalized metric space and let $f,g: X \to X$ be two self mappings such that $f(X) \subseteq g(X)$ satisfying*

$$d(fx,fy) \le M(gx,gy) - \phi(M(gx,gy))$$

for all $y \in X$ where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function which satisfies $\theta(t) = 0 \iff t = 0$, and

$$M(gx,gy) = \max\{d(gx,gy), d(gx,fx), d(gy,fy)\}.$$

Then f and g have unique common fixed point.

Theorem 3.45 ([105]). *Let (X,d) be a Hausdorff and complete generalized metric space and $S,T,F,G : X \to X$ be four single-valued function such that for all $x,y \in X$,*

(a) $d(Sx,Ty) \le \phi(\max\{d(Fx,Gy), d(fx,Sx), d(Gy,Ty)\})$ *where ϕ is a $\mathscr{W}$ function, or*

(b) $d(Sx,Ty) \le \phi(\max\{d(Fx,Gy), d(fx,Sx), d(Gy,Ty)\})$ *where ϕ is a $\mathscr{S}$ function.*

Assume that $T(X) \subset F(X)$ and $S(X) \subset G(X)$ and the pairs $\{S,F\}$ and $\{T,G\}$ are weakly compatible. If one of the subsets $F(X)$, $G(X), T(X), S(X)$ is complete, then S,T,F, and G have a unique common fixed point in X.

Kumar *et al*. [105] proved some results on generalized metric spaces based on EA and CLR_g properties introduced by Aamri *et al*. [1], and Sintunavarat *et al*. [154].

Definition 3.46. Let (X,d) be a metric space and $g,f : X \to X$ be two mappings.

(a) [1] f and g satisfy the property EA if there exist a sequence $\{x_n\}$ such that for some $\alpha \in X$, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = \alpha$.

(b) [154] f and g satisfy the property CLR_g if there exist a sequence $\{x_n\}$ such that for some $\alpha \in X$, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(\alpha)$. Similarly CLR_f property can be defined.

Theorem 3.47 ([105]). *Let (X,d) be a Hausdorff and complete generalized metric space and ϕ is a $\mathscr{W}$-function. Let S,T,F,G be four single-valued self-mappings such that for all $x,y \in X$, they satisfies either of the conditions:*

$$d(Sx,Ty) \le \phi(\max\{d(Fx,Gy), d(fx,Sx), d(Gy,Ty)\})$$

(a) *Assume that $T(X) \subset F(X)$ and $S(X) \subset G(X)$ and the pairs $\{S,F\}$ and $\{T,G\}$ satisfy the property EA and one of the subsets $F(X)$, $G(X), T(X), S(X)$ is complete.*

(b) *Assume that*

$T(X) \subset F(X)$ and the pair $\{T,G\}$ satisfy the property CLR_G or

$S(X) \subset G(X)$ and the pair $\{S,F\}$ satisfy the property CLR_F.

Then S,T,F, and G have a unique common fixed point in X.

In 2017, Budhia *et al*. [38] proved that if rectangular metric spaces were considered as Hausdorff spaces then α-ψ type contractive mapping in such spaces ensure the existence of solution of a nonlinear fractional differential equation satisfying integral boundary conditions.

On the other hand, some researchers moves in another direction to develop fixed point theorems and they use the concept of orbitally completeness in generalized metric spaces to avoid the lack of the properties (i)-(v).

First, they consider a self mapping T on a generalized metric space (X,d) and for a fixed $x_0 \in X$, construct an iterative sequence $\{x_0, Tx_0, T^2x_0, \cdots\}$, named as T-orbit. The contraction condition satisfied by T forced the sequence $\{T^n x_0\}$ to be Cauchy. Finally, the T-orbitally completeness ensures the existence of limit of the sequence which becomes the fixed point for T. Following are collection of such fixed point theorems. First, we give the definition of T-orbit in a proper way.

Definition 3.48 ([47]). Let (X,d) be a generalized metric space and T be a self mapping on X. For each $x \in X$, $O(x,\infty)$ or $O(x) = \{x_0, Tx_0, T^2x_0, \cdots\}$ is called the T-orbit of $x \in X$. The space X is said to be orbitally complete if every Cauchy sequence in $O(x,\infty)$ converges to some $x \in X$.

Remark 3.49 ([43]). A T-orbitally complete generalized metric space may not be complete [108].

Theorem 3.50. *(X,d) be a generalized metric space and T be a self mapping on X which satisfies*

$$d(Tx,Ty) \le q d(x,y)$$

for all $x,y \in X$ where $0 < q < 1$. If X is T-orbitally complete then T has a unique fixed point in X.

Miheţ [115] shown that the existence of a fixed point for a Kannan contraction in an orbitally complete generalized metric space as a consequence of Kannan contraction theorem in a generalized metric space for which they used a lemma ([115, Lemma 2.2]) which was proved by induction on n without involving the triangle inequality.

Lemma 3.51 ([115]). *If (X,d) is a generalized metric space and $T : X \to X$ is a mapping such that, for all $x,y \in X$ T satisfies*

$$d(Tx,Ty) \le \frac{1}{2}[d(x,Tx) + d(y,Ty)]$$

for some $0 < \beta < \frac{1}{2}$, then for every $x \in X$,

$$d(T^n x; T^{n+1}x) \le \left(\frac{\beta}{1-\beta}\right)^n d(x,Tx), \quad \forall\, n \in \mathbb{N}.$$

Akram *et al.* [3] extended the notion of $\mathscr{A}$-contractions of metric spaces to generalized metric spaces.

Definition 3.52 ([6]). Let $\mathscr{A}$ be the set consisting the mappings $\alpha : \mathbb{R}^3_{\ge 0} \to \mathbb{R}_{\ge 0}$ satisfying

(i) α is continuous,

(ii) $a \le kb$ for some $k \in [0,1)$ whenever $a \le \alpha(a,b,b)$ or $a \le \alpha(b,a,b)$ or $a \le \alpha(b,b,a)$, for all a,b.

Definition 3.53 ([3]). Let (X,d) be a generalized metric space. A self mapping T on X is said to be $\mathscr{A}$-contraction if it satisfies the condition

$$d(Tx,Ty) \le \alpha(d(x,y), d(x,Tx), d(y,Ty))$$

for all $x,y \in X$ and for some $\alpha \in X$.

[3]M. Akram and A. A. Siddiqui, A fixed point theorem for A-contractions on a class of generalized metric spaces, *Korean Journal of Mathematics* **10**(2) (2003), 1 – 5.

Theorem 3.54 ([3]). *Let T be an $\mathscr{A}$-contraction on orbitally complete generalized metric space. Then T has a unique fixed point in X.*

Corollary 3.55 ([3]). *On an orbitally complete generalized metric space,*

(i) *Every Bianchini's contraction* [33] *has a unique fixed point.*

(ii) *Every Reich's contraction* [138] *has a unique fixed point.*

Definition 3.56 ([49]). A generalized metric space (X,d) is said to be ϵ-chainable if for any two points $x,y\in X$, there exist a finite set of points $x=x_0,x_1,x_2,\cdots,x_n=y$ such that $d(x_{i-1},x_i)\le\epsilon$, for $i=1,2,\cdots,n$ where $\epsilon>0$.

Definition 3.57 ([49]). A mapping $T:X\to X$ is called locally contractive if for every $x\in X$ there exists an $\epsilon_x>0$ and $\lambda_x\in[0,1)$ such that for all $p,q\in\{y:d(x,y)\le\epsilon_x\}$ the relation $d(T(p),T(q))\le\lambda_x d(p,q)$ holds.

Definition 3.58 ([49]). A mapping $T:X\to X$ is called (ϵ,λ) uniformly locally contractive if it is locally contractive at all points $x\in X$ and ϵ,λ do not depend on x that is

$$d(x,y)<\epsilon \implies d(Tx,Ty)<\lambda d(x,y), \quad \forall\, x,y\in X.$$

Remark 3.59. From the definition it is clear that a uniformly locally contractive mapping is continuous (in the usual sense).

Theorem 3.60 ([49]). *If T is an (ϵ,λ) uniformly locally contractive mapping defined on a T-orbitally complete, $\frac{\epsilon}{2}$-chainable generalized metric space X satisfying the following condition*

$$\forall\, x,y,z\in X,\ d(x,y)<\frac{\epsilon}{2} \text{ and } d(y,z)<\frac{\epsilon}{2} \implies d(x,z)<\epsilon$$

then T has a unique fixed point in X.

Remark 3.61. Das *et al.* [49] shown that the condition in Theorem 3.60 is strictly weaker than the requirement of a generalized metric space to be a metric space.

Example 3.62 ([49]). Let $X=\{a,b,c,e\}$ and $d:X\times X\to R^+$ be defined by

$$d(a,b)=.25,\ d(a,c)=d(b,c)=.1,\ d(a,e)=d(b,e)=d(c,e)=.2;\ d(x,x)=0, \quad \forall x\in X$$

$T:X\to X$ be a mapping defined by

$$Tx=\begin{cases} c, & \text{if } x\in\{a,b,c\},\\ a, & \text{if } x=e.\end{cases}$$

Then (X,d) is a $\frac{\epsilon}{2}$-chainable generalized metric space where $\epsilon=.4$ satisfying the condition of Theorem 3.60 but it is not a metric space since $d(a,b)=.25>d(a,c)+d(b,c)=.2$. It can also be noticed that T is a (ϵ,λ) uniformly locally contractive mapping with $\lambda=\frac{1}{2}$ and T has a unique fixed point c.

Theorem 3.63 ([65]). *Let (X,d) be a generalized metric space and $T:X\to X$ be a mapping such that*

$$d(Tx,Ty)\le\phi(\max\{d(x,y),d(x,Tx),d(y,Ty),d(y,Tx)\}),$$

where $\phi\in\varphi$, the set of nondecreasing upper semi-continuous functions $\phi:[0,\infty)\to[0,\infty)$ such *that* $\sum_{n=1}^{\infty}\phi^n(t)<\infty$, $\forall\ t>0$. *If there exist* $x\in X$ *such that* $O(x)$ *is orbitally, then* T *has a unique fixed point in* X.

Corollary 3.64 ([65]). *Let* (X,d) *be a generalized metric space and* $T:X\to X$ *be a mapping such that*

$$d(Tx,Ty)\le k\max\{d(x,y),d(x,Tx),d(y,Ty),d(y,Tx)\},$$

where $0\le k<1$ *and if there exist* $x\in X$ *such that* $O(x)$ *is orbitally, then* T *has a unique fixed point in* X.

Theorem 3.65 ([65]). *Let* (X,d) *be a generalized metric space and* $T:X\to X$ *be a continuous mapping such that*

$$d(Tx,T^2x)\le\psi(d(x,Tx));\ \ d(Tx,T^3x)\le\psi(d(x,T^2x)),$$

where $\psi\in\varphi$, *the set of non-decreasing functions* $\psi:[0,\infty)\to[0,\infty)$ *satisfying*

$$\sum_{n=1}^{\infty}\sum\psi^n(t)<\infty,\quad\forall\ t\ge0$$

and if there exist $x\in X$ *such that* $O(x)$ *is orbitally, then* T *has a fixed point in* X.

Corollary 3.66 ([65]). *Let* (X,d) *be a generalized metric space and* $T:X\to X$ *be a continuous mapping such that*

$$\min\{d(Tx,Ty),\max\{d(x,Tx),d(y,Ty)\}\}\le\psi(d(x,y));\ \ d(Tx,T^2x)\le d(x,Tx),$$

where $\psi\in\varphi$, *and if there exist* $x\in X$ *such that* $O(x)$ *is orbitally, then* T *has a fixed point in* X.

To generalize the Popa's Theorem [131] in generalized metric spaces, Kikina and Kikina [99] introduced a class of function which gave a general structure to the main theorems.

Definition 3.67 ([99]). Let $f:\mathbb{R}^{+3}\to\mathbb{R}$ be a upper semi-continuous function with 4 variables satisfying the properties:

(i) f is non decreasing in respect with each variable;

(ii) $f(t,t,t)\le t$, $t\in\mathbb{R}^+$;

and $\mathscr{F}_3$ denotes the set of all such functions and the functions will be called a $\mathscr{F}_3$-function.

Theorem 3.68 ([99]). *Let* (X,d), (Y,ρ) *be two generalized metric spaces and* $T:X\to Y$, $S:Y\to X$ *be two mappings such that they satisfies the inequalities:*

$$d(Sy,STx)\le cf_1(d(x,Sy),d(x,STx),\rho(y,Tx)),$$

$$\rho(Tx,TSy)\le cf_2(\rho(y,Tx),\rho(y,TSy),d(x,Sy))$$

for all $x\in X$ *and* $y\in Y$, *where* $0\le c<1$ *and* $f_1,f_2\in\mathscr{F}_3$. *If there exists* $x_0\in X$ *such that* $O(x_0)$ *is* ST-*orbitally complete in* X *and* $O(Tx_0)$ *is* TS-*orbitally complete in* Y, *then* ST *has a unique fixed point* $\alpha\in X$ *and* TS *has a unique fixed point* $\beta\in Y$. *Further,* $T\alpha=\beta$ *and* $S\beta=\alpha$.

Corollary 3.69 ([99]). *Let* (X,d), (Y,ρ) *be two generalized metric spaces and* $T:X\to Y$, $S:Y\to X$ *be two mappings satisfying the inequalities:*

$$d(Sy,STx)\le c\max\{d(x,Sy),d(x,STx),\rho(y,Tx)\},$$

$$\rho(Tx,TSy)\le c\max\{\rho(y,Tx),\rho(y,TSy),d(x,Sy)\}$$

for all $x \in X$ and $y \in Y$, where $0 \leq c < 1$. If there exists $x_0 \in X$ such that $O(x_0)$ is ST-orbitally complete in X and $O(Tx_0)$ is TS-orbitally complete in Y, then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further, $T\alpha = \beta$ and $S\beta = \alpha$.

Corollary 3.70 ([99]). *Let (X,d), (Y,ρ) be two generalized metric spaces and $T : X \to Y$, $S : Y \to X$ be two mappings they satisfies*

$$d^2(Sy,STx) \leq c_2 \max\{\rho(y,Tx)d(x,Sy), \rho(y,Tx)d(x,STx), d(x,Sy)d(x,STx)\},$$

$$\rho^2(Tx,TSy) \leq c_1 \max\{\rho(y,Tx)d(x,Sy), d(x,Sy)\rho(y,TSy), \rho(y,Tx)\rho(y,TSy)\}$$

for all $x \in X$ and $y \in Y$, where $0 \leq c_1, c_2 < 1$. If there exists $x_0 \in X$ such that $O(x_0)$ is ST-orbitally complete in X and $O(Tx_0)$ is TS-orbitally complete in Y, then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further, $T\alpha = \beta$ and $S\beta = \alpha$.

Corollary 3.71 ([99]). *Let (X,d), (Y,ρ) be two generalized metric spaces and $T : X \to Y$, $S : Y \to X$ be two mappings satisfying the inequalities:*

$$d^p(Sy,STx) \leq c \max\{d^p(x,Sy), d^p(x,STx), \rho^p(y,Tx)\},$$

$$\rho^p(Tx,TSy) \leq c \max\{\rho^p(y,Tx), \rho^p(y,TSy), d^p(x,Sy)\}$$

for all $x \in X$ and $y \in Y$, where $0 \leq c < 1$. If there exists $x_0 \in X$ such that $O(x_0)$ is ST-orbitally complete in X and $O(Tx_0)$ is TS-orbitally complete in Y, then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further, $T\alpha = \beta$ and $S\beta = \alpha$.

Corollary 3.72 ([99]). *Let (X,d), (Y,ρ) be two generalized metric spaces and $T : X \to Y$, $S : Y \to X$ be two mappings which satisfies*

$$d^2(Sy,STx) \leq a_1\rho(y,Tx)d(x,Sy) + b_1\rho(y,Tx)d(x,STx) + c_1 d(x,Sy)d(x,STx),$$

$$\rho^p(Tx,TSy) \leq a_2\rho(y,Tx)d(x,Sy) + b_2 d(x,Sy)\rho(y,TSy) + c_2\rho(y,Tx)\rho(y,TSy)$$

for all $x \in X$ and $y \in Y$, $a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4 \geq 0$ such that $a_1 + b_1 + c_1 < 1$, $a_2 + b_2 + c_2 < 1$. If there exists $x_0 \in X$ such that $O(x_0)$ is ST-orbitally complete in X and $O(Tx_0)$ is TS-orbitally complete in Y, then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further, $T\alpha = \beta$ and $S\beta = \alpha$.

Remark 3.73. Many other similar results can be obtained for different $\mathscr{F}_3$-function f.

Before going to the main results of Kikina and Kikina [97] which are extension of [64] and [138] in generalized metric spaces, we recall a definition of the family of mapping, called $\mathscr{F}_4$-function [97], introduced by them. Then they extended the theorems of Kikina and Kikina [99] for $\mathscr{F}_4$-function.

Definition 3.74 ([97]). Let $f : \mathbb{R}^{+4} \to \mathbb{R}$ be a upper semi-continuous function with 4 variables satisfying the properties:

(i) f is non decreasing in respect with each variable;

(ii) $f(t,t,t,t) \leq t$, $t \in \mathbb{R}^+$;

and $\mathscr{F}_4$ denotes the set of all such functions and the functions will be called a $\mathscr{F}_4$-function.

Theorem 3.75 ([97]). *Let (X,d), (Y,ρ) be two generalized metric spaces and $T : X \to Y$, $S : Y \to X$ be two mappings, atleast one of them is continuous such that they satisfies*

$$d(STx,STx') \le cf_1(d(x,x'),d(x,STx),d(x',STx'),\rho(Tx,Tx')),$$

$$\rho(TSy,TSy') \le cf_2(\rho(y,y'),\rho(y,TSy),\rho(y',TSy'),d(Sy,Sy'))$$

for all $x,x' \in X$ and $y,y' \in Y$, where $0 \le c < 1$ and $f_1,f_2 \in \mathscr{F}_4$. If there exists $x_0 \in X$ such that $O(x_0)$ is ST-orbitally complete in X and $O(Tx_0)$ is TS-orbitally complete in Y, then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further, $T\alpha = \beta$ and $S\beta = \alpha$.

Corollary 3.76 ([97]). *Let (X,d), (Y,ρ) be two generalized metric spaces and $T : X \to Y$, $S : Y \to X$ be two mappings, atleast one of them is continuous such that they satisfies*

$$d(STx,STx') \le c\max\{d(x,x'),d(x,STx),d(x',STx'),\rho(Tx,Tx')\},$$

$$\rho(TSy,TSy') \le c\max\{\rho(y,y'),\rho(y,TSy),\rho(y',TSy'),d(Sy,Sy')\}$$

for all $x,x' \in X$ and $y,y' \in Y$, where $0 \le c < 1$. If there exists $x_0 \in X$ such that $O(x_0)$ is ST-orbitally complete in X and $O(Tx_0)$ is TS-orbitally complete in Y, then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further, $T\alpha = \beta$ and $S\beta = \alpha$.

Corollary 3.77 ([97]). *Let (X,d), (Y,ρ) be two generalized metric spaces and $T : X \to Y$, $S : Y \to X$ be two mappings, atleast one of them is continuous such that they satisfies*

$$d(STx,STx') \le a_1 d(x,x') + a_2 d(x,STx) + a_3 d(x',STx') + a_4 \rho(Tx,Tx'),$$

$$\rho(TSy,TSy') \le b_1\rho(y,y') + b_2\rho(y,TSy) + b_3\rho(y',TSy') + b_4 d(Sy,Sy')$$

for all $x,x' \in X$, and $a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4$ are nonnegative numbers such that $0 \le a_1+a_2+a_3+a_4 < 1$, $0 \le b_1+b_2+b_3+b_4 < 1$. If there exists $x_0 \in X$ such that $O(x_0)$ is ST-orbitally complete in X and $O(Tx_0)$ is TS-orbitally complete in Y, then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further, $T\alpha = \beta$ and $S\beta = \alpha$.

Corollary 3.78 ([97]). *Let (X,d), (Y,ρ) be two generalized metric spaces and $T : X \to Y$, $S : Y \to X$ be two mappings, atleast one of them is continuous such that they satisfies*

$$d(STx,STx') \le c[d(x,STx) + d(x',STx') + \rho(Tx,Tx')],$$

$$\rho(TSy,TSy') \le c[\rho(y,TSy) + \rho(y',TSy') + d(Sy,Sy')]$$

for all $x,x' \in X$ and $y,y' \in Y$, where $0 < c < \frac{1}{3}$. If there exists $x_0 \in X$ such that $O(x_0)$ is ST-orbitally complete in X and $O(Tx_0)$ is TS-orbitally complete in Y, then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further, $T\alpha = \beta$ and $S\beta = \alpha$.

Corollary 3.79 ([97]). *Let (X,d) be a generalized metric spaces and T be a self map which satisfies the condition*

$$d(Tx,Ty) \le cf(d(x,y),d(x,Tx),d(y,Ty)),$$

for all $x,y \in X$ where $0 \le c < 1$ and $f \in \mathscr{F}_4$. If there exists $x_0 \in X$ such that $O(x_0)$ is T-orbitally complete in X, then T has a unique fixed point in X.

In 2015, Kikina and Kikina [98] also proved some fixed point theorems in generalized metric spaces for self maps in a class of almost contractions defined by an implicit relation.

Lemma 3.80 ([98])**.** *Let (X,d) be a generalized metric space, and let $\{x_n\}$ be a sequence of distinct points in X and $l \geq 0$. If $d(x_n,x_{n+1}) \leq \delta^n l$, $0 \leq \delta < 1$, $\forall\ n \in \mathbb{N}$ and $\lim_{n\to\infty} d(x_n,x_{n+2}) = 0$, then $\{x_n\}$ is a Cauchy sequence.*

Definition 3.81 ([98])**.** The set of real functions $\phi : \mathbb{R^+}^6 \to \mathbb{R}$ are called ϕ_6-function, which are upper semi-continuous in each coordinate variable and satisfy atleast one of the following conditions:

(i) if $\phi(u,v,v,u,u,0) \leq 0$, $\forall\ u,v \geq 0$, then there exists a real constant $h \in [0,1)$ such that $u \leq hv$,

(ii) if $\phi(u,v_1,v_2,v_3,0,v_4) \leq 0$, $\forall\ u,v_1,v_2,v_3,v_4 \geq 0$, then there exists a real constant $\delta \in [0,,1)$ and some $L \geq 0$ such that $u \leq \delta \max\{v_1,v_2,v_3,v_4\} + Lv_4$,

(iii) $\phi(u,u,0,0,u,u) \leq 0 \implies u = 0$.

Definition 3.82 ([98])**.** Let (X,d) be a generalized metric space, and $\phi \in \phi_6$. A self mapping T is called an almost ϕ-contraction if $\forall x,y \in X$,

$$\phi[d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(y,T2x),d(y,Tx)] \leq 0.$$

Theorem 3.83 ([98])**.** *Let (X,d) be a generalized metric space, $\phi \in \phi_6$, and T be an almost ϕ-contraction on X. If ϕ satisfies the first and second conditions of ϕ_6-function and (X,d) is T-orbitally complete, then*

(i) *$Fix(T) = x \in X : Tx = x \neq \phi$;*

(ii) *for any $x_0 \in X$, the Picard iteration $\{x_n\}$ defined by $x_n = Tx_{n-1}$, $n = 1,2,\cdots$ converges to some $\alpha \in Fix(T)$.*

Theorem 3.84 ([98])**.** *Let (X,d) be a generalized metric space, $\phi \in \phi_6$, and T be an almost ϕ-contraction on X. If ϕ satisfies the first, second, and third conditions of ϕ_6-function and (X,d) is T-orbitally complete, then*

(i) *T has a unique fixed point α in X;*

(ii) *for any $x_0 \in X$, the Picard iteration $\{x_n\}$ defined by $x_n = Tx_{n-1}$, $n = 1,2,\cdots$ converges to α.*

As we have discussed earlier, Sharma *et al*. [150] shown that a convergent sequence in generalized metric spaces may have more than one limit. But later Kadelburg and Radenovic [90] proved that this ambiguity can be removed in some special cases and this is very useful to some proofs. Turinici [[4]] shown that if a sequence in a generalized metric space is both Cauchy and convergent then limit of that sequence must be unique.

Lemma 3.85 ([90])**.** *Let (X,d) be a generalized metric space and let $\{x_n\}$ be a Cauchy sequence in X such that $x_n \neq x_m$ for $m \neq n$. Then $\{x_n\}$ converges to atmost one point.*

Lemma 3.86 ([[4]])**.** *Let (X,d) be a generalized metric space and let $\{x_n\}$ be a sequence in X which is both Cauchy and convergent. Then the limit x of $\{x_n\}$ is unique. Moreover, if $z \in X$ is arbitrary, then $\lim_{n\to\infty} d(x_n,z) = d(x,z)$.*

[4] M. Turinici, Functional contractions in local Branciari metric spaces, *arXiv:1208.4610v1* [math.GN], August 22, 2012, DOI: 10.48550/arXiv.1208.4610.

Lemma 3.87 ([89], [90]). *Let (X,d) be a generalized metric space and let $\{y_n\}$ be a sequence in X with distinct elements ($y_n \neq y_m$ for $n \neq m$). Suppose that $d(y_n, y_{n+1})$ and $d(y_n, y_{n+2})$ tend to 0 as $n \to \infty$ and that $\{y_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences tend to ϵ as $k \to \infty$:*

$$d(y_{m_k}, y_{n_k}),\ d(y_{m_k}, y_{n_k+1}),\ d(y_{m_k-1}, y_{n_k}),\ d(y_{m_k-1}, y_{n_k+1}).$$

Using these lemmas, the results of ([37], [23], [3],[108]) can be modified. For further details, see [90]. Later, common fixed point results under Geraghty-type conditions are established. On the other hand Meir-Keeler and Boyd-Wong-type results are proved in [89] and [5] without additional assumptions.

Theorem 3.88 ([89]). *Let (X,d) be a generalized metric space and let $f,g : X \to X$ be two self mappings such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. If $S = \{\beta : [0,\infty) \to [0,1)$ be a function such that $\beta(t_n) \to 1$ as $n \to \infty$ implies $t_n \to 0$ as $n \to \infty\}$, and for some function $\beta \in S$,*

$$d(fx, fy) \le \beta(d(gx, gy))d(gx, gy)$$

holds for all $x, y \in X$, then f and g have a unique point of coincidence y. Moreover, for each $x_0 \in X$, a corresponding sequence $\{y_n\}$ can be chosen such that $\lim_{n\to\infty} y_n = y$.
If, moreover, f and g are weakly compatible, then they have a unique common fixed point.

Corollary 3.89 ([89]). *Let (X,d) be a complete generalized metric space and let $f : X \to X$ be a self mapping. If for some function $\beta \in S$, and $\forall x, y \in X$, f satisfies the condition*

$$d(fx, fy) \le \beta(d(x, y))d(x, y)$$

then f has a unique fixed point.

Theorem 3.90 ([89]). *Let (X,d) be a generalized metric space and let $f,g : X \to X$ be two self mappings such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. If for some alternating distance function ψ and some $c \in (0,1)$,*

$$\psi(d(fx, fy)) \le c\psi(d(gx, gy))$$

holds for all $x, y \in X$, then f and g have a unique point of coincidence y.
If, moreover, f and g are weakly compatible, then they have a unique common fixed point.

Corollary 3.91 ([89]). *Let (X,d) be a generalized metric space and let $f : X \to X$ be a self mapping. If for some alternating distance function ψ and some $c \in (0,1)$,*

$$\psi(d(fx, fy)) \le c\psi(d(x, y))$$

holds for all $x, y \in X$, then f has a unique fixed point.

As a modification, Kadelburg and Radenović [89] proved a fixed point theorem for α-admissible mappings.

[5]Z. Kadelburg, S. Radenovic and S. Shukla, Boyd-Wong and Meir-Keeler type theorems in generalized metric spaces, *preprint*.

Theorem 3.92 ([89]). *Let (X,d) be a generalized metric space and let $f : X \to X$ be a α-ψ contractive which satisfies:*

(a) *f is α-admissible;*

(b) *$\exists\ x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(x_0, f^2x_0) \geq 1$;*

(c) *f is continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, $\forall\ n$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$, $\forall n$.*

Then f has a fixed point.

Following are some common fixed point results for compatible and weakly compatible mappings on generalized metric spaces given by Kadelburg *et al.* [5].

Theorem 3.93 ([5]). *Let (X,d) be a generalized metric space and let $f, g : X \to X$ be two self mappings such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. Let φ denote the set of all functions $\phi : [0,\infty) \to [0,\infty)$ satisfying*

(i) *$\phi(0) = 0$;*

(ii) *$\phi(t) < t$, $\forall\ t > 0$;*

(iii) *ϕ is upper semi continuous from the right that is, for any sequence $\{t_n\} \in [0,\infty)$, $t_n \to t$ as $n \to \infty$ implies $\lim_{n\to\infty} \phi(t_n) \leq \phi(t)$.*

If for $\phi \in \varphi$, $d(fx, fy) \leq \phi(d(gx, gy))$ holds for all $x, y \in X$, then f and g have a unique point of coincidence y. Moreover, for each $x_0 \in X$, a corresponding Jungck sequence $\{y_n\}$ can be chosen such that $\lim_{n\to\infty} y_n = y$.

If, moreover, f and g are weakly compatible, then they have a unique common fixed point.

Theorem 3.94 ([5], Meir-Keeler type). *Let (X,d) be a generalized metric space and let $f, g : X \to X$ be two self mappings such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. Suppose, for each $\epsilon > 0$, $\exists\ \delta > 0$ such that*

$$\epsilon \leq d(g(x), g(y)) < \epsilon + \delta \text{ implies } d(fx, fy) < \epsilon \text{ and } fx = fy \text{ whenever } g(x) = g(y).$$

Then f and g have a unique point of coincidence y. Moreover, for each $x_0 \in X$, a corresponding Jungck sequence $\{y_n\}$ can be chosen such that $\lim_{n\to\infty} y_n = y$.

If, moreover, f and g are weakly compatible, then they have a unique common fixed point.

Next they improve the result of Aydi *et al.* [21](Theorem 2.1).

Theorem 3.95 ([5]). *Let (X,d) be a generalized metric space and let $f, g : X \to X$ be two self mappings such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. Assume that the condition hold:*

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Tx) + d(y, Ty)) - \phi(d(x, Tx), d(y, Ty)),$$

where $\phi : [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous, and $\phi(a,b) = 0$ if and only if $a = b = 0$. Then f and g have a unique point of coincidence y. Moreover, for each $x_0 \in X$, a corresponding Jungck sequence $\{y_n\}$ can be chosen such that $\lim_{n\to\infty} y_n = y$.

If, moreover, f and g are weakly compatible, then they have a unique common fixed point.

They also prove the Theorem 1.3 of Jleli *et al*. [85] without the additional assumption (1.1) [85] on function ϑ.

Theorem 3.96 ([5]). *Let* (X,d) *be a complete generalized metric space and* $f : X \to X$ *be a mapping. Suppose that there exist* $\theta \in \vartheta$ *and* $k \in (0,1)$ *such that* $\forall\ x,y \in X$,

$$d(fx,fy) \neq 0 \implies \theta(d(fx,fy)) \leq [\theta(M(x,y))]^k,$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$. *Then* T *has a unique fixed point.*

Next results are on compact generalized metric spaces. Kadelburg and Radenovic [90] proved Nemytzki and Edelstien type results in compact generalized metric spaces.

Theorem 3.97 ([90]). *Let* (X,d) *be a compact generalized metric space with continuous generalized metric* d *and let* $f,g : X \to X$ *be two self mappings such that* $f(X) \subseteq g(X)$*, one of these two subsets of* X *being closed. Suppose that the following conditions hold:*

$$d(fx,fy) < d(gx,gy) \text{ for } gx \neq gy \text{ and } fx = fy \text{ whenever } gx = gy.$$

Then f *and* g *have a unique point of coincidence* y*. Moreover, for each* $x_0 \in X$*, a corresponding sequence* $\{y_n\}$ *can be chosen such that* $\lim\limits_{n\to\infty} y_n = y$.
If, moreover, f *and* g *are weakly compatible, then they have a unique common fixed point.*

Theorem 3.98 ([90]). *Let* (X,d) *be a generalized metric space with continuous generalized metric* d *and* $f : X \to X$ *be a contractive self mapping. If there exists a point* $x_0 \in X$ *such that the corresponding sequence of iterates* $\{f^n x_0\}$ *contains a convergent subsequence* $\{f^{n_i} x_0\}$*, then* $u = \lim\limits_{i\to\infty} f^{n_i} x_0$ *is a unique fixed point of* f.

Theorem 3.99 ([90]). *Let* (X,d) *be a compact generalized metric space with continuous generalized metric* d *and let* $f : X \to X$ *be a continuous mapping. Assume that, for all* $x, yinX$,

$$\frac{1}{2} d(x,fx) < d(x,y) \text{ implies } d(fx,fy) < d(x,y).$$

Then f *has a unique fixed point.*

In 2021, Dung [56] used a different technique and shown that the Proposition 4.2-4.4 of Kadelburg and Radenovic [90] can be proved without the continuity condition of d.

Proposition 3.100 ([56]). *Suppose that*

(i) (X,d) *is a generalized metric space;*

(ii) $f : X \to X$ *such that* $d(fx,fy) < d(x,y)$ *for all* $x \neq y$;

(iii) *there exists a point* $x_0 \in X$ *such that the corresponding sequence of iterates* $\{f^n(x_0)\}$ *contains a convergent subsequence* $\{f^{k_n} x_0\}$.

Then $x^* = \lim\limits_{n\to\infty} f^{k_n}(x_0)$ *is a unique fixed point of* f.

Proposition 3.101 ([56]). *Suppose that*

(i) (X,d) *is a generalized metric space;*

(ii) $f : X \to X$ *satisfying* $\exists > 0$ *such that* $0 < d(x,y) < \epsilon$ *implies* $d(fx,fy) < d(x,y)$;

(iii) *there exists a point* $x_0 \in X$ *such that the corresponding sequence of iterates* $\{f^n(x_0)\}$ *contains*

a convergent subsequence $\{f^{k_n}x_0\}$.
Then $x^* = \lim_{n\to\infty} f^{k_n}(x_0)$ *is a unique fixed point of* f.

Proposition 3.102 ([56]). *Suppose that*

(i) (X,d) *is a compact generalized metric space;*

(ii) $f : X \to X$ *such that* $\frac{1}{2}d(x,fx) < d(x,y)$ *implies* $d(fx,fy) < d(x,y)$ *for all* $x,y \in X$.

Then T *has a unique fixed point.*

Remark 3.103. There are a lot of fixed point results for multivalued mappings in metric space which are based on contractive condition that use Hausdorff-Pompeiu metric. If $CB(X)$, be the collection of closed and bounded subsets of a metric space (X,d), then the mapping $\mathscr{H}$ defined by,

$$\mathscr{H}(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A),\}, \quad \forall\ A,B \in CB(X)$$

is a metric on $CB(X)$.
Moreover, $(CB(X),\mathscr{H})$ is a complete metric space iff (X,d) is complete. But an analogous construction is not possible in generalized metric spaces.

Example 3.104 ([90]). Let $X = \{a,b,c\}$ and let $d : X \times X \to [0,\infty)$ be defined by

$$d(a,b) = 4,\ d(a,c) = d(b,c) = 1, \text{ and } d(x,x) = 0,\ d(x,y) = d(y,x), \quad \forall\ x,y \in X.$$

Clearly, (X,d) is a generalized metric space but is not a metric space.
Let $\mathscr{H}$ be defined by above, and consider the quadrilateral $(\{a\},\{b\},\{a,c\},\{c\})$, with d-closed and d-bounded vertices. It is easy to see that

$$\mathscr{H}(\{a\},\{b\}) = 4 > 1+1+1 = \mathscr{H}(\{a\},\{a,c\}) + \mathscr{H}(\{a,c\},\{c\}) + \mathscr{H}(\{c\},\{b\}).$$

Hence, rectangular inequality is not satisfied, and $(CB(X),H)$ is not a generalized metric space.

Henceforth, to develop multivalued fixed point results in generalized metric spaces, another idea of a notion of distance on $CB(X)$ is essential.

Remark 3.105. It is well-known that, in most of the cases the coupled fixed point results of metric spaces can be deduced from some definite known results for mappings with one variable with the help of the metrics

$$d_+((x,y),(u,v)) = d(x,u) + d(y,v),$$

$$d_{\max}((x,y),(u,v)) = \max\{d(x,u), d(y,v)\}$$

on the set $X \times X$ where (X,d) is a metric space.

Kadelburg and Radenovic [90] shown that an analogous development is not possible in generalized metric spaces.

Example 3.106 ([90]). Consider the generalized metric space (X,d) defined in Example 3.104, and the quadrilateral $((a,b),(b,c),(a,c),(c,c)) \in X \times X$. Then

$$d_+((a,b),(b,c)) = 5 > 1+1+1 = d_+((a,b),(a,c)) + d_+((a,c),(c,c)) + d_+((c,c),(b,c)),$$

$$d_{\max}((a,b),(b,c)) = 4 > 1+1+1 = d_{\max}((a,b),(a,c)) + d_{\max}((a,c),(c,c)) + d_{\max}((c,c),(b,c)).$$

Hence, in both cases, rectangular inequality is not satisfied and $(X \times X, d_+)$ and $(X \times X, d_{\max})$ are not generalized metric spaces.

In between those developments, Suzuki [163] in 2014, shown that generalized metric spaces did not have necessarily compatible topology. On the same time Kumam and Dung [104] exercised on the topology of generalized metric spaces.

For a generalized metric space (X,d), let τ_d and τ^d be the topologies induced by the convergence on (X,d) and by the family of open balls in (X,d) respectively. Kumam and Dung [104] established some results on the relation of τ_d and τ^d.

Proposition 3.107 ([104]). *Let (X,d) be a generalized metric space. Then $\tau_d \subset \tau^d$.*

Remark 3.108. There exists a generalized metric space (X,d) such that $\tau^d \not\subset \tau_d$. By an example they justified it.

Example 3.109 ([104]). Let (X,d) be the generalized metric space of Example 3.25. Then

$$\lim_{n\to\infty} d\left(\frac{1}{n},0\right) = 0$$

implies that $\lim_{n\to\infty} \frac{1}{n} = 0$ in (X,τ_d). Again $B_{\frac{2}{3}}\left(\frac{1}{3}\right)$is a neighborhood of 0 in (X,τ^d). Since $\{\frac{1}{n}\}$ is not eventually in $B_{\frac{2}{3}}\left(\frac{1}{3}\right)$, $\{\frac{1}{n}\}$ is not convergent to 0 in (X,τ^d). Therefore, $\tau^d \not\subset \tau_d$.

Remark 3.110. They also showed that there exists a generalized metric space (X,d) such that

$$D((x_1,y_1),(x_2,y_2)) = d(x_1,x_2)+d(y_1,y_2)$$

for all $x_1,x_2,y_1,y_2 \in X$, is not a generalized metric on $X \times X$.

Proposition 3.111 ([104]). *Let (X,d) be a generalized metric space. If (X,τ_d) has no isolated point and d is a sequentially continuous function of its variables on $(X,\tau_d)\times(X,\tau_d)$, then d is a metric on X.*

Proposition 3.112 ([104]). *Let (X,d) be a generalized metric space. For each $x, y \in X$, put*

$$\rho_d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ \inf\{\max\{d(x,u_1),d(u_1,u_2),\cdots,d(u_n,y)\} : u_1,u_2,\cdots,u_n \in X,\ n \in \mathbb{N}\}, & \text{if } x \neq y. \end{cases}$$

Then,

(i) *ρ_d is a metric on X.*

(ii) *If $\lim_{n\to\infty} x_n = x$ in (X,d), then $\lim_{n\to\infty} x_n = x$ in (X,ρ_d).*

Proposition 3.113 ([104]). *Let (X,d) be a generalized metric space. Then,*

(i) *d is sequentially continuous in its variables on $(X,\tau_d)\times(X,\tau_d)$, then (X,τ_d) is Hausdorff.*

(ii) *(X,τ^d) is Hausdorff.*

Remark 3.114. In Example 2.9 [104], Kumam and Dung showed that there exists a Hausdorff, generalized metric space (X,d) such that d is not sequentially continuous in its variables on

$$(X,\tau_d)\times(X,\tau_d).$$

To overcome this problem, they established an equivalent condition for a generalized metric d to be sequentially continuous in its variables on $(X,\tau_d)\times(X,\tau_d)$.

Proposition 3.115 ([104]). *Let (X,d) be a generalized metric space. Then d is sequentially continuous in its variables on $(X,\tau_d)\times(X,\tau_d)$ if and only if every convergent sequence on (X,τ_d) is a Cauchy sequence on (X,d).*

In 2017, Suzuki [162] proved that every generalized metric space has a sequentially compatible topology. To discuss the developments on this topic, first we recollect some notions from [162].

Define $S(x,\delta)$ and $T(x,\delta)$ by $S(x,\delta)=\{y\in X : d(x,y)<\delta\}$ and $T(x,\delta)=S(x,\delta)\setminus\{x\}$ for $x\in X$ and $\delta>0$. Define a set $F(x,\delta)$ as follows:
$f\in F(x,\delta)$ iff f is a function from $T(x,\delta)$ into $(0,\infty)$ satisfying $d(x,y)+f(y)<\delta$ for any $y\in T(x,\delta)$.
For $x\in X$, $\delta>0$ and $f\in F(x,\delta)$, we define $U(x,\delta,f)$ by $U(x,\delta,f)=\{x\}\cup\cup[S(y,f(y)) : y\in T(x,\delta)]$
Let τ be a topology on X induced by a sub-base $\{U(x,\delta,f) : x\in X,\ \delta>0,\ f\in F(x,\delta)\}$.

Lemma 3.116 ([162]). *Let $x\in X$, $\delta>0$ and $f\in F(x,\delta)$. Then the following hold:*

(i) *For any $y\in U(x,\delta,f)$, $\exists\ \epsilon>0$ satisfying $S(y,\epsilon)\subset U(x,\delta,f)$.*

(ii) *For any $y\in U(x,\delta,f)$, $\exists\ \epsilon>0$ and $g\in F(y,\epsilon)$ satisfying $U(y,\epsilon,g)\subset U(x,\delta,f)$.*

Lemma 3.117 ([162]). *Let U be an open subset of (X,τ). Then the following hold:*

(i) *For any $x\in U$, $\exists\ \delta>0$ satisfying $S(x,\delta)\subset U$.*

(ii) *For any $x\in U$, $\exists\ \delta>0$ and $f\in F(x,\delta)$ satisfying $S(x,\delta,f)\subset U$.*

Lemma 3.118 ([162]). *Let U be a subset of X. Then U is open in τ iff for any $x\in U$, $\exists\ \delta>0$ satisfying $S(x,\delta)\subset U$.*

Theorem 3.119 ([162]). (a) *τ is sequentially compatible with d.*

(b) *Every generalized metric space (X,d) has a sequentially compatible topology with d.*

(c) *(X,τ) is T_1.*

Theorem 3.120 ([162]). *Let (X,d) be a generalized metric space. Then the following are equivalent:*

(i) *(X,τ) is T_2.*

(ii) *If a sequence $\{x_n\}$ in X converges to x in (X,d), then $\liminf d(x_n,y)>0$ holds for any $y\in X\setminus\{x\}$.*

(iii) *If a sequence $\{x_n\}$, X converges to x and y in (X,d), then $x=y$ holds.*

4. 3-Generalized Metric Spaces

As a result of continue research, in 2016, Suzuki *et al.* [175] established that only 3-generalized metric spaces have a compatible topology, moreover a compatible symmetric topology. He proved that every 3-generalized metric space is metrizable. Also, shown that not every v-generalized metric spaces has a compatible symmetric topology. From which we enrich with the fact that only

1-generalized and 3-generalized metric spaces always have a compatible symmetric topology. Let us recollect all those established results on compatible topology.

Theorem 4.1 ([175]). *Let (X,d) be a 3-generalized metric space. Define a function $\rho : X \times X \to [0,\infty)$ by*

$$\rho(x,y) = \inf\left\{\sum_{j=0}^{n} d(u_j,u_{j+1}) : n \in \mathbb{N}\cup\{0\},\ u_0 = x, u_1,\cdots,u_n \in X,\ u_{n+1} = y\right\}.$$

Then (X,ρ) is a metric space, and for every $x \in X$ and for every net $\{x_\alpha\} \subset X$, $\lim_\alpha d(x_\alpha,x) = 0$ if and only if $\lim_\alpha \rho(x_\alpha,x) = 0$.

Lemma 4.2 ([175]). *Let (X,d) be a 3-generalized metric space. Define subsets A and B of X such that $x \in A$ iff there exists a sequence $\{x_n\}$ in $X\setminus\{x\}$ converging to x, and $x \in B$ iff there exists a sequence $\{x_n\}$ in $A\setminus\{x\}$ converging to x, then*

$$d(u_1,u_n) \le \sum_{j=1}^{n-1} d(u_j,u_{j+1})$$

for $u_1,u_2,\cdots,u_n \in X$ with $\{u_1,u_2,\cdots,u_n\}\cap B \neq \phi$.

Theorem 4.3 ([175]). *Let (X,d) be a 3-generalized metric space. Let A and B as in Lemma 4.2. Define $\delta_x > 0$ by*

$$\delta_x = \begin{cases} \inf\{d(x,y) : y \in X\setminus\{x\}\}, & \text{if } x \in X\setminus A,\\ \inf\{d(x,y) : y \in A\setminus\{x\}\}, & \text{if } x \in A\setminus B,\\ \infty, & \text{if } x \in B \end{cases}$$

for $x \in X$. Define a subset N_x of X by $N_x = \{S(x,r) : 0 < r < \delta_x\}$ where $S(x,r) = \{y \in X : d(x,y) < r$. Then the topology induced by a sub-base $\cup\{N_x : x \in X\}$ is compatible with d.

By an example they justified that for $v \ge 4$, v-generalized metric spaces does not have a compatible topology.

Example 4.4 ([175]). Let B and C be two nonempty subsets of X such that $X = \{a\}\cup B\cup C$, $a \notin B$, $a \notin C$, and $B\cap C = \phi$. Let M be a positive real number and $S : C \to B$, $S : C \to C$, $f : B\cup C \to (0,M]$ be two mappings. Define a function $d : X\times X \to [0,\infty)$ by

$$d(x,x) = 0,\ d(a,x) = d(x,a) = f(x) \quad \text{if } x \in B,$$

$$d(Sx,x) = d(x,Sx) = f(x) \quad \text{if } x \in C,\quad d(x,y) = M, \quad \text{otherwise.}$$

Then (X,d) is a v-generalized metric space for $v \ge 4$.

Example 4.5 ([175]). Let $X = \{(0,0)\}\cup((0,2]\times[0,2])$. Define a function $d : X\times X \to [0,\infty)$ by

$$d(x,x) = 0,\ d((0,0),(x,0)) = d((x,0),(0,0)) = x \quad \text{if } x \in (0,2],$$

$$d((x,0),(x,y)) = d((x,y),(x,0)) = y \quad \text{if } x,y \in (0,2],\quad d(x,y) = 6, \quad \text{otherwise.}$$

Then (X,d) is not a v-generalized metric space for $v = 1,2,3$, but a v-generalized metric space for $v \ge 4$. Furthermore, X does not have a topology which is compatible with d.

Next, they discussed on symmetric and semimetric spaces [175]. For details on symmetric and semimetric spaces (see [74]). They concluded that:

(i) v-generalized metric spaces (X,d) are symmetrizable. d is a symmetric on X.

(ii) Let (X,d) be a v-generalized metric space. Then X has a topology which is compatible with d in the sense of Definition 2.2 iff X is semimetrizable.

Suzuki continued his study on 3-generalized metric spaces and in 2016, he established more results on 3-generalized metric spaces [161].

Definition 4.6 ([161])**.** Let (X,d) be a 3-generalized metric space and ρ is the function defined in the Theorem 4.1. A sequence $\{x_n\}$ in X is said to be d-Cauchy if $\lim_{n\to\infty}\sup_{m>n} d(x_n,x_m)=0$ and $\{x_n\}$ in X is said to be ρ-Cauchy if $\lim_{n\to\infty}\sup_{m>n} \rho(x_n,x_m)=0$.

Lemma 4.7 ([161])**.** *If a sequence $\{x_n\}$ in X is d-Cauchy, then $\{x_n\}$ is ρ-Cauchy.*

Lemma 4.8 ([161])**.** *Let $\{x_n\}$ be a sequence in X such that $\{x_n\}$ is ρ-Cauchy and $\{x_n\}$ does not converge in (X,ρ). Define a function $g: X \to (0,\infty)$ by $g(x)=\lim_{n\to\infty}\rho(x_n,x)$. Then,*

(i) *There exists a subsequence $\{x_{k_n}\}$, of $\{x_n\}$ such that $\{x_{k_n}\}$ is d-Cauchy.*

(ii) *$g(x)=\lim_{n\to\infty} d(x_{k_n},x)$ holds for any $x\in X$.*

(iii) *$|g(x)-g(y)|\le d(x,y)\le g(x)+g(y)$ holds for any $x,y\in X$.*

(iv) *$\{x_n\}$ is d-Cauchy.*

(v) *$g(x)=\lim_{n\to\infty} d(x_n,x)$ holds for any $x\in X$.*

Suzuki concluded his work [161] with the following results.

Theorem 4.9 ([161])**.** *Let (X,d) be a 3-generalized metric space and ρ is the function defined in Theorem 4.1. Then the following are equivalent:*

(i) *(X,d) is complete.*

(ii) *(X,ρ) is complete.*

By the next example Suzuki justified his established results.

Example 4.10 ([161])**.** Define a complete subset $X=\{0\}\cup\{x_n : n\in\mathbb{N}\}\subset l^1(\mathbb{N})$, where $x_n=\left(\frac{1}{n}\right)e_n$ and $\{e_n\}$ is the canonical basis of $l^1(\mathbb{N})$. Define a metric ρ on X by $\rho(x,y)=\|x-y\|$, that is

$$\rho(x,y)=\begin{cases}\frac{1}{n}+\frac{1}{m}, & \text{if } x=x_m,\ y=x_n,\ m<n,\\ \frac{1}{n}, & \text{if } x=0, y=x_n,\\ 0, & \text{if } x=y,\\ \rho(y,x), & \text{otherwise.}\end{cases}$$

A and B be two subsets of X by $A=\{0\}$ and $B=\{x_n : n\in\mathbb{N}\}$.

Define a function $d : X \times X \to [0,\infty)$ by

$$d(x,y) = \begin{cases} 2, & \text{if } x = x_m,\ y = x_n,\ m < n, \\ \frac{1}{n}, & \text{if } x = 0,\ y = x_n, \\ 0, & \text{if } x = y, \\ d(y,x), & \text{otherwise.} \end{cases}$$

Then,

(i) (X,d) is a v-generalized metric space for any $v \geq 2$.

(ii) ρ coincides with the ρ defined by d in Theorem 4.1.

(iii) There does not exist $L \in \mathbb{R}$ such that $d(x,y) \leq Ld(x,y)$ for any $x,y \in X$.

(iv) $\{x_n\}$ converges to 0 in (X,d) and (X,ρ).

(v) $\{x_n\}$ is ρ-Cauchy, however, $\{x_n\}$ is not d-Cauchy.

Theorem 4.11 ([161]). *Let (X,d) be a 3-generalized metric space and ρ is the function defined in Theorem* 4.1. *Then the following are equivalent:*

(i) (X,d) *is compact.*

(ii) (X,ρ) *is compact.*

5. Some 'Hybrid' v-Generalized Metric Spaces

'Hybrid' spaces are such spaces where several type axioms are considered at one time. As we mentioned earlier, several generalization of standard generalized(rectangular) metric spaces, such as cone rectangular metric spaces [25], partial rectangular metric spaces [152], rectangular b-metric spaces [72], rectangular S-metric spaces [4] have been introduced and studied.
In this section, we recall the definitions and basic results of such spaces.

5.1 Cone rectangular metric spaces

Replacing the set of non-negative real numbers by an ordered real Banach space, Huang and Zhang, in 2007, introduced cone metric space [76], which is completely different from other generalized spaces. Following the cone metric and rectangular metric, in 2009, Azam and his co-workers introduced cone rectangular metric spaces [25].

Definition 5.1 ([25]). Let X be a nonempty set, $(E,\|\cdot\|)$ be a real Banach space, and '$\preceq$' be the partial ordering with respect the cone P in E. Let $d : X \times X \to E$ be a mapping which satisfies the following conditions:

(i) $d(x,y) \succeq \theta$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,y) \preceq d(x,u) + d(u,v) + d(v,y)$;

for all $x,y \in X$ and for all distinct points $u,v \in X \setminus \{x,y\}$. Then d is called a cone rectangular metric and the pair (X,d) is called a cone rectangular metric space.

They defined convergence of sequence, Cauchy sequence, etc in this setting and proved some theorems (Theorem 1-2 of [25]) as a similar manner of cone metric spaces [76] and finally proved Banach contraction principle.

Details study have been done in cone rectangular metric spaces. Researchers have shown that a huge number of fixed point results can be reduced their metric using various methods, such as scalarization method [55] and Minkowski functional [91]. Kadelburg and Radenovic [89] proved that some fixed point results in cone rectangular metric spaces [84, 110, 111] can be deduced from the respective results in 2-generalized metric spaces using Minkowski functional. For further details on fixed point results on cone rectangular metric spaces, see [3, 32, 81, 110, 122, 125, 129, 133–135, 137, 141, 148, 177–179, 182].

5.2 Complex Valued Rectangular Metric Spaces

The idea of complex valued metric spaces was introduced by Azam *et al*. in 2011 [24]. They defined a partial order '$\precsim$' on $\mathbb{C}$ as in the following:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \le Re(z_2),\ Im(z_1) \le Im(z_2).$$

It follows that $z_1 \prec z_2$, if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,

(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, $z_1 \precnsim z_2$ if $z_1 \ne z_2$ and one of (i), (ii), and (iii) is satisfied and $z_1 \prec z_2$ if only (iii) is satisfied. $0 \precsim z_1 \precnsim z_2 \implies |z_1| < |z_2|$ and $z_1 \precsim z_2$, $z_2 \prec z_3 \implies z_1 \prec z_3$.

In 2013, Abbas *et al*. [2] introduced complex valued rectangular metric spaces as in the following:

Definition 5.2. Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{C}$ is called a complex valued generalized metric on X and the pair (X,d) is called a complex valued generalized metric space if d satisfies the following conditions:

(a) $0 \precsim d(x,y)$, $\forall\ x,y \in X$ and $d(x,y) = 0 \iff x = y$;

(b) $d(x,y) = d(y,x)$, $\forall\ x,y \in X$;

(c) $d(x,y) \precsim d(x,u) + d(u,v) + d(v,y)$, $\forall\ x,y \in X$ and all distinct $u,v \in X \setminus \{x,y\}$.

Basically to extend the concept of complex valued metric spaces, they replaced the triangular inequality in the complex valued metric by the rectangular inequality involving four points. After that, in 2020, Ullah and Shagari introduced complex valued extended rectangular b-metric spaces [149] followed by the concept of complex valued rectangular b-metric spaces, introduced by Ege [62] in 2015.

In the papers [68, 114, 128, 153, 156, 158, 159] more development has been done in complex valued rectangular metric spaces.

5.3 Partial Rectangular Metric Spaces

The concept of partial rectangular metric as a generalization of 2-generalized metric [37] and partial metric [112], was brought to light by Shukla [152] in 2014.

Definition 5.3. [152] A mapping $d : X \times X \to [0,\infty)$ is called a partial rectangular metric on a nonempty set X if $\forall\ x,y \in X$ it satisfies the followings:

(i) $d(x,y) \geq 0$;

(ii) $d(x,y) = d(x,x) = d(y,y)$ if and only if $x = y$;

(iii) $d(x,y) = d(y,x)$;

(iv) $d(x,x) \leq d(x,y)$;

(v) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y) - d(u,u) - d(v,v)$, for all distinct points $u,v \in X \setminus \{x,y\}$.

The pair (X,d) is called a partial rectangular metric space.

After defining the notion of convergence of a sequence, Cauchy sequences in partial rectangular metric space (see Definition 11 of [152]), Shukla proved some generalized version of well-known fixed point results.

Dung and Hang [58] exercised on the relation between partial rectangular metric spaces and rectangular metric spaces and proved that fixed point theorems on partial rectangular metric spaces can be deduced from certain fixed point theorems on rectangular metric spaces.

First, we recall the relation between partial rectangular metric and rectangular metric established by Shukla [152].

Theorem 5.4 ([152]). *Let (X,d) be a partial rectangular metric space and $d^r(x,y) = 2d(x,y) - d(x,x) - d(y,y)$, $\forall\ x,y \in X$. Then,*

(i) *d^r is a rectangular metric on X.*

(ii) *$\lim\limits_{n\to\infty} x_n = x$ in (X,d) if and only if $\lim\limits_{n\to\infty} x_n = x$ in (X,d^r).*

(iii) *A sequence $\{x_n\}$ is Cauchy in (X,d) if and only if it is Cauchy in (X,d^r).*

Dung and Hang [58] at first introduced some notations as follows.

Definition 5.5 ([58]). Let (X,d) be a partial rectangular metric space.

(i) A sequence $\{x_n\}$ is called 0-Cauchy if $\lim\limits_{m,n\to\infty} d(x_n,x_m) = 0$.

(ii) (X,d) is called 0-complete if for any 0-Cauchy sequence $\{x_n\}$ in X, $\exists\ x \in X$ such that $\lim\limits_{n\to\infty} d(x_n,x) = d(x,x) = \lim\limits_{mn\to\infty} d(x_n,x_m) = 0$.

Lemma 5.6 ([58]). *Let (X,d) be a partial rectangular metric space.*

(i) *If (X,d) is complete then it is 0-complete.*

(ii) *If $x_n \neq x$, $y_n \neq y$, $x_n \neq y_n$, $\forall\ n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$, $d(x,x) = d(y,y) = 0$, then $\lim\limits_{n\to\infty} d(x_n,y_n) = d(x,y)$.*

The following example shows that the converse implication of Lemma 5.6(i) is not hold.

Example 5.7. Let $X = (0,1)$ and $d(x,y) = |x-y| + 2$, $\forall\ x,y \in X$. Then (X,d) is a 0-complete partial rectangular metric space but not complete.

Theorem 5.8 ([58]). *Let (X,p) be a partial rectangular metric space and define a function d_p on X by*

$$d_p(x,y) = \begin{cases} 0, & \text{if } x = y, \\ p(x,y), & \text{if } x \neq y. \end{cases}$$

Then the followings are hold:

(i) d_p *is a rectangular metric on* X.

(ii) *The partial rectangular metric space* (X,p) *is 0-complete iff the rectangular metric space* (X,d_p) *is complete.*

Theorem 5.9 ([58]). *Let* (X,p) *be a partial rectangular metric space and* T *be a quasi contraction on* X. *Then* T *is a quasi contraction on the rectangular metric space* (X,d_p).

Using the above theorem (Theorem 5.9), Dung and Hang have shown that Theorem 6 of Shukla's [152], can be proved easily from the result of rectangular metric spaces. By this argument, many fixed results of rectangular metric spaces ([97, 100, 102]) can be transformed to partial rectangular metric spaces. Shukla later proved some fixed point on G-F contraction principle on this space [151].

5.4 Rectangular b-Metric Spaces

In 2015, George *et al*. [72] introduced rectangular b-metric spaces generalizing the concept of b-metric [45, 46] and 2-generalized metric [37] and proved a generalized version of Banach and Kannan fixed point theorems in rectangular b-metric spaces [72].

Definition 5.10 ([72]). Let X be a nonempty set and a mapping $d : X \times X \to [0,\infty)$ is called a rectangular b-metric on X if it satisfies the followings:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $\exists$ a real number $s \geq 1$ such that $d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$, $\forall\ x,y \in X$ and all distinct points $u,v \in X \setminus \{x,y\}$.

The pair (X,d) is called a rectangular b-metric space (in short RbMS) with coefficient s.

Remark 5.11. Every metric space is a rectangular metric space and every rectangular metric space is a rectangular b-metric space. But the converse implication is not necessarily true.

They defined open ball, convergence of a sequence, Cauchy sequence (Definition 1.6 [72]), etc. in rectangular b-metric spaces in a similar way. Like generaized metric spaces, in rectangular b-metric spaces also the induced topology is not necessarily Hausdorff, the limit of a sequence is not necessarily unique and every convergent sequence is not necessarily a Cauchy sequence. The following example is given by George *et al*. [72] to justify the results.

Example 5.12 ([72]). Let $X = A \cup B$ where $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, $B = \mathbb{N}$ and define a function $d : X \times X \to [0,\infty)$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 2\alpha, & \text{if } x,y \in A, \\ \frac{\alpha}{2n}, & \text{if } x \in A,\ y \in \{2,3\}, \\ \alpha, & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ is a constant. Then (X,d) is a rectangular b-metric space with coefficient $s = 2$. However, not a rectangular metric space or b-metric space, the open ball $B_{\frac{\alpha}{2}}\left(\frac{1}{2}\right)$ is not an open set, the sequence $\frac{1}{n}$ converges to both 2 and 3, and hence not a Cauchy sequence.

The concept of $b_v(s)$-metric space [116] was introduced by Mitrović and Radenović as a generalization of metric space, rectangular metric space, b-metric space, rectangular b-metric space, and v-generalized metric space. They just multiplied the constant coefficient ($s \geq 1$) of the b-metric on the right-hand side of the inequality of the v-generalized metric. Using some new results they proved some fixed point theories in a new approach.

In 2019, Mustafa *et al*. [121], proposed the notion of extended rectangular b-metric spaces as in the following.

Definition 5.13 ([121])**.** Let X be a nonempty set. $\Omega : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with $t \leq \Omega(t)$, $\forall\ t > 0$ and $\Omega(0) = 0$ and $d : X \times X \to [0,\infty)$ be a mapping such that for all $a, b \in X$ all distinct points $u, v \in X$, each distinct from a and b satisfies the following conditions:

(i) $d(a,b) = 0 \iff a = b$;

(ii) $d(a,b) = d(b,a)$;

(iii) $d(a,b) \leq \Omega[d(a,u) + d(u,v) + d(v,b)]$;

for all distinct points $u, v \in X \setminus \{a,b\}$. Then the pair (X,d) is called a extended rectangular b-metric space (ERbMS).

Remark 5.14. We have $t \geq \Omega^{-1}(t)$, $\forall\ t > 0$ and $\Omega^{-1}(0) = 0$. Each rectangular b-metric space is an ERbMS with $\Omega(t) = st$, $s \geq 1$.

As a generalization of cone metric space, cone b-metric space and cone rectangular metric space, the concept of generalized cone b-metric spaces was introduced by George *et al*. [71]. There is another one generalized space called partial rectangular b-metric spaces which was introduced by Asim *et al*. [17].

In the papers [9, 10, 16, 22, 28, 29, 42, 53, 57, 70, 72, 88, 117, 118, 121, 124, 127, 130, 132, 145, 155, 181, 184],[6], several fixed point results in rectangular b-metric, extended rectangular b-metric, cone rectangular b-metric, partial rectangular b-metric spaces have been developed. However, most of those results can be easily derived from the concept of classical generalized metric spaces and 2-generalized metric spaces.

5.5 Rectangular M-metric Spaces

Rectangular M-metric spaces are generalization of M-metric spaces [14] and rectangular metric spaces, was brought to light by Özgür *et al*. [126] in 2018.

Definition 5.15. Let X be a nonempty set and $m_r : X \times X \to [0,\infty)$ be a function. Then m_r is said to be a rectangular M-metric if the following conditions are satisfied for all $x, y \in X$

(RM1) $m_r(x,y) = m_{r_{x,y}} = M_{r_{x,y}} \iff x = y$;

(RM2) $m_{r_{x,y}} \leq m_r(x,y)$;

[6] M. Rossafi and A. Kari, New fixed point theorems for (ϕ, F)-contraction on rectangular b-metric spaces, *arXiv preprint*, arXiv:2201.05689, January 11, 2022, DOI: 10.48550/arXiv.2201.05689.

(RM3) $m_r(x,y) = m_r(y,x)$;

(RM4) $m_r(x,y) - m_{r_{x,y}} \leq m_r(x,y) - m_{r_{x,u}} + m_r(u,v) - m_{r_{u,v}} + m_r(v,y) - m_{r_{v,y}}$,
where $m_{r_{x,y}} = \min\{m_r(x,x), m_r(y,y)\}$ and $M_{r_{x,y}} = \max\{m_r(x,x), m_r(y,y)\}$.
The pair (X, m_r) is called a rectangular M-metric space.

For more fixed point results in this setting, see [15, 18, 80].

5.6 Rectangular S-metric Spaces

Recently in 2022, Adewale and Iluno [4] extended the concept of S-metric to rectangular metric and introduced rectangular S-metric spaces.

Definition 5.16 ([4]). A mapping $S : X \times X \to [0,\infty)$ on a nonempty set X is called a rectangular S-metric if it satisfies the following conditions:

(i) $S(x,y,z) = 0$ if and only if $x = y = z$;

(ii) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$;

for all $x,y,z \in X$ and for all $a \in X \setminus \{x,y,z\}$. Then the pair (X,S) is called a rectangular S-metric space.

In Examples 2.2-2.4 of [4], it is shown that rectangular S-metric is a proper generalization of rectangular metric. Like others, in this new setting, they also defined open ball, rectangular S-metric topology, the convergence of a sequence, Cauchy sequence, etc. and finally proved Banach, Kannan and Zamfirescu's fixed point theorem in rectangular S-metric spaces whose reduced results are the corresponding theorems in rectangular metric spaces (see [4]).

6. Conclusion

The main motive behind this review article is to generalize, complement, unify, and enrich the already established results in v-generalized metric spaces. In this review article, we considered only those aspects of v-generalized metric spaces from beginning to the recent time which are appeared in the publications. v-generalized metric spaces was introduced by Branciari in 2000. Day by day, various fixed point results on v-generalized metric spaces and its various hybrid generalized spaces (viz. cone rectangular metric spaces, partial rectangular metric spaces, rectangular b-metric spaces, extended rectangular b-metric spaces, rectangular S-metric spaces, rectangular M-metric spaces) were provided by several authors. In future, we think more interesting results will be obtained regarding those spaces. This survey will help to enrich the concept of generalization of metric spaces, the development and the mode of construction of fixed point results in v-generalized metric spaces and its hybrid spaces, metrization problems, etc. There are large number of researchers involving themselves in the metrization problem and completeness of 2-generalized, 3-generalized metric spaces. As a result we become enriched by a lot of remarkable publications. We hope motivated researchers on this particular field will be highly encouraged and enriched their works through this review article.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

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