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Absolute Convergence of Multiple Series of Fourier-Haar Coefficients of Functions of Bounded p^{th}-Power Hardy Type Variation

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Abstract. For functions of two variables having bounded p^{th}-power variation of Hardy type on unit square the sufficient condition for absolute convergence of double series of Fourier-Haar coefficients with power type weights is obtained. From this condition we obtain two corollaries for absolute convergence of the series of Fourier-Haar coefficients of functions of one variable of bounded Wiener p^{th}-power variation or belonging to the class $\operatorname{Lip}\alpha$. The main result and all corollaries are sharp. N-dimensional analogs of main result and corollaries are formulated.

1. Introduction

The Haar orthonormal system $\{\chi_n\}_{n=1}^{\infty}$ had been defined in 1909 (see [10], [11]). By this system A. Haar gave positive answer on the question of D. Hilbert: is there an orthogonal system such that Fourier series with respect to this system of any continuous function converges uniformly to that function.

The functions of the system $\{\chi_n\}_{n=1}^{\infty}$ are step functions on the interval $[0,1]$. Therefore this system does not be a basis in the space $C[0,1]$ of continuous functions. But G. Faber [4] in 1910 proved that each continuous function on the interval $[0,1]$ can be uniquely represented by the series with respect to the system $\{1,\int_0^x \chi_n(t)dt\}_{n=1}^{\infty}$. Consequently the last system is a basis in the space $C[0,1]$. Therefore G. Faber constructed a basis in the space $C[0,1]$ on 17 years earlier than J. Shauder [17]. In 1928 J. Shauder [18] proved that the Haar system is a basis in the spaces $L^p[0,1]$ for $1 \le p < \infty$.

After introducing the notion of the wavelet system it was observed that the Haar system is the most simple wavelet system (see the book of I.Ya. Novikov, V.Yu. Protasov and M.A. Skopina [16, p. 39]).

The Haar system is orthogonal basis in the space $L^2[0,1]$.

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We consider the absolute convergence of the series of Fourier-Haar coefficients. For functions of several variables having bounded p^{th}-power variation of Hardy type on the unit N-dimensional cube $[0,1]^N$ the sufficient condition for absolute convergence of multiple series of Fourier-Haar coefficients with power type weights is obtained. From this condition follow some corollaries on absolute convergence of the series of Fourier-Haar coefficients of functions of one variable of bounded Wiener p^{th}-power variation or belonging to the class $\operatorname{Lip}\alpha$, $0<\alpha\le 1$.

The main result and all corollaries are sharp.

2. **Auxiliary results**

We remind the definition of Haar system. Let us set $\chi_1(x)\equiv 1$ on $[0,1]$. We introduce the open dyadic intervals $I_i^k = \left(\frac{i-1}{2^k}, \frac{i}{2^k}\right)$, $i=1,\dots,2^k$, $k=0,1,\dots$. Let us represent the natural number $n\ge 2$ in the form $n=2^k+i$, $i=1,\dots,2^k$, $k=0,1,\dots$. Then we set $\chi_n(x)=\sqrt{2^k}$ for $x\in I_{2i-1}^{k+1}$, $\chi_n(x)=-\sqrt{2^k}$ for $x\in I_{2i}^{k+1}$ and $\chi_n(x)=0$ for $x\in[0,1]\setminus\bar{I}_i^k$, where $\bar{I}_i^k$ is closure of the interval I_i^k. The Haar functions $\chi_n(x)$ are step functions. If the function $\chi_n(x)$ has a jump in some point $x\in(0,1)$, then we set

$$\chi_n(x)=\frac{1}{2}[\chi_n(x-0)+\chi_n(x+0)].$$

At the end points of the interval $[0,1]$ we set $\chi_n(0)=\chi_n(+0)$ and $\chi_n(1)=\chi_n(1-0)$.

The information on Fourier-Haar series one can find in the book of B.S. Kashin and A.A. Sahakyan [14] and in the survey paper of the author [8].

We denote by $V[0,1]$ the class of functions of bounded variation on the interval $[0,1]$, and by $\operatorname{Lip}\alpha$ the class of functions satisfying the Lipschitz condition of order $\alpha\in(0,1]$.

Absolute convergence of the series of Fourier-Haar coefficients for the first time was studied by Z. Cisielski and J. Musielak [3]. They proved the following result.

Theorem 2.1. *If $f\in V[0,1]\cap\operatorname{Lip}\alpha$ $(0<\alpha\le 1)$, then $\sum_{n=1}^{\infty}|\hat{f}(n)|<\infty$, where*

$$\hat{f}(n)=\int_0^1 f(x)\chi_n(x)dx$$

are the Fourier-Haar coefficients of f.

P.L. Ulyanov [21] proved the following sharp theorem.

Theorem 2.2. *For functions $f\in V[0,1]$ the series*

$$\sum_{n=1}^{\infty}|\hat{f}(n)|^{\beta} \quad \text{and} \quad \sum_{n=1}^{\infty}n^{\gamma-1/2}|\hat{f}(n)| \tag{2.1}$$

converge, if $\beta>2/3$ or $\gamma<1$ accordingly. But this statement does not hold, if $\beta=2/3$ or $\gamma=1$.

It follows from this theorem that the condition $f \in \mathrm{Lip}\,\alpha$ in Theorem 2.1 is unnecessary. Moreover, the convergence of the first series in (2.1) for any $\beta > 2/3$ is more strong condition then convergence of the series $\sum_{n=1}^{\infty} |\hat{f}(n)|$.

The author [9] generalized the Theorem 2.2 on the class $V_p[0,1]$, $1 \le p < \infty$, which was introduced by N. Wiener [23].

Definition 2.3. The function f is said to be a function of bounded p^{th}-power variation on the unit interval $[0,1]$, if

$$V(f)_p = \sup_{\tau} \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p \right\}^{1/p} < \infty, \ 1 \le p < \infty,$$

where $\tau = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$ is arbitrary partition of the interval $[0,1]$. The class of all such functions is denoted by $V_p[0,1]$.

Let us note that the imbedding $\mathrm{Lip}(1/p) \subset V_p[0,1]$ holds, if $1 \le p < \infty$, and $V_q[0,1] \subset V_p[0,1]$, if $1 \le q < p < \infty$.

Theorem 2.4 (see [9]). (i) *For a function $f \in V_p[0,1]$, $1 \le p < \infty$, the series* (2.1) *converge, if $\beta > 2p/(p+2)$ or $\gamma < 1/p$ respectively.*

(ii) *The statement of the item (i) does not hold, if $\beta = 2p/(p+2)$ or $\gamma = 1/p$ respectively.*

Let us note that the second part of Theorem 2.4 is proved by means of a function belonging to the class $\mathrm{Lip}(1/p) \subset V_p[0,1]$. More exactly, we consider the Weierstrass nondifferentiable function

$$f_\alpha(x) = \sum_{m=0}^{\infty} 2^{-\alpha m} \cos 2^{m+1}\pi x, \quad 0 < \alpha < 1,$$

on the interval $[0,1]$. G.H. Hardy [12] proved that $f_\alpha \in \mathrm{Lip}\,\alpha$. The statement of the second part of Theorem 2.4 follows from

Lemma 2.5. *For the functions $f_{1/p}(x)$, $1 < p < \infty$, and $f_1(x) = 1 - 2x$, the first series in (2.1) diverge if $\beta = 2p/(p+2)$ and the second one diverges if $\gamma = 1/p$, where $1 \le p < \infty$.*

Therefore from the Theorem 2.4 and Lemma 2.5 taking in account the imbedding $\mathrm{Lip}(1/p) \subset V_p[0,1]$, $1 \le p < \infty$, we obtain

Theorem 2.6 (see [9]). (i) *Let $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$, and $\beta > 2/(2\alpha+1)$ or $\gamma < \alpha$. Then both series in* (2.1) *converge.*

(ii) *The statement of the item* (i) *does not hold, if $\beta = 2/(2\alpha+1)$ or $\gamma = \alpha$ respectively.*

The statement of Theorem 2.6 is an analog of a theorem of O. Szasz (see the book [2, p. 647]) and of a theorem of G. Hardy [12] related to trigonometric Fourier coefficients.

We give sharp sufficient conditions on the parameters $\beta > 0$ and γ for convergence of the series

$$\sum_{n=1}^{\infty} n^{\gamma} |\hat{f}(n)|^{\beta}$$

of functions $f \in V_p[0,1]$, $1 \le p < \infty$, or $f \in \operatorname{Lip} \alpha$, $0 < \alpha \le 1$.

Similar results are given for functions of N variables having bounded p^{th}-power variation of Hardy type on the unit cube $[0,1]^N$.

3. **The results for functions of two variables**

Let us define the class of functions of bounded p^{th}-power variation of Hardy type on the unit square $\Delta = [0,1] \times [0,1]$. For two partitions $\tau_1 = \{0 = x_0 < x_1 < \ldots < x_r = 1\}$ and $\tau_2 = \{0 = y_0 < y_1 < \ldots < y_l = 1\}$ of the interval $[0,1]$ and for a function $f(x,y)$ defined on the unit square Δ we set

$$V_{1,0}(f)_p = \sup_{0 \le y \le 1} \sup_{\tau_1} \left\{ \sum_{i=1}^{r} |f(x_i, y) - f(x_{i-1}, y)|^p \right\}^{1/p}, \tag{3.1}$$

$$V_{0,1}(f)_p = \sup_{0 \le x \le 1} \sup_{\tau_2} \left\{ \sum_{j=1}^{l} |f(x, y_j) - f(x, y_{j-1})|^p \right\}^{1/p}, \tag{3.2}$$

$$V_{1,1}(f)_p = \sup_{\tau_1, \tau_2} \left\{ \sum_{i=1}^{r} \sum_{j=1}^{l} |\Delta_{1,1} f(x_i, y_j)|^p \right\}^{1/p} \tag{3.3}$$

where $1 \le p < \infty$ and

$$\Delta_{1,1} f(x_i, y_j) = f(x_i, y_j) - f(x_{i-1}, y_j) - f(x_i, y_{j-1}) + f(x_{i-1}, y_{j-1}).$$

Definition 3.1. The function $f(x,y)$ is said to be a function of bounded p^{th}-power variation of Hardy type on the unit square Δ, if the values (3.1)-(3.3) are finite for a given $p \in [1, \infty)$. The class of all such functions is denoted by $H_p(\Delta)$.

The class $H_1(\Delta)$ had been defined by G.H. Hardy [13]. Let us remark that in the case $1 \le p' < p'' < \infty$ the imbedding $H_{p'}(\Delta) \subset H_{p''}(\Delta)$ is valid.

For a function $f \in L(\Delta)$ we set

$$\hat{f}(m,n) = \iint_{\Delta} f(x,y) \chi_m(x) \chi_n(y) \, dx \, dy,$$

i.e. $\hat{f}(m,n)$ are Fourier coefficients of the function f with respect to double Haar system $\{\chi_m(x)\chi_n(y)\}_{m,n=1}^{\infty}$.

Theorem 3.2. (i) *Let us assume $f \in H_p(\Delta)$, $1 \le p < \infty$, and $\gamma + 1 < \beta(1/p + 1/2)$, $\beta > 0$. Then the series $\sum\limits_{m=1}^{\infty} \sum\limits_{n=1}^{\infty} (mn)^{\gamma} |\hat{f}(m,n)|^{\beta}$ converges.*

(ii) *The statement of the item* (i) *does not hold, if $\gamma + 1 = \beta(1/p + 1/2)$, $\beta > 0$, $1 \le p < \infty$.*

Taking $\gamma = 0$ or $\beta = 1$ we obtain from Theorem 3.2 two corollaries.

Corollary 3.3. (i) *Let* $f \in H_p(\Delta)$, $p \in [1, \infty)$. *Then* $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\hat{f}(m,n)|^{\beta} < \infty$, *if* $\beta > \frac{2p}{p+2}$.

(ii) *The statement of the item* (i) *does not hold, if* $\beta = \frac{2p}{p+2}$.

Corollary 3.4. (i) *Let* $f \in H_p(\Delta)$, $p \in [1, \infty)$. *Then* $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{\gamma} |\hat{f}(m,n)| < \infty$, *if* $\gamma < 1/p - 1/2$.

(ii) *The statement of the item* (i) *does not hold, if* $\gamma = 1/p - 1/2$.

The Corollaries 3.3 and 3.4 are two dimensional analogs of one dimensional result of the author [9] (see Theorem 2.4 above).

If $\|f(\cdot + h, \cdot + \eta) - f(\cdot, \cdot)\|_C = O((h^2 + \eta^2)^{\alpha/2})$, $(h, \eta) \in \Delta$, then we shall write $f \in \mathrm{Lip}(\alpha, \Delta)$, $0 < \alpha \leq 1$.

It is easy to prove the imbedding $\mathrm{Lip}(\alpha, \Delta) \subset H_{2/\alpha}(\Delta)$, $0 < \alpha \leq 1$. Therefore it follows from the Theorem 3.2

Corollary 3.5. *Let* $f \in \mathrm{Lip}(\alpha, \Delta)$, $0 < \alpha \leq 1$. *Then* $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{\gamma} |\hat{f}(m,n)|^{\beta} < \infty$, *if* $\gamma + 1 < \beta(\alpha + 1)/2$, $\beta > 0$.

The statement of this corollary is known (see [19], where also it is proved that in the case $\gamma + 1 = \beta(\alpha + 1)/2$, $\beta > 0$, $0 < \alpha < 1$, the statement of the Corollary 3.5 does not hold). In the case $\gamma = 0$ Corollary 3.5 was proved earlier in the paper [20]. Moreover, in this paper it is proved that the statement of the Corollary 3.5 is not true, if $\gamma = 0$ and $\beta(\alpha + 1)/2 = 1$.

Let us observe that in the papers of U. Goginava [5] (for $p = 1$) and [6] (for $1 < p < \infty$) the classes $PBV_p(\Delta)$, $1 \leq p < \infty$, of functions of bounded partial p^{th}-power variation on the unit square $\Delta = [0,1] \times [0,1]$ were considered.

Definition 3.6. A function f is said to be of bounded partial p^{th}-power variation on the unit square $\Delta = [0,1] \times [0,1]$, if the values (3.1) and (3.2) are finite.

It is obvious that the imbedding $H_p(\Delta) \subset PBV_p(\Delta)$, $1 \leq p < \infty$, holds. This imbedding is sharp.

In the paper [1] the following result is proved.

Theorem 3.7. *If* $f \in PBV_p(\Delta)$, $p \in [1, \infty)$, *and* $\gamma + 1 < \beta(1/2p + 1/2)$, $\beta > 0$, *then* $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{\gamma} |\hat{f}(m,n)|^{\beta} < \infty$.

From this theorem we obtain the following corollaries.

Corollary 3.8. *If* $f \in PBV_p(\Delta)$, $p \in [1, \infty)$, *then* $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\hat{f}(m,n)|^{\beta} < \infty$ *for* $\beta > \frac{2p}{p+1}$.

Corollary 3.9. *If $f \in PBV_p(\Delta)$, $p \in [1,\infty)$, then* $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\gamma}|\hat{f}(m,n)| < \infty$ *for* $\gamma < 1/2p - 1/2$.

From the papers [19] and [20] it follows that the statement of the Theorem 3.7 does not hold, if $\gamma + 1 = \beta(1/2p + 1/2)$, $\beta > 0$, $1 < p < \infty$.

Let us observe that each function $f(x)$ of one variable we may consider as a function $f(x,y)$ of two variables. In this case we have obviously $V_{0,1}(f)_p = V_{1,1}(f)_p = 0$ (see (3.2) and (3.3)). Therefore, each function $f(x)$ of one variable belonging to the Wiener class $V_p[0,1]$, $p \in [1,\infty)$, also belongs to the class $H_p(\Delta)$. In this case the series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\gamma}|\hat{f}(m,n)|^{\beta}$ has the form $\sum_{m=1}^{\infty}(m)^{\gamma}|\hat{f}(m)|^{\beta}$ since $\hat{f}(m,n) = 0$ if $n \geq 2$ and $\hat{f}(m,1) = \hat{f}(m)$, $m \geq 1$.

Therefore, from the Theorem 3.2 it follows

Theorem 3.10. (i) *Let $f \in V_p[0,1]$, $1 \leq p < \infty$, and $\gamma + 1 < \beta(1/p + 1/2)$, $\beta > 0$. Then* $\sum_{n=1}^{\infty} n^{\gamma}|\hat{f}(n)|^{\beta} < \infty$.

(ii) *The statement of the item* (i) *does not hold, if $\gamma + 1 = \beta(1/p + 1/2)$, $\beta > 0$, $1 \leq p < \infty$.*

In the cases $p = 1$, $\gamma = 0$ and $p = 1$, $\beta = 1$ Theorem 3.10 was proved by P.L. Ulyanov [21] (see Theorem 2.2 above) and in the cases $\gamma = 0$ or $\beta = 1$ it was proved by the author [9] (see Theorem 2.4 above).

Taking in account the imbedding $\operatorname{Lip}\alpha \subset V_{1/\alpha}[0,1]$, $0 < \alpha \leq 1$, and the fact that the second statement of Theorem 3.10 is proved by means of functions from the class $\operatorname{Lip}(1/p)$, we can obtain from Theorem 3.10 as a corollary the following

Theorem 3.11. (i) *Let $f \in \operatorname{Lip}\alpha$, $0 < \alpha \leq 1$, and $\gamma + 1 < \beta(\alpha + 1/2)$, $\beta > 0$. Then* $\sum_{n=1}^{\infty} n^{\gamma}|\hat{f}(n)|^{\beta} < \infty$.

(ii) *The statement of the item (i) does not hold, if $\gamma + 1 = \beta(\alpha + 1/2)$, $\beta > 0$, $0 < \alpha \leq 1$.*

Let us observe that Theorem 3.11 in the cases $\gamma = 0$ or $\beta = 1$ was proved in our paper [9]. In the case $\gamma = 0$ Theorem 3.11 is an analog of a theorem of O. Szasz (see [2, p. 647]), and in the case $\beta = 1$ it is an analog of a theorem of G. Hardy [12] related to trigonometric Fourier coefficients.

4. Multidimensional results

In this section we shall formulate the multidimensional analog of the Theorem 3.2. To this aim we introduce the class $V_p([0,1]^N)$, $1 \leq p < \infty$, of functions of N variables having bounded p^{th}-power Vitali type variation on the N-dimensional unit cube $[0,1]^N$. Let us consider the partition Π of the cube $[0,1]^N$ by hyperplanes $x_s = x_s^{(r_s)}$, where $x_s^{(r_s)} < x_s^{(r_s+1)}$, $r_s = 0,\dots,l_s$, $x_s^{(0)} = 0$, $x_s^{(l_s)} = 1$, $s = 1,\dots,N$.

For a function $f(x) \equiv f(x_1, \ldots, x_N)$, which is defined on the cube $[0,1]^N$, we introduce the difference

$$\Delta_s^{(r_s)}(f,x) = f(x_1, \ldots, x_s^{(r_s+1)}, \ldots, x_N) - f(x_1, \ldots, x_s^{(r_s)}, \ldots, x_N)$$

with respect to the variable x_s, $s = 1, \ldots, N$. After that we consider the iterated difference $\Delta_N^{(r_N)} \ldots \Delta_1^{(r_1)}(f, \circ)$ of order N. This difference in fact does not depend on the point x.

Definition 4.1. The function $f(x) \equiv f(x_1, \ldots, x_N)$ is said to be of bounded p^{th}-power Vitali type variation on the N-dimensional unit cube $[0,1]^N$, if

$$V^{(N)}(f)_p \equiv \sup_{\Pi} \left\{ \sum_{r_1=0}^{l_1-1} \cdots \sum_{r_N=0}^{l_N-1} |\Delta_N^{(r_N)} \ldots \Delta_1^{(r_1)}(f, \circ)|^p \right\}^{1/p} < \infty,$$

for some $p \in [1, \infty)$. The class of all such functions we denote by $V_p([0,1]^N)$.

The class $V_1([0,1]^N)$ for the case $N \geq 2$ had been introduced by Vitali [22] in 1908, and later in 1910 this class was also introduced by Lebesgue [15].

Now we define the class $H_p([0,1]^N)$, $N \geq 2$, of functions of bounded p^{th}-power Hardy type variation on the unit cube $[0,1]^N$. Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ be s-dimensional integer-valued vector whose coordinates satisfy the inequalities $1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_s \leq N$, where $1 \leq s < N$. By $\bar{\alpha} = (\bar{\alpha}_1, \ldots, \bar{\alpha}_{N-s})$ we denote the integer-valued vector, whose coordinates consist of those integers $1, 2, \ldots, N$ which do not belong to the sequence $\alpha_1, \alpha_2, \ldots, \alpha_s$ and $\bar{\alpha}_1 < \ldots < \bar{\alpha}_{N-s}$. After that each point $x \in \mathbb{R}^N$ can be written by convention in the form $x = (x^\alpha, x^{\bar{\alpha}})$, where $x^\alpha = (x_{\alpha_1}, \ldots, x_{\alpha_s})$, $x^{\bar{\alpha}} = (x_{\bar{\alpha}_1}, \ldots, x_{\bar{\alpha}_{N-s}})$. If the point $x^{\bar{\alpha}}$ is fixed then the function $f(x^\alpha, x^{\bar{\alpha}})$ be a function of s variables $x_{\alpha_1}, \ldots, x_{\alpha_s}$.

Definition 4.2. The function $f \in V_p([0,1]^N)$ is said to be of bounded p^{th}-power Hardy type variation on the unit cube $[0,1]^N$, if

$$H^{(N)}(f)_p \equiv \sup_{\bar{\alpha}, \dim \bar{\alpha} < N} \ \sup_{x^{\bar{\alpha}} \in [0,1]^{N-s}} V^{(s)}(f(\circ, x^{\bar{\alpha}}))_p < \infty$$

for some $p \in [1, \infty)$. The class of all such functions we denote by $H_p([0,1]^N)$.

In the case $N = 2$, $p = 1$ this definition was introduced by Hardy [13].

In order to formulate the N-dimensional analog of the Theorem 3.2 let us consider N-multiple Haar system $\{\chi_n(x)\}$. Each function $\chi_n(x)$ of this system is defined by the N-dimensional integer-valued vector $n = (n_1, \ldots, n_N)$ and $\chi_n(x) = \chi_{n_1}(x_1) \cdots \chi_{n_N}(x_N)$. For a function $f \in L([0,1]^N)$ its Fourier-Haar coefficients are defined by the equality

$$\hat{f}(n) = \int_{[0,1]^N} f(x) \chi_n(x) dx.$$

Theorem 4.3. *Let $f \in H_p([0,1]^N)$, $1 \le p < \infty$. Then*

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_N=1}^{\infty} (n_1 \cdots n_N)^{\gamma} |\hat{f}(n)|^{\beta} < \infty, \quad \text{if } \gamma + 1 < \beta(1/p + 1/2), \ \beta > 0.$$

This statement does not hold, if $\gamma + 1 = \beta(1/p + 1/2)$, where $\beta > 0$, $1 \le p < \infty$.

Theorem 4.3 is proved by induction based on the Theorem 3.2. From this theorem by setting $\gamma = 0$ or $\beta = 1$, we obtain the following corollaries.

Corollary 4.4. *If $f \in H_p([0,1]^N)$ for some $p \in [1,\infty)$, then*

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_N=1}^{\infty} |\hat{f}(n_1, \dots, n_N)|^{\beta} < \infty \ \text{ for } \ \beta > \frac{2p}{p+2}.$$

This statement does not hold, if $\beta = \frac{2p}{p+2}$.

Corollary 4.5. *If $f \in H_p([0,1]^N)$ for some $p \in [1,\infty)$, then*

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_N=1}^{\infty} (n_1 \cdots n_N)^{\gamma} |\hat{f}(_1, \dots, n_N)| < \infty \ \text{ for } \ \gamma < 1/p - 1/2.$$

This statement does not hold, if $\gamma = 1/p - 1/2$.

Let us formulate the N-dimensional analog of the Corollary 3.5. By definition $f \in \mathrm{Lip}(\alpha, [0,1]^N)$, $0 < \alpha \le 1$, if $\sup\limits_{|x-y| \le \delta, x,y \in [0,1]^N} |f(x) - f(y)| = O(\delta^{\alpha})$.

It is easy to prove the imbedding $\mathrm{Lip}(\alpha, [0,1]^N) \subset V_{N/\alpha}([0,1]^N)$, $0 < \alpha \le 1$. Taking in account this imbedding we obtain the N-dimensional analog of the Corollary 3.5.

Corollary 4.6. *Let $f \in \mathrm{Lip}(\alpha, [0,1]^N)$, $0 < \alpha \le 1$ and $\gamma + 1 < \beta(\alpha/N + 1/2)$, $\beta > 0$. Then*

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_N=1}^{\infty} (n_1 \cdots n_N)^{\gamma} |\hat{f}(n)|^{\beta} < \infty.$$

In the case $\gamma = 0$ the statement of this Corollary is known (see [7], where also it is proved, that in this case the statement of this Corollary does not hold, if $\beta\,(\alpha/N + 1/2) = 1$).

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References

[1] A. Aplakov, Absolute convergence of the double series of Fourier-Haar coefficients, *Acta Math. Acad. Paedag. Nyiregyhaz.* **22** (2006), 33–39.

[2] N.K. Bari, *Trigonometric Series* (in Russian), Fizmatgiz Publisher, Moscow, 1961.

[3] Z. Ciesielski and J. Musielak, On absolute convergence of Haar series, *Colloq. Math.* **7**(1) (1959), 61–65.

[4] G. Faber, Über die Orthogonalfunktionen des Herrn Haar, *Jahresberichte Deutsch. Math. Verein.* **19** (1910), 104–112.

[5] U. Goginava, On the uniform summability of multiple Walsh-Fourier series, *Anal. Math.* **26**(3) (2000), 209–226.

[6] U. Goginava, Uniform convergence of N-dimensional trigonometric Fourier series, *Georgian Math. J.* **7**(4) (2000), 665–676.

[7] L. Gogoladze and V. Tsagareishvili, Absolute convergence of multiple Fourier series, *Georgian Mathematical Union, First Intern. Conf.*, Batumi, September 12-19, Book of Abstracts, pp. 24–25 (2010).

[8] B.I. Golubov, Series with respect to Haar system (in Russian), *Mathematical Analysis*, 1970, pp. 109–146. Acad. Sci. USSR, Allunion Inst. Sci. and Techn. Information, Moscow, 1971.

[9] B.I. Golubov, On Fourier series of continuous functions with respect to Haar system (in Russian), *Izv. Akad. Nauk SSSR. Ser. Math.* **28**(6) (1964), 1271–1296.

[10] A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Inauguraldissertation (Göttingen Univ., 1909).

[11] A. Haar, Zur Theorie der orthogonalen Funktionensysteme, *Math. Annalen* **69** (1910), 331–371.

[12] G.H. Hardy, Weierstrass nondifferentiable function, *Trans. Amer. Math. Soc.* **17** (1916), 301–325.

[13] G.H. Hardy, On double Fourier series and especially those which represent the double zeta-function with real and incommensurable parameters, *Quart. J. Math.* **37**(1) (1905), 57–79.

[14] B.S. Kashin and A.A. Sahakyan, *Orthogonal series* (in Russian), Fizmatlit Publisher, Moscow, 1984.

[15] A. Lebesgue, Sur l'integration des functions discontinue, *Ann. Ecol. Normal Super.*, Ser. 3, **27** (1910), 361–450.

[16] I.Ya. Novikov, V.Yu. Protasov and M.A. Skopina, Theory of wavelets (in Russian), Fizmatlit Publisher, Moscow, 2005.

[17] J. Schauder, Zur Theorie stetiger Abbildungen in Funktionalräumen, *Math. Zeit.* **26** (1927), 47–65.

[18] J. Schauder, Eine Eigenschaft des Haarschen Orthogonalsystems, *Math. Zeit.* **28** (1928), 317–320.

[19] G.Z. Tabatadze, On absolute convergence of Fourier-Haar series (in Russian), *Proc. Georg. Acad. Sci.* **103**(3) (1981), 541–543.

[20] V.Sh. Tsagareishvili, On Fourier-Haar coefficients (in Russian), *Proc. Georg. Acad. Sci.* **81**(1) (1976), 29–31.

[21] P.L. Ulyanov, On the series with respect to Haar system (in Russian), *Mat. Sb.* **63** (3) (1964), 356–391.

[22] G. Vitali, Sui gruppi di punti e sulle funzioni di variable, *Atti Accad. Sci. Torino* **43** (1908), 75–92.

[23] N. Wiener, The quadratic variation of a function and its Fourier coefficients, *Massachusetts J. Math.* **3** (1924), 72–94.

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