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Research Article

Some Identities on Sums of Finite Products of Chebyshev Polynomials of the Third and Fourth Kinds

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Abstract. In this paper, we will introduce some identities involving sums of the finite products of Chebyshev polynomials of the third and fourth kinds, Fibonacci, and Lucas numbers in terms of the derivatives of Pell, Fibonacci, Jacobi, Gegenbauer, Vieta-Fibonacci, Vieta-Pell, and second-kind Chebyshev polynomials.

Keywords. Chebyshev polynomials, Pell polynomials, Fibonacci polynomials, Jacobi polynomials, Gegenbauer polynomials, Vieta-Fibonacci polynomials, Vieta-Pell polynomials

Mathematics Subject Classification (2020). 11B39, 11B83, 33C47

1. Introduction

For all integers, $\alpha \ge 2$ and $|\xi| < 1$, the Fibonacci numbers ($\mathcal{F}_\alpha$), Lucas numbers ($\mathcal{L}_\alpha$), Fibonacci polynomials ($\mathcal{F}_\alpha(\xi)$), Pell polynomials ($\mathcal{P}_\alpha(\xi)$), Vieta-Fibonacci polynomials ($\mathcal{S}_\alpha(\xi)$), Vieta-Pell polynomials ($\mathcal{R}_\alpha(\xi)$), Chebyshev polynomials of the first ($\mathcal{T}_\alpha(\xi)$), second ($\mathcal{U}_\alpha(\xi)$), third ($\mathcal{V}_\alpha(\xi)$), and fourth ($\mathcal{W}_\alpha(\xi)$) kinds, Gegenbauer polynomials ($\mathcal{C}(\alpha : \mu)(\xi)$), and Jacobi polynomials ($\mathcal{P}(\alpha : \mu, \lambda)(\xi)$) [1, 4, 5, 7] are defined recursively as follows:

$$\mathcal{F}_\alpha = \mathcal{F}_{\alpha-1} + \mathcal{F}_{\alpha-2},\ \mathcal{F}_0 = 0,\ \mathcal{F}_1 = 1, \tag{1.1}$$

$$\mathcal{L}_\alpha = \mathcal{L}_{\alpha-1} + \mathcal{L}_{\alpha-2},\ \mathcal{L}_0 = 2,\ \mathcal{L}_1 = 1, \tag{1.2}$$

$$\mathcal{F}_\alpha(\xi) = \xi\mathcal{F}_{\alpha-1}(\xi) + \mathcal{F}_{\alpha-2}(\xi),\ \mathcal{F}_0(\xi) = 0,\ \mathcal{F}_1(\xi) = 1, \tag{1.3}$$

$$\mathcal{P}_\alpha(\xi) = 2\xi\mathcal{P}_{\alpha-1}(\xi) + \mathcal{P}_{\alpha-2}(\xi),\ \mathcal{P}_0(\xi) = 0,\ \mathcal{P}_1(\xi) = 1, \tag{1.4}$$

$$\mathcal{S}_\alpha(\xi) = \xi\mathcal{S}_{\alpha-1}(\xi) - \mathcal{S}_{\alpha-2}(\xi),\ \mathcal{S}_0(\xi) = 0,\ \mathcal{S}_1(\xi) = 1, \tag{1.5}$$

$$\mathcal{R}_\alpha(\xi) = 2\xi\mathcal{R}_{\alpha-1}(\xi) - \mathcal{R}_{\alpha-2}(\xi),\ \mathcal{R}_0(\xi) = 0,\ \mathcal{R}_1(\xi) = 1, \tag{1.6}$$

$$\mathcal{T}_\alpha(\xi) = 2\xi\mathcal{T}_{\alpha-1}(\xi) - \mathcal{T}_{\alpha-2}(\xi),\ \mathcal{T}_0(\xi) = 1,\ \mathcal{T}_1(\xi) = \xi, \tag{1.7}$$

$$\mathcal{U}_\alpha(\xi) = 2\xi\mathcal{U}_{\alpha-1}(\xi) - \mathcal{U}_{\alpha-2}(\xi),\ \mathcal{U}_0(\xi) = 1,\ \mathcal{U}_1(\xi) = 2\xi, \tag{1.8}$$

$$\mathcal{V}_\alpha(\xi) = 2\xi\mathcal{V}_{\alpha-1}(\xi) - \mathcal{V}_{\alpha-2}(\xi),\ \mathcal{V}_0(\xi) = 1,\ \mathcal{V}_1(\xi) = 2\xi - 1, \tag{1.9}$$

$$\mathcal{W}_\alpha(\xi) = 2\xi\mathcal{W}_{\alpha-1}(\xi) - \mathcal{W}_{\alpha-2}(\xi),\ \mathcal{W}_0(\xi) = 1,\ \mathcal{W}_1(\xi) = 2\xi + 1, \tag{1.10}$$

$$\begin{aligned}&\mathcal{C}(\alpha:\mu)(\xi) = \frac{1}{\alpha}[2\xi(\alpha+\mu-1)\mathcal{C}(\alpha-1:\mu)(\xi) - (\alpha+2\mu-2)\mathcal{C}(\alpha-2:\mu)(\xi)],\\ &\mathcal{C}(0:\mu)(\xi) = 1,\ \mathcal{C}(1:\mu)(\xi) = 2\mu\xi,\ \mu > -\frac{1}{2},\end{aligned} \tag{1.11}$$

$$\begin{aligned}&2(\alpha+1)(\mu+\lambda+\alpha+1)(\mu+\lambda+2\alpha)\mathcal{P}(\alpha+1:\mu,\lambda)(\xi)\\ &\quad = (\mu+\lambda+2\alpha+1)[(\mu^2-\lambda^2)+(\mu+\lambda+2\alpha)(\mu+\lambda+2\alpha+2)]\mathcal{P}(\alpha:\mu,\lambda)(\xi),\\ &-2(\mu+\alpha)(\lambda+\alpha)(\mu+\lambda+2\alpha+2)\mathcal{P}(\alpha-1:\mu,\lambda)(\xi),\\ &\mathcal{P}(0:\mu,\lambda)(\xi) = 1,\ \mathcal{P}(1:\mu,\lambda)(\xi) = \frac{1}{2}[\mu-\lambda+(\mu+\lambda+2)\xi],\quad \mu,\lambda > -1.\end{aligned} \tag{1.12}$$

Many authors have studied the elementary properties of the Chebyshev polynomials and obtained many interesting results. For instance, Zhang [7] has studied the finite sums of the products of Chebyshev polynomials, Fibonacci numbers, and Lucas numbers and derived interesting results, particularly

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{U}_{a_1}(\xi).\mathcal{U}_{a_2}(\xi).\mathcal{U}_{a_3}(\xi).\dots.\mathcal{U}_{a_{r+1}}(\xi) = \frac{1}{2^r r!}\mathcal{U}^r_{\alpha+r}(\xi),$$

where $\mathcal{U}^r_\alpha(\xi)$ denotes the r^{th} derivative of $\mathcal{U}_\alpha(\xi)$ with respect to ξ and the summation runs over all the $r+1$-dimension non-negative integer coordinates $(a_1, a_2, \dots, a_{r+1})$ such that $a_1 + a_2 + \cdots + a_{r+1} = \alpha$.

In the same line, this paper will attempt to introduce some more identities involving sums of the finite products of Chebyshev polynomials of third and fourth kind, Fibonacci numbers, and Lucas numbers in terms of pell, Fibonacci, Jacobi, Gegenbauer, Vieta-Fibonacci, and Vieta-Pell polynomials and second-kind Chebyshev polynomials. The main results of this paper are:

Theorem 1.1. *For $\alpha, r \geq 0$, we have*

$$\begin{aligned}&\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{V}_{a_1}(i\xi).\mathcal{V}_{a_2}(i\xi).\mathcal{V}_{a_3}(i\xi).\dots.\mathcal{V}_{a_{r+1}}(i\xi)\\ &\quad = \frac{1}{2^r r!}\sum_{t=0}^{\alpha}(-1)^t\binom{r+1}{t} i^{\alpha-t}\mathcal{P}^r_{\alpha-t+r+1}(\xi)\end{aligned} \tag{1.13}$$

$$
\begin{aligned}
&= \frac{1}{r!}\sum_{t=0}^{\alpha}(-1)^t\binom{r+1}{t} i^{\alpha-t}\mathcal{F}^r_{\alpha-t+r+1}(2\xi) \\
&\quad\cdot \sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{W}_{a_1}(i\xi).\mathcal{W}_{a_2}(i\xi).\mathcal{W}_{a_3}(i\xi).\ldots.\mathcal{W}_{a_{r+1}}(i\xi) \qquad (1.14)\\
&= \frac{1}{2^r r!}\sum_{t=0}^{\alpha}\binom{r+1}{t} i^{\alpha-t}\mathcal{P}^r_{\alpha-t+r+1}(\xi) \qquad (1.15)\\
&= \frac{1}{r!}\sum_{t=0}^{\alpha}\binom{r+1}{t} i^{\alpha-t}\mathcal{F}^r_{\alpha-t+r+1}(2\xi), \qquad (1.16)
\end{aligned}
$$

where $i=\sqrt{-1}$ *and* $\mathcal{P}^r_\alpha(\xi)$ *and* $\mathcal{F}^r_\alpha(\xi)$ *are the* r^{th} *derivatives of the Pell polynomial and Fibonacci polynomials.*

Theorem 1.2. *For* $\alpha, r \geq 0$, *we have*

$$
\begin{aligned}
&\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{V}_{a_1}(\xi).\mathcal{V}_{a_2}(\xi).\mathcal{V}_{a_3}(\xi).\ldots.\mathcal{V}_{a_{r+1}}(\xi) \\
&= \frac{1}{2^r r!}\frac{(\alpha+1)!\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}\right)}\sum_{t=0}^{\alpha}(-1)^t\binom{r+1}{t}\mathcal{P}^r\left(\alpha-t+r:\frac{1}{2},\frac{1}{2}\right)(\xi) \qquad (1.17)\\
&= \frac{1}{2^r r!}\sum_{t=0}^{\alpha}(-1)^t\binom{r+1}{t}\mathcal{C}^r(\alpha-t+r:1)(\xi) \\
&\quad\cdot \sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{W}_{a_1}(\xi).\mathcal{W}_{a_2}(\xi).\mathcal{W}_{a_3}(\xi).\ldots.\mathcal{W}_{a_{r+1}}(\xi) \qquad (1.18)\\
&= \frac{1}{2^r r!}\frac{(\alpha+1)!\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}\right)}\sum_{t=0}^{\alpha}\binom{r+1}{t}\mathcal{P}^r\left(\alpha-t+r:\frac{1}{2},\frac{1}{2}\right)(\xi) \qquad (1.19)\\
&= \frac{1}{2^r r!}\sum_{t=0}^{\alpha}\binom{r+1}{t}\mathcal{C}^r(\alpha-t+r:1)(\xi), \qquad (1.20)
\end{aligned}
$$

where $\mathcal{P}^r(\alpha:\mu,\lambda)(\xi)$ *and* $\mathcal{C}^r(\alpha:\mu)(\xi)$ *are the* r^{th} *derivative of Jacobi polynomials and Gagenbauer polynomials, respectively.*

Theorem 1.3. *For* $\alpha, r \geq 0$, *we have*

$$
\begin{aligned}
&\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{V}_{a_1}\left(\frac{\xi}{2}\right).\mathcal{V}_{a_2}\left(\frac{\xi}{2}\right).\mathcal{V}_{a_3}\left(\frac{\xi}{2}\right).\ldots.\mathcal{V}_{a_{r+1}}\left(\frac{\xi}{2}\right) \\
&= \frac{1}{r!}\sum_{t=0}^{\alpha}(-1)^t\binom{r+1}{t}\mathcal{S}^r_{\alpha-t+r}(\xi), \qquad (1.21)\\
&\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{W}_{a_1}\left(\frac{\xi}{2}\right).\mathcal{W}_{a_2}\left(\frac{\xi}{2}\right).\mathcal{W}_{a_3}\left(\frac{\xi}{2}\right).\ldots.\mathcal{W}_{a_{r+1}}\left(\frac{\xi}{2}\right) \\
&= \frac{1}{r!}\sum_{t=0}^{\alpha}\binom{r+1}{t}\mathcal{S}^r_{\alpha-t+r}(\xi), \qquad (1.22)
\end{aligned}
$$

where $\mathcal{S}^r_\alpha(\xi)$ *is* r^{th} *derivative of Vieta-Fibonacci polynomial.*

Theorem 1.4. *For* $\alpha, r \geq 0$, *we have*

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{V}_{a_1}(\xi).\mathcal{V}_{a_2}(\xi).\mathcal{V}_{a_3}(\xi).\dots.\mathcal{V}_{a_{r+1}}(\xi)$$

$$= \frac{1}{2^r r!}\sum_{t=0}^{\alpha}\binom{r+1}{t}(-1)^t \mathcal{R}^r_{\alpha-t+r+1}(\xi), \tag{1.23}$$

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{W}_{a_1}(\xi).\mathcal{W}_{a_2}(\xi).\mathcal{W}_{a_3}(\xi).\dots.\mathcal{W}_{a_{r+1}}(\xi)$$

$$= \frac{1}{2^r r!}\sum_{t=0}^{\alpha}\binom{r+1}{t}\mathcal{R}^r_{\alpha-t+r+1}(\xi), \tag{1.24}$$

where $\mathcal{R}^r_\alpha(\xi)$ *is* r^{th} *derivative of Vieta-Pell polynomials.*

Corollary 1.1. *For* $\alpha, r \geq 0$, *we have*

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{F}_{a_1}.\mathcal{F}_{a_2}.\mathcal{F}_{a_3}.\dots.\mathcal{F}_{a_{r+1}}$$

$$= \frac{1}{2^r r!}\sum_{t=0}^{\alpha}(-1)^t\binom{r+1}{t} i^{\alpha-t}\mathcal{P}^r_{\alpha-t+r+1}\left(-\frac{3}{2}i\right) \tag{1.25}$$

$$= \frac{1}{r!}\sum_{t=0}^{\alpha}(-1)^t\binom{r+1}{t} i^{\alpha-t}\mathcal{F}^r_{\alpha-t+r+1}(-3i) \tag{1.26}$$

$$= \frac{1}{2^r r!}\frac{(\alpha+1)}{\mathcal{P}_\alpha\left(\alpha:\frac{1}{2},\frac{1}{2}\right)(1)}\sum_{t=0}^{\alpha}(-1)^t\binom{r+1}{t}\mathcal{P}^r\left(\alpha-t+r:\frac{1}{2},\frac{1}{2}\right)\left(\frac{3}{2}\right) \tag{1.27}$$

$$= \frac{1}{2^r r!}\sum_{t=0}^{\alpha}(-1)^t\binom{r+1}{t}\mathcal{C}^r(\alpha-t+r:1)\left(\frac{3}{2}\right) \tag{1.28}$$

$$= \frac{1}{r!}\sum_{t=0}^{\alpha}(-1)^t\binom{r+1}{t}\mathcal{S}^r_{\alpha-t+r}(3) \tag{1.29}$$

$$= \frac{1}{2^r r!}\sum_{t=0}^{\alpha}\binom{r+1}{t}(-1)^t\mathcal{R}^r_{\alpha-t+r+1}\left(\frac{3}{2}\right) \tag{1.30}$$

$$= \frac{1}{2^r r!}\sum_{t=0}^{\alpha}(-1)^t\binom{r+1}{t}\mathcal{U}^r_{\alpha-t+r}\left(\frac{3}{2}\right). \tag{1.31}$$

Corollary 1.2. *For* $\alpha, r \geq 0$, *we have*

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{L}_{a_1}.\mathcal{L}_{a_2}.\mathcal{L}_{a_3}.\dots.\mathcal{L}_{a_{r+1}}$$

$$= \frac{1}{2^r r!}\sum_{t=0}^{\alpha}\binom{r+1}{t} i^{\alpha-t}\mathcal{P}^r_{\alpha-t+r+1}\left(-\frac{3}{2}i\right) \tag{1.32}$$

$$= \frac{1}{r!}\sum_{t=0}^{\alpha}\binom{r+1}{t} i^{\alpha-t}\mathcal{F}^r_{\alpha-t+r+1}(-3i) \tag{1.33}$$

$$= \frac{1}{2^r r!}\frac{(\alpha+1)}{\mathcal{P}_\alpha\left(\alpha:\frac{1}{2},\frac{1}{2}\right)(1)}\sum_{t=0}^{\alpha}\binom{r+1}{t}\mathcal{P}^r\left(\alpha-t+r:\frac{1}{2},\frac{1}{2}\right)\left(\frac{3}{2}\right) \tag{1.34}$$

$$= \frac{1}{2^r r!} \sum_{t=0}^{\alpha} \binom{r+1}{t} \mathcal{C}^r(\alpha - t + r : 1)\left(\frac{3}{2}\right) \tag{1.35}$$

$$= \frac{1}{r!} \sum_{t=0}^{\alpha} \binom{r+1}{t} \mathcal{S}^r_{\alpha-t+r}(3) \tag{1.36}$$

$$= \frac{1}{2^r r!} \sum_{t=0}^{\alpha} \binom{r+1}{t} \mathcal{R}^r_{\alpha-t+r+1}\left(\frac{3}{2}\right) \tag{1.37}$$

$$= \frac{1}{2^r r!} \sum_{t=0}^{\alpha} \binom{r+1}{t} \mathcal{U}^r_{\alpha-t+r}\left(\frac{3}{2}\right). \tag{1.38}$$

2. Proof of Theorems and Corollaries

In this section, we will discuss the proof of theorems. Utilizing the above-discussed recurrence relations from (1.1)-(1.12), for any positive integer α, it is not difficult to deduce the following identities ([1, 2, 5–7]),

$$\mathcal{U}_\alpha\left(\frac{3}{2}\right) = \mathcal{F}_{2\alpha+2}, \tag{2.1}$$

$$\mathcal{V}_\alpha(\xi) = \mathcal{U}_\alpha(\xi) - \mathcal{U}_{\alpha-1}(\xi), \tag{2.2}$$

$$\mathcal{W}_\alpha(\xi) = \mathcal{U}_\alpha(\xi) + \mathcal{U}_{\alpha-1}(\xi), \tag{2.3}$$

$$\mathcal{W}_\alpha(\xi) = (-1)^\alpha \mathcal{V}_\alpha(-\xi), \tag{2.4}$$

$$\mathcal{V}_\alpha\left(\frac{3}{2}\right) = \mathcal{F}_{2\alpha+1}, \tag{2.5}$$

$$\mathcal{W}_\alpha\left(\frac{3}{2}\right) = \mathcal{L}_{2\alpha+1}, \tag{2.6}$$

$$\mathcal{P}_\alpha(\xi) = \mathcal{F}_\alpha(2\xi), \tag{2.7}$$

$$\mathcal{P}_{\alpha+1}(\xi) = \frac{1}{\sqrt{(-1)^\alpha}} \mathcal{U}_\alpha(\sqrt{-1}\xi) = \frac{1}{i^\alpha} \mathcal{U}_\alpha(i\xi), \tag{2.8}$$

$$\mathcal{S}_\alpha(\xi) = \mathcal{U}_\alpha\left(\frac{1}{2}\xi\right), \tag{2.9}$$

$$\mathcal{U}_\alpha(\xi) = \mathcal{R}_{\alpha+1}(\xi), \tag{2.10}$$

$$\mathcal{U}_\alpha(\xi) = \mathcal{C}(\alpha : 1)(\xi), \tag{2.11}$$

$$\mathcal{U}_\alpha(\xi) = \frac{(\alpha+1)!\,\Gamma\left(\frac{3}{2}\right)}{\Gamma(\alpha + \frac{3}{2})} \mathcal{P}\left(\alpha : \frac{1}{2}, \frac{1}{2}\right)(\xi). \tag{2.12}$$

Also, from [3], it can be seen that

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{V}_{a_1}(\xi).\mathcal{V}_{a_2}(\xi).\mathcal{V}_{a_3}(\xi)\ldots.\mathcal{V}_{a_{r+1}}(\xi) = \frac{1}{2^r r!} \sum_{t=0}^{\alpha} (-1)^t \binom{r+1}{t} \mathcal{U}^r_{\alpha-t+r}(\xi), \tag{2.13}$$

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{W}_{a_1}(\xi).\mathcal{W}_{a_2}(\xi).\mathcal{W}_{a_3}(\xi)\ldots.\mathcal{W}_{a_{r+1}}(\xi) = \frac{1}{2^r r!} \sum_{t=0}^{\alpha} \binom{r+1}{t} \mathcal{U}^r_{\alpha-t+r}(\xi), \tag{2.14}$$

where all the sums in (2.13) and (2.14) runs over all non-negative integers $a_1, a_2, \dots, a_{r+1}$ with $a_1 + a_2 + \dots + a_{r+1} = \alpha$ with $\binom{r+1}{t} = 0$ for $t > r+1$.

Proof of Theorem 1.1. Replacing ξ by $i\xi$ in (2.13) and (2.14), we have

$$\begin{aligned}
&\sum_{a_1+a_2+a_3+\dots+a_{r+1}=\alpha} \mathcal{V}_{a_1}(i\xi).\mathcal{V}_{a_2}(i\xi).\mathcal{V}_{a_3}(i\xi).\dots.\mathcal{V}_{a_{r+1}}(i\xi) \\
&\quad = \frac{1}{2^r r!}\sum_{t=0}^{\alpha}(-1)^t \binom{r+1}{t} \mathcal{U}^r_{\alpha-t+r}(i\xi) \\
&\quad \cdot \sum_{a_1+a_2+a_3+\dots+a_{r+1}=\alpha} \mathcal{W}_{a_1}(i\xi).\mathcal{W}_{a_2}(i\xi).\mathcal{W}_{a_3}(i\xi).\dots.\mathcal{W}_{a_{r+1}}(i\xi) &(2.15)\\
&\quad = \frac{1}{2^r r!}\sum_{t=0}^{\alpha} \binom{r+1}{t} \mathcal{U}^r_{\alpha-t+r}(i\xi). &(2.16)
\end{aligned}$$

Differentiating (2.7) and (2.8), r-times with respect to ξ, we get

$$\mathcal{P}^r_\alpha(\xi) = 2^r F^r_\alpha(2\xi), \tag{2.17}$$

$$\mathcal{U}^r_\alpha(i\xi) = i^{\alpha-r}\mathcal{P}^r_{n+1}(\xi). \tag{2.18}$$

Using (2.17) and (2.18) in (2.15) and (2.16), we have

$$\begin{aligned}
&\sum_{a_1+a_2+a_3+\dots+a_{r+1}=\alpha} \mathcal{V}_{a_1}(i\xi).\mathcal{V}_{a_2}(i\xi).\mathcal{V}_{a_3}(i\xi).\dots.\mathcal{V}_{a_{r+1}}(i\xi) \\
&\quad = \frac{1}{2^r r!}\sum_{t=0}^{\alpha}(-1)^t \binom{r+1}{t} i^{\alpha-t}\mathcal{P}^r_{\alpha-t+r+1}(\xi) \\
&\quad = \frac{1}{r!}\sum_{t=0}^{\alpha}(-1)^t \binom{r+1}{t} i^{\alpha-t}\mathcal{F}^r_{\alpha-t+r+1}(2\xi) \\
&\quad \cdot \sum_{a_1+a_2+a_3+\dots+a_{r+1}=\alpha} \mathcal{W}_{a_1}(i\xi).\mathcal{W}_{a_2}(i\xi).\mathcal{W}_{a_3}(i\xi).\dots.\mathcal{W}_{a_{r+1}}(i\xi) \\
&\quad = \frac{1}{2^r r!}\sum_{t=0}^{\alpha} \binom{r+1}{t} i^{\alpha-t}\mathcal{P}^r_{\alpha-t+r+1}(\xi) \\
&\quad = \frac{1}{r!}\sum_{t=0}^{\alpha} \binom{r+1}{t} i^{\alpha-t}\mathcal{F}^r_{\alpha-t+r+1}(2\xi).
\end{aligned}$$

Hence Theorem 1.1 is established. □

Proof of Theorem 1.2. Differentiating (2.11) and (2.12) r-times, we have

$$\mathcal{U}^r_\alpha(\xi) = \mathcal{C}^r(\alpha:1)(\xi), \tag{2.19}$$

$$\mathcal{U}^r_\alpha(\xi) = \frac{(\alpha+1)!\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}\right)}\mathcal{P}^r\left(\alpha:\frac{1}{2},\frac{1}{2}\right)(\xi). \tag{2.20}$$

Using (2.19) and (2.20) in (2.13) and (2.14), we have

$$\begin{aligned}
&\sum_{a_1+a_2+a_3+\dots+a_{r+1}=\alpha} \mathcal{V}_{a_1}(\xi).\mathcal{V}_{a_2}(\xi).\mathcal{V}_{a_3}(\xi).\dots.\mathcal{V}_{a_{r+1}}(\xi) \\
&\quad = \frac{1}{2^r r!}\frac{(\alpha+1)!\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}\right)}\sum_{t=0}^{\alpha}(-1)^t \binom{r+1}{t}\mathcal{P}^r\left(\alpha-t+r:\frac{1}{2},\frac{1}{2}\right)(\xi)
\end{aligned}$$

$$= \frac{1}{2^r r!} \sum_{t=0}^{\alpha} (-1)^t \binom{r+1}{t} \mathcal{C}^r(\alpha - t + r : 1)(\xi)$$
$$\cdot \sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{W}_{a_1}(\xi).\mathcal{W}_{a_2}(\xi).\mathcal{W}_{a_3}(\xi).\dots.\mathcal{W}_{a_{r+1}}(\xi)$$
$$= \frac{1}{2^r r!} \frac{(\alpha+1)!\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}\right)} \sum_{t=0}^{\alpha} \binom{r+1}{t} \mathcal{P}^r\left(\alpha - t + r : \frac{1}{2}, \frac{1}{2}\right)(\xi)$$
$$= \frac{1}{2^r r!} \sum_{t=0}^{\alpha} \binom{r+1}{t} \mathcal{C}^r(\alpha - t + r : 1)(\xi)$$

Hence Theorem 1.2 is established. □

Proof of Theorem 1.3. Replacing ξ by $\frac{\xi}{2}$ in (2.13) and (2.14), we have

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{V}_{a_1}\left(\frac{\xi}{2}\right).\mathcal{V}_{a_2}\left(\frac{\xi}{2}\right).\mathcal{V}_{a_3}\left(\frac{\xi}{2}\right).\dots.\mathcal{V}_{a_{r+1}}\left(\frac{\xi}{2}\right) = \frac{1}{2^r r!} \sum_{t=0}^{\alpha} (-1)^t \binom{r+1}{t} \mathcal{U}^r_{\alpha-t+r}\left(\frac{\xi}{2}\right), \tag{2.21}$$

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{W}_{a_1}\left(\frac{\xi}{2}\right).\mathcal{W}_{a_2}\left(\frac{\xi}{2}\right).\mathcal{W}_{a_3}\left(\frac{\xi}{2}\right).\dots.\mathcal{W}_{a_{r+1}}\left(\frac{\xi}{2}\right) = \frac{1}{2^r r!} \sum_{t=0}^{\alpha} \binom{r+1}{t} \mathcal{U}^r_{\alpha-t+r}\left(\frac{\xi}{2}\right). \tag{2.22}$$

Differentiating (2.9) r-times,we have

$$\mathcal{S}^r_\alpha(\xi) = \frac{1}{2^r} \mathcal{U}^r_\alpha\left(\frac{1}{2}\xi\right). \tag{2.23}$$

Using (2.23) in (2.21) and (2.22), we get

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{V}_{a_1}\left(\frac{\xi}{2}\right).\mathcal{V}_{a_2}\left(\frac{\xi}{2}\right).\mathcal{V}_{a_3}\left(\frac{\xi}{2}\right).\dots.\mathcal{V}_{a_{r+1}}\left(\frac{\xi}{2}\right) = \frac{1}{r!} \sum_{t=0}^{\alpha} (-1)^t \binom{r+1}{t} \mathcal{S}^r_{\alpha-t+r}(\xi), \tag{2.24}$$

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{W}_{a_1}\left(\frac{\xi}{2}\right).\mathcal{W}_{a_2}\left(\frac{\xi}{2}\right).\mathcal{W}_{a_3}\left(\frac{\xi}{2}\right).\dots.\mathcal{W}_{a_{r+1}}\left(\frac{\xi}{2}\right) = \frac{1}{r!} \sum_{t=0}^{\alpha} \binom{r+1}{t} \mathcal{S}^r_{\alpha-t+r}(\xi). \tag{2.25}$$

Hence Theorem 1.3 is established. □

Proof of Theorem 1.4. Differentiating (2.10) r-times, we have

$$U^r_\alpha(\xi) = \mathcal{R}^r_{\alpha+1}(\xi). \tag{2.26}$$

Using (2.26) in (2.13) and (2.14), we have,

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{V}_{a_1}(\xi).\mathcal{V}_{a_2}(\xi).\mathcal{V}_{a_3}(\xi).\dots.\mathcal{V}_{a_{r+1}}(\xi) = \frac{1}{2^r r!} \sum_{t=0}^{\alpha} \binom{r+1}{t} (-1)^t \mathcal{R}^r_{\alpha-t+r+1}(\xi),$$

$$\sum_{a_1+a_2+a_3+\cdots+a_{r+1}=\alpha} \mathcal{W}_{a_1}(\xi).\mathcal{W}_{a_2}(\xi).\mathcal{W}_{a_3}(\xi).\dots.\mathcal{W}_{a_{r+1}}(\xi) = \frac{1}{2^r r!} \sum_{t=0}^{\alpha} \binom{r+1}{t} \mathcal{R}^r_{\alpha-t+r+1}(\xi).$$

Hence Theorem 1.4 is established. □

Proof of Corollary 1.1. By putting $\xi = \frac{3}{2i}$ in (1.13),(1.14); $\xi = \frac{3}{2}$ in (1.17),(1.18); $\xi = 3$ in (1.21); $\xi = \frac{3}{2}$ in (1.23),(2.13); and using (2.5) establishes the Corollary 1.1. □

Proof of Corollary 1.2. Similarly, by putting $\xi = \frac{3}{2i}$ in (1.15),(1.16); $\xi = \frac{3}{2}$ in (1.19),(1.20); $\xi = 3$ in (1.22); $\xi = \frac{3}{2}$ in (1.24),(2.14); and using (2.6) establishes the Corollary 1.2. □

3. Conclusion

In this paper, we attempt to study some basic properties of the Chebyshev polynomials of third and fourth kind and establish some identities of the sums of finite products involving these polynomials in terms of Fibonacci polynomials, Pell polynomials, Vieta-Fibonacci polynomials,Vieta-Pell polynomials, Gegenbauer polynomials, and Jacobi polynomials. Similar identities for the sums of finite products of Fibonacci numbers and Lucas numbers are deduced.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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