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Research Article

Analysis of a Markovian Retrial Queue With Working Vacation Under *N*-Control Pattern

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Abstract. A Markovian retrial queue with working vacation under N-control pattern is investigated in this article. To describe the system, we employ a QBD analogy. The model's stability condition is deduced. The stationary probability distribution is generated by utilizing the matrix-analytic technique. The conditional stochastic decomposition of the line length in the orbit is calculated. The performance measures and special cases are designed. The model's firmness is demonstrated numerically.

Keywords. Markovian retrial queue, Working vacation, N-control pattern, Conditional stochastic decomposition

Mathematics Subject Classification (2020). 60K25, 90B22

1. Introduction

Wallace [14] investigated the *Quasi Birth-Death* process (QBD) in Queueing Theory using a Markov chain with a tridiagonal generator. Numerical techniques can be used to analyze the congestion situations when it is impossible to achieve a explicit solution for queueing problems. The Matrix Geometric technique is ideal for this type of solutions. Neuts [10], Latouche and Ramaswami [5] proposed the matrix geometric solution to the QBD process. Control policies are important for managing queue levels at different epochs. Yadin and Naor [16] first propose the N-policy.

The queueing system with attendant vacation is noteworthy, and can be refer in Tian and Zhang [13]. Servi and Finn [11] created a modern vacation policy, termed as *Working Vacation*

(WV), where the attendant delivers a lesser rate of service than during the engaged period. Wu and Takagi [15] worked on M/G/1/MWV. Kalyanaraman and Murugan [4] have worked on the retrial queue with vacation, Murugan and Santhi [9] have worked on WV.

Liu *et al*. [7] analysed the stochastic decompositions in the M/M/1/WV queue. The M/M/1/WV queue and WV interruptions was analysed by Li and Tian [6]. Analysis for the M/M/1/MWV queue and N-policy was studied by Zhang and Xu [19]. Ye and Liu [17] discussed the analysis of the $M/M/1$ queue with two vacation policies.

Recently, retrial queues have been studied widely and it was different from normal queues. Due to limited waiting space in the retrial queue the customers are forced to stay in the orbit. Whenever the approaching customers finds that the attendant is engaged they join the orbit and requests service from the orbit. An $M/M/1$ retrial queue with general retrial times was studied by Choi *et al*. [2]. The retrial queue and WV was simultaneously considered by Do [3]. Tao *et al*. [12] discussed the M/M/1 retrial queue with collisions and WV interruption under N-policy. We consider a Markovain retrial queue with WV under N-control pattern.

The following are the categories for this article. We present the model description and find the infinitesimal generator in Section 2. The stability condition and Rate matrix (R) is computed in Section 3. In Section 4, we use a matrix-analytic technique to derive the stationary probability distribution. The line length's conditional stochastic decomposition is computed in Section 5. In Section 6, we calculate performance measures. The special cases is presented in Section 7, and Section 8 has a firmness of the model. The conclusion is given in Section 9.

2. QBD Process Model

We examine a Markovian retrial queue with WV under N-control pattern. With the parameter λ, the customer's inter-arrival times are exponentially distributed. A Poisson process with rate α governs request retrials from the infinite-sized orbit. The attendant will take a WV when the system gets clear, which is exponentially distributed with parameter θ. The service is exponentially distributed with parameters μ at the time of the regular busy period. When comparing to the service offered throughout engaged period, the service provided at the time of the WV is at a slower rate. WV service is exponentially distributed with parameters η ($\eta < \mu$). When a WV ends, if the attendant identifies not less than N customers in the orbit, the attendant will terminates WV and return to engaged period. Otherwise, the attendant will start another WV. Inter-arrival times, inter-retrial periods, service periods, and vacation periods are all presumed to be independent of one another.

Let the number of customers in the orbit at time t is indicated by $Q(t)$ and $H(t)$ represent attendant's condition at time t. The single attendant might exist in four different states at time t.

$$H(t) = \begin{cases} 0 & \text{attendant is on WV and is unoccupied,} \\ 1 & \text{attendant is on WV and is engaged,} \\ 2 & \text{attendant is on engaged period and is unoccupied,} \\ 3 & \text{attendant is on engaged period and is engaged.} \end{cases}$$

Evidently, $\{(Q(t),H(t)); t \geq 0\}$ is a Markov process with state space

$$\Omega = \{(m,h) : m \geq 0, h = 0,1,2,3\}.$$

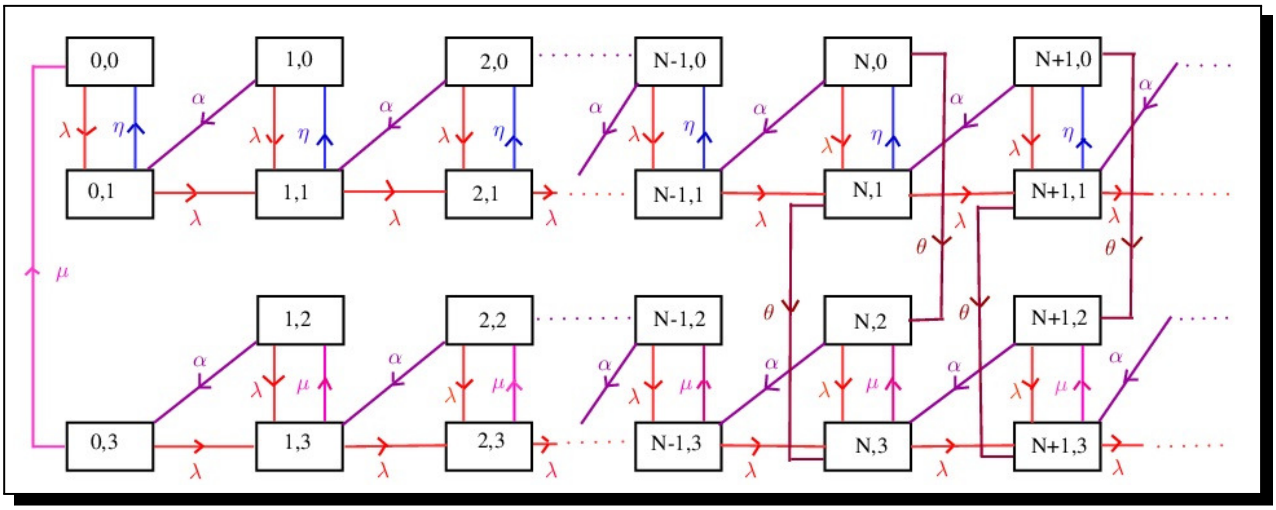

Figure 1. Transition between the states

The states infinitesimal generator can be described by employing *lexicographical sequence* as follows:

$$\widetilde{Q} = \begin{bmatrix} D_0 & F & & & & & & & & \\ E & D_1 & F & & & & & & & \\ & E & D_1 & F & & & & & & \\ & & E & D_1 & F & & & & & \\ & & & \vdots & \vdots & \vdots & & & & \\ & & & E & D_1 & F & & & & \\ & & & & E & D & F & & & \\ & & & & & E & D & F & & \\ & & & & & & \vdots & \vdots & \vdots & \end{bmatrix},$$

where

$$D_0 = \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ \eta & -\eta-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & -\mu-\lambda \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -\alpha-\lambda & \lambda & 0 & 0 \\ \eta & -\eta-\lambda & 0 & 0 \\ 0 & 0 & -\alpha-\lambda & \lambda \\ 0 & 0 & \mu & -\mu-\lambda \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} -\alpha-\lambda-\theta & \lambda & \theta & 0 \\ \eta & -\lambda-\eta-\theta & 0 & \theta \\ 0 & 0 & -\alpha-\lambda & \lambda \\ 0 & 0 & \mu & -\mu-\lambda \end{bmatrix}.$$

Due to the block structure of matrix $\widetilde{Q}$, $\{(Q(t),H(t)); t \geq 0\}$ is called a *QBD* process. Pr\{that the attendant is engaged and does not offer a service to a customer while there is no customer in the orbit\}$= 0$.

3. The Model's Stability Condition and R

Theorem 3.1. *The QBD process* $\{(Q(t),H(t)); t \geq 0\}$ *is* $(+)^{ve}$ *recurrent* $\Leftrightarrow \alpha(\mu-\lambda) > \lambda^2$.

Proof. Consider

$$S_m = E + D + F = \begin{bmatrix} -\alpha-\lambda-\theta & \alpha+\lambda & \theta & 0 \\ \eta & -\theta-\eta & 0 & \theta \\ 0 & 0 & -\alpha-\lambda & \alpha+\lambda \\ 0 & 0 & \mu & -\mu \end{bmatrix}.$$

In [5, Theorem 7.3.1] offers requirement for $(+)^{ve}$ recurrence of the QBD process, because matrix S_m is reducible. After permutation of rows and columns and hence the *QBD* is $(+)^{ve}$ recurrent $\Leftrightarrow \pi \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} e > \pi \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} e$.

Here all the elements of the column vector $e = 1$ and π is the unique solution of the system $\pi \begin{bmatrix} -\alpha-\lambda & \alpha+\lambda \\ \mu & -\mu \end{bmatrix} = 0$, $\pi e = 1$. The *QBD* process is $(+)^{ve}$ recurrent $\Leftrightarrow \alpha(\mu-\lambda) > \lambda^2$ after some algebraic manipulations. □

Theorem 3.2. *If* $\alpha(\mu-\lambda) > \lambda^2$, *the matrix quadratic equation* $R^2E + RD + F = 0$ *has the minimal non-negative solution*

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ r_1 & r_2 & r_3 & r_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_5 & r_6 \end{bmatrix},$$

where

$$r_1 = \frac{r_2\eta}{(\lambda+\alpha+\theta)},$$

$$r_2 = \frac{t - \sqrt{t^2 - 4\alpha\lambda\eta(\lambda+\alpha+\theta)}}{2\alpha\eta}$$

and

$$t = [(\lambda+\alpha+\theta)(\lambda+\theta+\eta) - \eta\lambda],$$

$$r_3 = \frac{r_1\theta + r_4\mu}{(\lambda+\alpha)},$$

$$r_4 = \frac{\alpha r_2 r_1\theta + r_1\theta\lambda + r_2\theta(\lambda+\alpha)}{(\lambda+\mu)(\lambda+\alpha) - \alpha r_2\mu - \alpha r_5(\lambda+\alpha) - \mu\lambda},$$

$$r_5 = \frac{\lambda}{\alpha},$$
$$r_6 = \frac{\lambda(\lambda+\alpha)}{\mu\alpha}.$$

Proof. We can consider $R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$, from the matrices E,D,F where R_{11}, R_{12} and R_{22} are all 2×2 matrices. Substituting R into $R^2E + RD + F = 0$, we get

$$R_{11}^2 \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} + R_{11} \begin{bmatrix} (-\alpha-\lambda-\theta) & \lambda \\ \eta & (-\lambda-\eta-\theta) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$(R_{11}R_{12} + R_{12}R_{22}) \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} + R_{11} \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} + R_{12} \begin{bmatrix} (-\alpha-\lambda) & \lambda \\ \mu & (-\mu-\lambda) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$R_{22}^2 \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} + R_{22} \begin{bmatrix} (-\alpha-\lambda) & \lambda \\ \mu & (-\mu-\lambda) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

From the above set of equations with some computations, we get R_{11}, R_{22} and R_{12}, respectively, as $R_{11} = \begin{bmatrix} 0 & 0 \\ r_1 & r_2 \end{bmatrix}$, $R_{22} = \begin{bmatrix} 0 & 0 \\ r_5 & r_6 \end{bmatrix}$ and $R_{12} = \begin{bmatrix} 0 & 0 \\ r_3 & r_4 \end{bmatrix}$.

4. Stationary Probability Distribution

If $\alpha(\mu-\lambda) > \lambda^2$, assign (Q,H) be the stationary probability distribution of the process $\{(Q(t),H(t));\, t\ge 0\}$. Represent,

$$\pi_m = (\pi_{m,0}, \pi_{m,1}, \pi_{m,2}, \pi_{m,3}), \quad m \ge 0;$$
$$\pi_{m,h} = P\{Q=m, H=h\} = \lim_{t\to\infty} P\{Q(t)=m, \quad H(t)=h\}, (m,h)\in\Omega.$$

It is worth noting that $\pi_{0,2} = 0$ from states we discussed earlier.

Theorem 4.1. *If $(\mu-\lambda)\alpha > \lambda^2$, the stationary probability distribution of (Q,H) is indicated by*

$$\pi_{m,0} = \pi_{N-1,1} r_1 r_2^{m-N}, \quad m \ge N, \tag{4.1}$$
$$\pi_{m,1} = \pi_{N-1,1} r_2^{m+1-N}, \quad m \ge N, \tag{4.2}$$
$$\pi_{m,2} = \pi_{N-1,1}\left[r_3 r_2^{m-N} + \frac{r_4 r_5}{r_6 - r_2}\left(r_6^{m-N} - r_2^{m-N}\right)\right] + \pi_{N-1,3} r_5 r_6^{m-N}, \quad m \ge N, \tag{4.3}$$
$$\pi_{m,3} = \pi_{N-1,1}\frac{r_4}{r_6-r_2}(r_6^{m+1-N} - r_2^{m+1-N}) + \pi_{N-1,3} r_6^{m+1-N}, \quad m \ge N, \tag{4.4}$$
$$\pi_{m,0} = \frac{\eta}{\lambda+\alpha}\pi_{0,1} + \frac{\eta}{\lambda+\alpha}(\pi_{1,1} - \pi_{0,1})\frac{1-q_1^m}{1-q_1}, \quad 2 \le m \le N-2, \tag{4.5}$$
$$\pi_{m,1} = \pi_{0,1} + (\pi_{1,1} - \pi_{01})\frac{1-q_1^m}{1-q_1}, \quad 2 \le m \le N-2, \tag{4.6}$$
$$\pi_{m,2} = \frac{\mu}{\lambda+\alpha}\pi_{0,3} + \frac{\mu}{\lambda+\alpha}(\pi_{1,3} - \pi_{0,3})\frac{1-q_2^m}{1-q_2}, \quad 2 \le m \le N-2, \tag{4.7}$$
$$\pi_{m,3} = \pi_{0,3} + (\pi_{1,3} - \pi_{0,3})\frac{1-q_2^m}{1-q_2}, \quad 2 \le m \le N-2, \tag{4.8}$$

$$\pi_{N-1,0} = \frac{-\lambda\eta}{[\lambda\eta + (r_1\alpha - \lambda - \eta)(\lambda+\alpha)]}\pi_{N-2,1}, \tag{4.9}$$

$$\pi_{N-1,1} = \frac{\lambda+\alpha}{\eta}\pi_{N-1,0}, \tag{4.10}$$

$$\pi_{N-1,2} = r_3\pi_{N-1,1} + \frac{\lambda}{\alpha}\pi_{N-2,3}, \tag{4.11}$$

$$\pi_{N-1,3} = \frac{\lambda+\alpha}{\mu}\pi_{N-1,2}, \tag{4.12}$$

$$\pi_{1,1} = -K^{-1}\left[\frac{\lambda(\lambda+\alpha+\eta)}{\lambda+\alpha} + \Delta - K\right]\pi_{0,1}, \tag{4.13}$$

$$\pi_{1,0} = \frac{\eta}{\lambda+\alpha}\pi_{1,1}, \tag{4.14}$$

$$\pi_{0,0} = \frac{\lambda+\eta}{\lambda}\pi_{0,1} - \frac{\alpha}{\lambda}\pi_{1,0}, \tag{4.15}$$

$$\pi_{0,3} = \frac{\lambda}{\mu}\pi_{0,0} - \frac{\eta}{\mu}\pi_{0,1}, \tag{4.16}$$

$$\pi_{1,2} = \frac{\lambda+\mu}{\alpha}\pi_{0,3}, \tag{4.17}$$

$$\pi_{1,3} = \frac{\lambda+\alpha}{\mu}\pi_{1,2}, \tag{4.18}$$

where

$$q_1 = \frac{\lambda(\lambda+\alpha)}{\alpha\eta},$$

$$q_2 = \frac{\lambda(\lambda+\alpha)}{\alpha\mu},$$

$$\Delta = \frac{-\lambda\alpha\eta}{[\lambda\eta + (r_1\alpha - \lambda - \eta)(\lambda+\alpha)]} - \lambda - \eta,$$

$$K = \left[\lambda\frac{1-q_1^{N-3}}{1-q_1} + \left(\Delta + \frac{\lambda\eta}{\lambda+\alpha}\right)\frac{1-q_1^{N-2}}{1-q_1}\right].$$

The normalization condition can finally be used to determine $\pi_{0,1}$.

Proof. Using the technique from [10], we have

$$\begin{aligned}\pi_m &= (\pi_{m,0}, \pi_{m,1}, \pi_{m,2}, \pi_{m,3}) = \pi_{N-1}R^{m+1-N} \\ &= (\pi_{N-1,0}, \pi_{N-1,1}, \pi_{N-1,2}, \pi_{N-1,3})R^{m+1-N}, \quad m \ge N.\end{aligned}$$

For $m \ge N$,

$$R^{m+1-N} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ r_1 r_2^{m-N} & r_2^{m+1-N} & r_3 r_2^{m-N} + \frac{r_4 r_5}{r_6 - r_2}(r_6^{m-N} - r_2^{m-N}) & \frac{r_4}{r_6-r_2}(r_6^{m+1-N} - r_2^{m+1-N}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_5 r_6^{m-N} & r_6^{m+1-N} \end{bmatrix}.$$

Substituting R^{m+1-N} into the above equation, we get (4.1)-(4.4).

However, $\pi_0, \pi_1, \dots, \pi_{N-1}$ satisfies the equation $(\pi_0, \pi_1, \dots, \pi_{N-1})B[R] = 0$, where

$$B[R] = \begin{bmatrix} D_0 & F & & & & \\ E & D_1 & F & & & \\ & E & D_1 & F & & \\ & & \vdots & \vdots & & \vdots \\ & & E & D_1 & & F \\ & & & E & & RE + D_1 \end{bmatrix}$$

and

$$RE + D_1 = \begin{bmatrix} -(\lambda+\alpha) & \lambda & 0 & 0 \\ \eta & r_1\alpha - \lambda - \eta & 0 & r_3\alpha \\ 0 & 0 & -(\lambda+\alpha) & \lambda \\ 0 & 0 & \mu & r_5\alpha - \lambda - \mu \end{bmatrix}.$$

The following equations are computed from $B[R]$

$$-\lambda\pi_{0,0} + \eta\pi_{0,1} + \mu\pi_{0,3} = 0, \tag{4.19}$$

$$\lambda\pi_{0,0} - (\lambda+\eta)\pi_{0,1} + \alpha\pi_{1,0} = 0, \tag{4.20}$$

$$-(\lambda+\mu)\pi_{0,3} + \alpha\pi_{1,2} = 0, \tag{4.21}$$

$$-(\lambda+\alpha)\pi_{m,0} + \eta\pi_{m,1} = 0, \quad 1 \le m \le N-2, \tag{4.22}$$

$$\lambda\pi_{m-1,1} + \lambda\pi_{m,0} - (\lambda+\eta)\pi_{m,1} + \alpha\pi_{m+1,0} = 0, \quad 1 \le m \le N-2, \tag{4.23}$$

$$-(\lambda+\alpha)\pi_{m,2} + \mu\pi_{m,3} = 0, \quad 1 \le m \le N-2, \tag{4.24}$$

$$\lambda\pi_{m-1,3} + \lambda\pi_{m,2} - (\lambda+\mu)\pi_{m,3} + \alpha\pi_{m+1,2} = 0, \quad 1 \le m \le N-2, \tag{4.25}$$

$$-(\lambda+\alpha)\pi_{N-1,0} + \eta\pi_{N-1,1} = 0, \tag{4.26}$$

$$\lambda\pi_{N-2,1} + \lambda\pi_{N-1,0} + (r_1\alpha - \lambda - \eta)\pi_{N-1,1} = 0, \tag{4.27}$$

$$-(\lambda+\alpha)\pi_{N-1,2} + \mu\pi_{N-1,3} = 0, \tag{4.28}$$

$$\lambda\pi_{N-2,3} + r_3\alpha\pi_{N-1,1} + \lambda\pi_{N-1,2} + (r_5\alpha - \lambda - \mu)\pi_{N-1,3} = 0. \tag{4.29}$$

From (4.19) to (4.29), we get (4.5) to (4.18), where $\sum_{h=0}^{3}\sum_{m=0}^{\infty}\pi_{m,h} = 1$, finally we can get $\pi_{0,1}$. □

5. Conditional Stochastic Decomposition

Lemma 5.1. *If $\alpha(\mu-\lambda) > \lambda^2$, let Q_0 be the conditional line length of an $M/M/1$ retrial queue in the orbit where the attendant is engaged, then Q_0 has a PGF*

$$G_{Q_0}(z) = \frac{1-r_6}{1-r_6 z}.$$

Proof. Consider a Markovian retrial queue. Two inter-valued random variables are used to explain the system at time t. Let $Q^\bullet(t)$ be the number of customers in the orbit at time t,

$$H^\bullet(t) = \begin{cases} 0 & \text{attendant is unoccupied,} \\ 1 & \text{attendant is engaged.} \end{cases}$$

Then $\{(Q^\bullet(t), H^\bullet(t)); t \ge 0\}$ is a Markov process with state space $\{(m,h) : m \ge 0, h = 0, 1\}$.

The *infinitesimal generator* can be expressed as

$$\widetilde{Q}^{\bullet}\begin{bmatrix} D_0 & F & & & \\ E & D & F & & \\ & E & D & F & \\ & & \vdots & \vdots & \vdots \end{bmatrix},$$

where

$$D_0=\begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu-\lambda \end{bmatrix},\quad F=\begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix},\quad E=\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix},\quad D=\begin{bmatrix} -\alpha-\lambda & \lambda \\ \mu & -\mu-\lambda \end{bmatrix}.$$

The QBD process $\{(Q^{\bullet}(t),H^{\bullet}(t));t\ge 0\}$ is $(+)^{ve}$ recurrent $\Leftrightarrow(\mu-\lambda)\alpha>\lambda^2$. Express

$$\pi_{m,h}=P\{Q^{\bullet}=m,H^{\bullet}=h\}=\lim_{t\to\infty}P\{Q^{\bullet}(t)=m,H^{\bullet}(t)=h\}.$$

The stationary probability distribution is

$$\widetilde{\pi}_{m,0}=\widetilde{\pi}_{0,1}r_5r_6^{m-1},\quad m\ge 1,$$
$$\widetilde{\pi}_{m,1}=\widetilde{\pi}_{0,1}r_6^{m},\quad m\ge 0,$$
$$\widetilde{\pi}_{0,0}=\left(1+\frac{1+r_5}{1-r_6}\frac{\lambda}{\mu}\right)^{-1},$$
$$\widetilde{\pi}_{0,1}=\frac{\lambda}{\mu}\widetilde{\pi}_{0,0}.$$

The normalization condition is used to determine the value of $\pi_{0,0}$.
Therefore,

$$G_{Q_0}(z)=\sum_{m=0}^{\infty}z^mP\{Q_0=m\}=\frac{\sum\limits_{m=0}^{\infty}\widetilde{\pi}_{0,1}r_6^mz^m}{\sum\limits_{m=1}^{\infty}\widetilde{\pi}_{0,1}r_6^{m-1}}=\frac{1-r_6}{1-r_6z}.$$

Establishing $Q^N=\{$difference of Q and N such that the state of the attendant is either 1 *or* 3 and $Q\ge N\}$ and Q^N is the line length which depends on the condition that the attendant is engaged and there are not less than N customers in the orbit.
Let $P_b^{\bullet}$ denotes that $Pr\{$the server is engaged given that atleast N customers present in the orbit$\}$.

$$\begin{aligned}
P_b^{\bullet}&=P\{Q\ge N,H=1\text{ or }3\}\\
&=\sum_{m=N}^{\infty}\pi_{m,1}+\sum_{m=N}^{\infty}\pi_{m,3}\\
&=\sum_{m=N}^{\infty}\pi_{N-1,1}r_2^{m+1-N}+\sum_{m=N}^{\infty}\frac{r_4}{r_6-r_2}(r_6^{m+1-N}-r_2^{m+1-N})\pi_{N-1,1}\\
&\quad+\sum_{m=N}^{\infty}r_6^{m+1-N}\pi_{N-1,3}\\
&=\frac{r_4+r_2(1-r_6)}{(1-r_2)(1-r_6)}\pi_{N-1,1}+\frac{r_6}{(1-r_6)}\pi_{N-1,3}.
\end{aligned}$$

□

Theorem 5.1. *If $\alpha(\mu-\lambda)>\lambda^2$, then we can disintegrate $Q^N=Q_0+Q_c$, where Q_0 go along with a geometric distribution with specification $1-r_6$. Subsidiary line length Q_c has a distribution*

$$P\{Q_c=0\}=\frac{1}{P_b^{\bullet}}\frac{(r_2+r_4)\pi_{N-1,1}+r_6\pi_{N-1,3}}{1-r_6},$$

$$P\{Q_c=m\}=\frac{\pi_{N-1,1}}{P_b^{\bullet}}\frac{r_2(r_2+r_4-r_6)}{1-r_6}r_2^{m-1}, \quad m\geq 1.$$

Proof. The PGF of Q^N is given below:

$$\begin{aligned}
G_{Q^N}(z)&=\sum_{m=0}^{\infty}z^m P\{Q^N=m\}\\
&=\frac{1}{p_b^{\bullet}}\left(\sum_{m=0}^{\infty}z^m\pi_{N+m,1}+\sum_{m=0}^{\infty}z^m\pi_{N+m,3}\right)\\
&=\frac{1}{p_b^{\bullet}}\left[\pi_{N-1,1}\frac{r_2}{1-r_2z}+\pi_{N-1,1}\frac{r_4}{(1-r_2z)(1-r_6z)}+\pi_{N-1,3}\frac{r_6}{1-r_6z}\right]\\
&=\frac{1}{p_b^{\bullet}}\frac{1-r_6}{1-r_6z}\left[\pi_{N-1,1}\frac{r_2(1-r_6z)}{(1-r_2z)(1-r_6)}+\pi_{N-1,1}\frac{r_4}{(1-r_2z)(1-r_6)}+\pi_{N-1,3}\frac{r_6}{1-r_6}\right]\\
&=\frac{1}{p_b^{\bullet}}\frac{1-r_6}{1-r_6z}\left[\frac{(r_2+r_4)\pi_{N-1,1}+r_6\pi_{N-1,3}}{1-r_6}+\pi_{N-1,1}\frac{r_2(r_2+r_4-r_6)z}{(1-r_2z)(1-r_6)}\right]\\
&=\frac{1-r_6}{1-r_6z}\left[\frac{1}{p_b^{\bullet}}\frac{(r_2+r_4)\pi_{N-1,1}+r_6\pi_{N-1,3}}{1-r_6}+\pi_{N-1,1}\frac{1}{p_b^{\bullet}}\frac{r_2(r_2+r_4-r_6)z}{(1-r_2z)(1-r_6)}\right]\\
&=G_{Q_0}(z)G_{Q_c}(z).
\end{aligned}$$

□

6. Performance Measures

From Theorem 4.1, we have

$$\begin{aligned}
Pr\{\text{that the attendant is engaged}\}&=P_b\\
&=\sum_{m=0}^{\infty}\pi_{m,1}+\sum_{m=0}^{\infty}\pi_{m,3}\\
&=(N-1)\left(\frac{\pi_{1,1}}{1-q_1}-\frac{q_1\pi_{0,1}}{1-q_1}\right)-\frac{\pi_{1,1}-\pi_{0,1}}{(1-q_1)^2}(1-q_1^{N-1})\\
&\quad+(N-1)\left(\frac{\pi_{1,3}}{1-q_2}-\frac{q_2\pi_{0,3}}{1-q_2}\right)-\frac{\pi_{1,3}-\pi_{0,3}}{(1-q_2)^2}(1-q_2^{N-1})\\
&\quad+\frac{1-r_6+r_4}{(1-r_2)(1-r_6)}\pi_{N-1,1}+\frac{1}{(1-r_6)}\pi_{N-1,3},
\end{aligned}$$

$$\begin{aligned}
Pr\{\text{that the attendant is unoccupied}\}&=P_f\\
&=\sum_{m=0}^{\infty}\pi_{m,0}+\sum_{m=1}^{\infty}\pi_{m,2}=1-P_b.
\end{aligned}$$

Assume that L *denotes the number of customers in the orbit*, subsequently

$$\begin{aligned}
E[L] &= \sum_{m=1}^{\infty} m(\pi_{m,0}+\pi_{m,1}+\pi_{m,2}+\pi_{m,3}) \\
&= \sum_{m=1}^{N-1} m(\pi_{m,0}+\pi_{m,2}) + \sum_{m=1}^{N-2} m(\pi_{m,1}+\pi_{m,3}) \\
&\quad + (N-1)\pi_{N-1,1}\frac{(1+r_1+r_3)(1-r_6)+r_4(1+r_5)}{(1-r_2)(1-r_6)} \\
&\quad + (N-1)\pi_{N-1,3}\frac{1+r_5}{1-r_6} + \pi_{N-1,3}\frac{r_5+r_6}{(1-r_6)^2} \\
&\quad + \pi_{N-1,1}\frac{(r_1+r_2+r_3)(1-r_6)^2+r_4r_5(2-r_2-r_6)+r_4(1-r_2r_6)}{(1-r_6)^2(1-r_2)^2}.
\end{aligned}$$

Let L_s *be the number of customers in the system*, subsequently

$$E[L_s] = \sum_{m=1}^{\infty} m(\pi_{m,0}+\pi_{m,2}) + \sum_{m=0}^{\infty}(m+1)(\pi_{m,1}+\pi_{m,3}).$$

We have the following assumptions and results.

Let

W	—	orbit customer's waiting time
$E[W_s]$	—	expected stopover time of orbit customer in the system
T	—	engaged period

Then, $E[W]=\frac{E[L]}{\lambda}$, $E[W_s]=\frac{E[L_s]}{\lambda}$ and $\pi_{0,0}=\frac{E[T_{0,0}]}{E[T]+1/\lambda}$, where $E[T_{0,0}]$ is the absolute time in the idle state throughout a *regenerative cycle*.

Also $E[T_{0,0}]=\dfrac{1}{\lambda}$, $E[T]=(\pi_{0,0}^{-1}-1)\lambda^{-1}$. □

7. Special Cases

(a) If $\alpha\to\infty$ this model is remodeled as "Analysis for the M/M/1 queue with multiple working vacations and N-policy".

(b) If $\alpha\to\infty$, $\eta=0$ this model is remodeled as "An M/M/1 queue with multiple vacation under N-policy".

(c) If $\alpha\to\infty$, $\eta=0$, $\theta=0$ this model is remodeled as "Standard M/M/1 queue under N-policy".

8. Numerical Results

By fixing the values of $N=2$, $\mu=7.5$, $\theta=1$, $\eta=0.5$ and extending the value of λ from 1.0 to 2.0 incremented with 0.2 and extending the values of α from 3.2 to 4.2 insteps of 0.5 subject to the stability condition the values of $E(L)$ are calculated and tabulated in Table 1 and the corresponding line graphs are drawn in Figure 2. From the graph it is inferred that as λ rises $E(L)$ rises as expected.

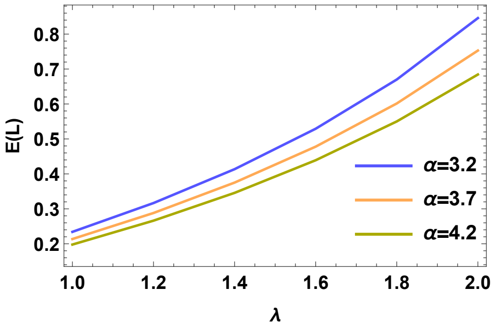

Figure 2. $E(L)$ with turn over of λ

λ	$\alpha = 3.2$	$\alpha = 3.7$	$\alpha = 4.2$
1.0	0.2341	0.2136	0.1997
1.2	0.3165	0.2881	0.2661
1.4	0.4135	0.3751	0.3457
1.6	0.5929	0.4779	0.4391
1.8	0.6701	0.6015	0.5501
2.0	0.8457	0.7528	0.6843

Table 1. $E(L)$ with turn over of λ

By fixing the values of $N = 2$, $\mu = 7.7$, $\theta = 1.7$, $\alpha = 3.5$ and extending the value of λ from 1.0 to 2.0 incremented with 0.5 and extending the values of η from 0.3 to 2.3 insteps of 1 subject to the stability condition the values of $E(L)$ are calculated and tabulated in Table 2 and the corresponding line graphs are drawn in Figure 3. From the graph it is inferred that as λ rises $E(L)$ rises as expected.

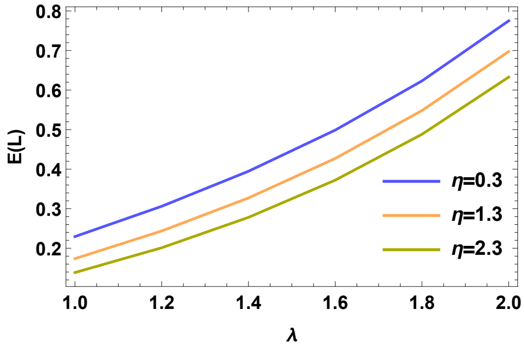

Figure 3. $E(L)$ with turn over of λ

λ	$\eta = 0.3$	$\eta = 1.3$	$\eta = 2.3$
1.0	0.2296	0.1738	0.1386
1.2	0.3061	0.2437	0.2012
1.4	0.3947	0.3272	0.2781
1.6	0.4987	0.4271	0.3722
1.8	0.6231	0.5482	0.4881
2.0	0.7750	0.6974	0.6327

Table 2. $E(L)$ with turn over of λ

By fixing the values of $N = 2$, $\mu = 7.9$, $\theta = 1.7$, $\eta = 2.3$ and extending the value of λ from 1.0 to 2.0 incremented with 0.2 and extending the values of α from 1.5 to 3.5 insteps of 1 subject to the stability condition the values of P_b are calculated and tabulated in Table 4 and the corresponding line graphs are drawn in Figure 3. From the graph it is inferred that as λ rises P_b rises as expected.

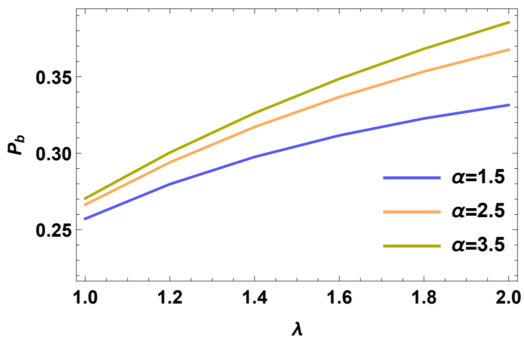

Figure 4. P_b with turn over of λ

λ	$\alpha = 1.5$	$\alpha = 2.5$	$\alpha = 3.5$
1.0	0.2751	0.2663	0.2704
1.2	0.2797	0.2941	0.3004
1.4	0.2975	0.3171	0.3263
1.6	0.3116	0.3367	0.3486
1.8	0.3227	0.3534	0.3682
2.0	0.3314	0.3676	0.3855

Table 3. P_b with turn over of λ

By fixing the values of $N = 2$, $\mu = 9.1$, $\theta = 3.6$, $\eta = 2.1$ and extending the values of λ from 1.0 to 2.0 incremented with 0.2 and extending the values α from 3 to 6 insteps of 1.5 subject to the stability condition the values of P_f are calculated and tabulated in Table 4 and the corresponding line graphs are drawn in Figure 5. From the graph it is inferred that as λ rises P_f falls as expected.

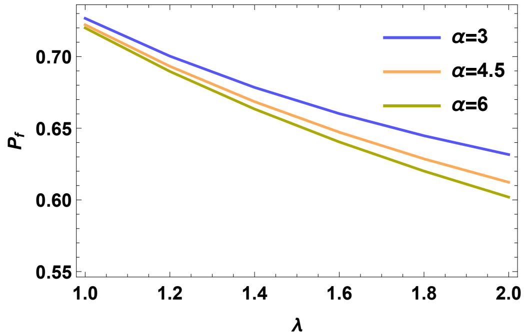

Figure 5. P_f with turn over of λ

λ	$\alpha = 3$	$\alpha = 4.5$	$\alpha = 6$
1.0	0.7266	0.7221	0.7199
1.2	0.7002	0.6932	0.6896
1.4	0.6784	0.6685	0.6633
1.6	0.6601	0.6472	0.6404
1.8	0.6447	0.6286	0.6201
2.0	0.6316	0.6123	0.6019

Table 4. P_f with turn over of λ

9. Conclusion

In this article, a Markovian retrial queue and WV under N-control pattern is evaluated. We calculate stability condition and rate matrix of the model. We went on the stationary probability distribution by adopting the matrix-analytic methods. We also derive the conditional stochastic decomposition and performance measures. We perform some special cases and illustrate some numerical examples under the stability condition.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] J. Artalejo and A. Gómez-Corral, *Retrial Queueing Systems*, Springer, Berlin (2008), DOI: 10.1007/978-3-540-78725-9.

[2] B. D. Choi, K. K. Park and C. E. M. Pearce, An M/M/1 retrial queue with control policy and general retrial times, *Queueing Systems* **14** (1993), 275 – 292, DOI: 10.1007/BF01158869.

[3] T. V. Do, M/M/1 retrial queue with working vacations, *Acta Informatica* **47** (2010), 67 – 75, DOI: 10.1007/s00236-009-0110-y.

[4] R. Kalyanaraman and S. P. B. Murugan, A single server retrial queue with vacation, *Journal of Applied Mathematics & Informatics* **26**(3-4) (2008), 721 – 732, URL: https://koreascience.kr/article/JAKO200822179193859.page.

[5] G. Latouche and V. Ramaswami, *Introduction to Matrix Analytic Methods in Stochastic Modelling*, xiv + 317, ASA-SIAM Series on Applied Probability, SIAM, USA (1999), DOI: 10.1137/1.9780898719734.

[6] J. Li and N. Tian, The M/M/1 queue with working vacations and vacation interruptions, *Journal of Systems Science and Systems Engineering* **16** (2007), 121 – 127, DOI: 10.1007/s11518-006-5030-6.

[7] W.-Y. Liu, X.-L. Xu and N.-S. Tian, Stochastic decompositions in the M/M/1 queue with working vacations, *Operation Research Letters* **35**(5) (2007), 595 – 600, DOI: 10.1016/j.orl.2006.12.007.

[8] P. Manoharan and T. Jeeva, An M/M/1 retrial queue with working vacation Interruption and setup time, *International Journal of Management Technology and Engineering* **IX**(VII) (2019), 204 – 211.

[9] S. P. B. Murugan and K. Santhi, An M/G/1 queue with two stage heterogeneous service and multiple working vacation, *International Journal of Mathematics Sciences and Applications*, **3**(1) (2013), 233 – 249.

[10] M. F. Neuts, *Matrix-Geometric Solutions in Stochastic Models*, 352 pages, Johns Hopkins University Press, Baltimore (1981).

[11] L. D. Servi and S. G. Finn, M/M/1 queue with working vacations (M/M/1/WV), *Performance Evaluation* **50**(1) (2002), 41 – 52, DOI: 10.1016/S0166-5316(02)00057-3.

[12] L. Tao, Z. Liu and Z. Wang, M/M/1 retrial queue with collisions and working vacation interruption under N-policy, *RAIRO – Operations Research* **46**(4) (2012), 355 – 371, DOI: 10.1051/ro/2012022.

[13] N. Tian and Z. G. Zhang, *Vacation Queueing Models: Theory and Applications*, Springer-Verlag, New York (2006), DOI: 10.1007/978-0-387-33723-4.

[14] V. L. Wallace, The solution of quasi birth and death processes arising from multiple access computer systems, *Technical Report* UMR4381, University of Michigan, (1969), URL: https://hdl.handle.net/2027.42/8180.

[15] D.-A. Wu and H. Takagi, M/G/1 queue with multiple working vacations, *Performance Evaluation* **63**(7) (2006), 654–681, DOI: 10.1016/j.peva.2005.05.005.

[16] M. Yadin and P. Naor, Queueing system with a removable service station, *Operational Research Quarterly* **14**(4) (1963), 393 – 405, DOI: 10.2307/3006802.

[17] Q. Ye and L. Liu, The analysis of the M/M/1 queue with two vacation policies (M/M/1/SWV+MV), *International Journal of Computer Mathematics* **94**(1) (2017), 115 – 134, DOI: 10.1080/00207160.2015.1091450.

[18] M. Zhang and Z. Hou, Performance analysis of M/G/1 queue with working vacations and vacation interruption, *Journal of Computational and Applied Mathematics* **234**(10) (2010), 2977 – 2985, DOI: 10.1016/j.cam.2010.04.010.

[19] Z. Zhang and X. Xu, Analysis for the M/M/1 queue with multiple working vacations and N-policy, *International Journal of Information and Management Sciences***19** (2008), 495 – 506.