Communications in Mathematics and Applications
Volume 3 (2012), Number 3, pp. 215–222

http://www.rgnpublications.com

Approximate Identities on Non-Euclidean Manifolds

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Abstract. We define a convolution and present a theory for approximate identity on the non-Euclidean manifolds. Here we focus on the hyperboloid and sphere.

1. Introduction

The set $L^1(\mathbb{R})$ of all complex Lebesgue integrable functions on the real line is a commutative Banach algebra, if multiplication is defined by convolution

$$(f*g)(x)=\int_{\mathbb{R}} f(x-y)g(y)dy, \quad x\in\mathbb{R}.$$

$L^1(\mathbb{R})$ has no identity, but approximate identities are available [6, 2]. Suppose F is a continuous Lebesgue integrable function on $\mathbb{R}$ so that $\int_{\mathbb{R}} F(x)dx = 1$ and set $K_\lambda(x)=\frac{1}{\lambda}F(\frac{x}{\lambda})$. Then $(K_\lambda)_{\lambda>0}$ is an approximate identity for $L^1(\mathbb{R})$ i.e., $\lim\limits_{\lambda\to 0} f*K_\lambda = f$ for all $f\in L^1(\mathbb{R})$. For example

(1) *Poisson kernel*: $K_\lambda(x)=\dfrac{\lambda}{\pi(\lambda^2+x^2)}$,

(2) *Fejèr kernel*: $K_\lambda(x)=\dfrac{1-\cos\lambda x}{\pi\lambda x^2}$,

(3) *Gaussian kernel*: $K_\lambda(x)=\dfrac{\lambda}{\sqrt{\pi}}e^{-\lambda^2x^2}$, are approximate identities on $L^1(\mathbb{R})$ [4].

In this paper we enter into the non-Euclidean world and concentrate on the sphere and hyperboloid. The Dilation operator acts on the sphere by Stereographic projection and on the hyperboloid by conic projection. We define convolution on these manifolds and then define the corresponding approximate identity.

Now, here and in what follows, we shall describe the notation and some classical group that will appear in this paper. Let $\mathbb{N}$, $\mathbb{R}$, be the sets of positive integers, real numbers, respectively. For $n\in\mathbb{N}$ we denote by $M_n(\mathbb{R})$ the space of all n-by-n matrices over $\mathbb{R}$. The general linear group $GL_n(\mathbb{R})$ consists of all invertible

2010 *Mathematics Subject Classification.* 41A65, 42A85, 44A35.
*Key words and phrases.*Convolution; Identity approximate; One-sheeted hyperboloid.

matrices in $M_n(\mathbb{R})$. The orthogonal group of degree n is a subgroup of the general linear group $GL_n(\mathbb{R})$ given by

$$O(n) = \{Q \in GL_n(\mathbb{R}) : Q^t Q = QQ^t = I_n\},$$

where Q^t is the transpose of Q and I_n denoting the n-by-n identity matrix. Every orthogonal matrix has determinant either 1 or -1. The orthogonal n-by-n matrices with determinant 1 form a normal subgroup of $O(n)$, known as the special orthogonal group, $SO(n)$.

The indefinite orthogonal group, $O(p,q)$, with $p+q=n$ and $p,q \in \mathbb{N}$, defined by

$$O(p,q) = \{M \in M_n(\mathbb{R}) : M^t I_{p,q} M = I_{p,q}\},$$

with $I_{p,q} = \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix}$. Note that for $M \in O(p,q)$, $\det M = +1$ or -1.

The indefinite special orthogonal group is defined by

$$SO(p,q) = \{M \in O(p,q) : \det M = 1\}.$$

Now, consider the indefinite orthogonal group $SO(1,q)$ and look at the $(0,0)$-entry of defining condition $M^t I_{1,q} M = I_{1,q}$ where $M = [a_{ij}]_{i,j=0}^q$. We get $a_{00}^2 = 1 + \sum_{j=1}^{q} a_{0j}^2 \geq 1$ and define $SO_0(1,q) = \{M = [a_{ij}]_{i,j=0}^q \in SO(1,q) : a_{00} \geq 1\}$, which is also the connected component of the identity in $O(1,q)$.

2. **Geometry of the one-sheeted hyperboloid**

The one-sheeted hyperboloid ($H^{1,1} \subseteq \mathbb{R}^3$) has cartesian equation $x_1^2 - x_2^2 - x_3^2 = -1$. In polar coordinates, $H^{1,1}$ may be parameterized as $x = (x_1, x_2, x_3) = x(\chi, \varphi)$, where

$$x_1 = \sinh \chi,$$

$$x_2 = \cosh \chi \cos \varphi, \quad \chi \in \mathbb{R}, \; \varphi \in [0, 2\pi),$$

$$x_3 = \cosh \chi \sin \varphi,$$

where χ is the arc length over meridians and φ is the arc length over equator. Consider the cone $C = \{x \in \mathbb{R}^3 : x_1^2 - x_2^2 - x_3^2 = 0\}$. All points of $H^{1,1}$ will be mapped onto C using a specific conic projection.

Dilations on $H^{1,1}$ are obtained by conic projection in three steps [1]:

(i) given a point $x = x(\chi, \varphi) \in H^{1,1}$, project it to the point

$$\xi = (\sinh \chi, \sinh \chi \cos \varphi, \sinh \chi \sin \varphi) \in C,$$

(ii) dilate ξ to $a\xi$ on cone C,

(iii) project back $x_a = x(\chi_a, \varphi)$ with $\sinh \chi_a = a \sinh \chi$.

3. **Convolution on the one-sheeted hyperboloid**

Since $H^{1,1}$ is a homogeneous space of $SO_0(1,2)$, $H^{1,1} = \frac{SO_0(1,2)}{SO_0(1,1)}$ [5], one can easily define a convolution. Let $L^1(H^{1,1}, d\mu)$ denote the space of integrable functions on $H^{1,1}$ with the $SO_0(1,2)$ invariant measure $d\mu(x) = \cosh\chi\, d\chi\, d\varphi$. Given $u, v \in L^2(H^{1,1})$ and define their convolution by:

$$(u * v)(g) = \int_{H^{1,1}} u(g^{-1}x)v(x)d\mu(x), \quad \text{for all } \ g \in SO_0(1,2).$$

A motion $g \in SO_0(1,2)$ can be factorized as $g = k_1 h k_2$, where $k_1, k_2 \in SO(2)$, $h \in SO_0(1,1)$, and the respective action of k_i's for $i = 1,2$ and h are as follows:

$$k_i(\varphi_0)\cdot x(\chi,\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_0 & -\sin\varphi_0 \\ 0 & \sin\varphi_0 & \cos\varphi_0 \end{pmatrix} \begin{pmatrix} \sinh\chi \\ \cosh\chi\cos\varphi \\ \cosh\chi\sin\varphi \end{pmatrix} = x(\chi, \varphi+\varphi_0),$$

and

$$h(\chi_0)\cdot x(\chi,0) = \begin{pmatrix} \cosh\chi_0 & \sinh\chi_0 & 0 \\ \sinh\chi_0 & \cosh\chi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sinh\chi \\ \cosh\chi \\ 0 \end{pmatrix} = x(\chi+\chi_0, 0).$$

Also we define:

$$\begin{aligned} h(\chi_o)\cdot x(\chi,\varphi) &= k(\varphi)h(\chi_0)k(-\varphi)\cdot x(\chi,\varphi) \\ &= x(\chi+\chi_o, \varphi). \end{aligned}$$

Now, we define a convolution on $H^{1,1}$, by a section

$$[\cdot] : H^{1,1} \to SO_0(1,2)$$

$$x = x(\chi,\varphi) \mapsto [x] = k(\varphi)h(\chi), k(\varphi) \in SO(2), h(\chi) \in SO_0(1,1).$$

Definition. Given $f, g \in L^1(H^{1,1}, d\mu)$ and define their convolution by

$$(f * g)(x) = \int_{H^{1,1}} f([y]^{-1}x)g(y)d\mu(y), \quad x \in H^{1,1}.$$

4. **Approximate identities on $H^{1,1}$**

Lemma 1. *Suppose $1 \le p < \infty$ and $f \in L^p(H^{1,1}, d\mu)$. The mapping $y \mapsto f_y = f([y]^{-1}\cdot)$ is a uniformly continuous mapping of $H^{1,1}$ into $L^p(H^{1,1}, d\mu)$.*

Proof. Let $\epsilon > 0$ be given. Since $\mathscr{C}_c(H^{1,1})$ is dense in $L^p(H^{1,1})$, there exists a continuous function g on $H^{1,1}$ and a positive constant A with

$$g(\chi,\varphi) = 0 \ \text{ if } \ |\chi| > A,$$

such that $\|g - f\|_p < \epsilon/3$. The uniform continuity of g implies that there is $0 < \delta < A$ such that for all $y(\chi,\varphi), \tilde{y}(\tilde{\chi},\tilde{\varphi}) \in H^{1,1}$ with $|\chi - \tilde{\chi}| < \delta$ we have $|g(y) - g(\tilde{y})| < (4\pi\sinh 2A)^{-\frac{1}{p}}\frac{\epsilon}{3}$ and so $\|g_y - g_{\tilde{y}}\|_p < \epsilon/3$. Hence

$$\|f_y - f_{\tilde{y}}\|_p \le \|f_y - g_y\|_p + \|g_y - g_{\tilde{y}}\|_p + \|g_{\tilde{y}} - f_{\tilde{y}}\|_p < \epsilon,$$

whenever $|\chi - \widetilde{\chi}| < \delta$. $\square$

Let ψ be a function on $H^{1,1}$. For $a > 0$ define

$$\psi_a(x) = \lambda(a,x)\psi(x_{\frac{1}{a}}),$$

where the Radon-Nikodym derivative $\lambda(a,x)$ is given by

$$\lambda(a,x) = \frac{d\mu(x_{\frac{1}{a}})}{d\mu(x)} = \frac{1}{a}.$$

Theorem 2. *Let $\psi \in L^1(H^{1,1}, d\mu)$ be rotation invariant with $\int_{H^{1,1}} \psi(x)dx = 1$. If $g \in L^\infty(H^{1,1}, d\mu)$ and g is continuous at $x \in H^{1,1}$, then $\lim_{a\to 0}(g * \psi_a)(x) = g(x)$.*

Proof. We have

$$\begin{aligned}(g * \psi_a)(x) - g(x) &= \int_{H^{1,1}} g([y]^{-1}x)\psi_a(y)d\mu(y) - \int_{H^{1,1}} g(x)\psi_a(y)d\mu(y)\\ &= \int_{H^{1,1}} [g([y]^{-1}x) - g(x)]\psi_a(y)d\mu(y)\\ &= \int_{H^{1,1}} [g([y_a]^{-1}x) - g(x)]\psi(y)d\mu(y),\end{aligned}$$

the last integrand is dominated by $2\|g\|_\infty \psi(y)$ and converges to 0 pointwise for every y as $a \to 0$. Hence the result follows from the Lebesgue dominated convergence theorem. $\square$

Theorem 3. *Let $\psi \in L^1(H^{1,1}, d\mu)$ be rotation invariant with $\int_{H^{1,1}} \psi(x)dx = 1$. If $1 \le p < \infty$ and $f \in L^p(H^{1,1}, d\mu)$, then $\lim_{a\to 0} \|f * \psi_a - f\|_p = 0$.*

Proof. We have

$$(f * \psi_a)(x) - f(x) = \int_{H^{1,1}} [f([y]^{-1}x) - f(x)]\psi_a(y)d\mu(y).$$

By Minkowski's inequality for integrals

$$\begin{aligned}\|f * \psi_a - f\|_p &\le \left(\int_{H^{1,1}} \left(\int_{H^{1,1}} |f([y]^{-1}x) - f(x)||\psi_a(y)|d\mu(y)\right)^p d\mu(x)\right)^{\frac{1}{p}}\\ &\le \int_{H^{1,1}} \left(\int_{H^{1,1}} |f([y]^{-1}x) - f(x)|^p |\psi_a(y)|^p d\mu(x)\right)^{\frac{1}{p}} d\mu(y)\\ &= \int_{H^{1,1}} \|f_y - f\|_p |\psi_a(y)| d\mu(y)\end{aligned}$$

If $g(y) = \|f_y - f\|_p$, then g is bounded and continuous, and $g(0) = 0$. Hence $\lim_{a\to 0} \|f * \psi_a - f\|_p = 0$ by previous theorem. $\square$

Example. (i) $\psi_a(\chi,\varphi)=\dfrac{\pi^{-\frac{3}{2}}}{2a}e^{-a^2\sinh^2\chi}$,

(ii) $\psi_a(\chi,\varphi)=\dfrac{1}{2a\pi^2(1+a^2\sinh^2\chi)}$,

are examples of sequence $(\psi_a)_a$ that satisfies in previous theorem.

5. **Approximate identity on the upper sheet of two sheeted hyperboloid**

We start by recalling the geometry of the upper sheet of two sheeted Hyperboloid. The two-sheeted hyperboloid, $H_+^2 \subseteq \mathbb{R}^3$, with Cartesian equation $x_0^2-x_1^2-x_2^2=1$ may be parameterized as

$$x_1=\cosh\chi,$$

$$x_2=\sinh\chi\cos\varphi,\quad \chi\geq 0,\quad \varphi\in[0,2\pi).$$

$$x_3=\sinh\chi\sin\varphi.$$

Consider the cone $C^+=\{x\in\mathbb{R}^3: x_1^2-x_2^2-x_3^2=0, x_1\geq 0\}$ and define dilation on H_+^2 by conic projection of H_+^2 onto C^+ [3]:

$$x(\chi,\varphi)\to x_a(\chi_a,\varphi)\ \text{ with }\ \sinh\frac{\chi_a}{2}=a\sinh\frac{\chi}{2}.$$

Since H_+^2 is a homogeneous space, $H_+^2=\frac{SO_0(1,2)}{SO(2)}$, we can define convolution for functions in $L^1(H_+^2,d\mu)$, the space of integrable functions on H_+^2 with the $SO_0(1,2)$ invariant measure $d\mu(x)=\sinh\chi\, d\chi\, d\varphi$, and show that the approximate identities exist on $L^1(H_+^2)$ in a similar fashion on $L^1(H^{1,1})$.

6. **Approximate identity on the sphere**

The two-dimensional sphere of radius 1 ($S^2\subseteq\mathbb{R}^3$) has cartesian equation $x_1^2+x_2^2+x_3^2=1$. In polar coordinates, S^2 may be parameterized as $x=(x_1,x_2,x_3)=x(\theta,\varphi)$, where

$$x_1=\sin\theta\cos\varphi$$

$$x_2=\sin\theta\sin\varphi,\quad \theta\in[0,\pi],\quad \varphi\in[0,2\pi).$$

$$x_3=\cos\theta$$

Dilations on S^2 are obtained by considering usual dilations in the tangent plane at the North Pole and lifting them to S^2 by inverse stereographic projection from the south Pole. Thus, in polar spherical coordinates, the dilation operator acts on a point (θ,φ) by:

$$D_a x(\theta,\varphi)=x_a(\theta_a,\varphi),\quad \text{with } \tan\frac{\theta_a}{2}=a\tan\frac{\theta}{2}.$$

Consider the group of proper rotations of $\mathbb{R}^3$ about the origin, denoted by $SO(3)$. Any element $\rho \in SO(3)$ may be expressed as the product of two rotations about the x_3-axis, and one about x_2-axis. Let

$$R_{x_3}(A) = \begin{pmatrix} \cos A & -\sin A & 0 \\ \sin A & \cos A & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } R_{x_2}(A) = \begin{pmatrix} \cos A & 0 & \sin A \\ 0 & 1 & 0 \\ -\sin A & 0 & \cos A \end{pmatrix},$$

so ρ has the Euler angle decomposition $\rho = \rho(\alpha, \beta, \gamma) = R_{x_3}(\alpha)R_{x_2}(\beta)R_{x_3}(\gamma)$, where $0 \le \alpha$, $\gamma < 2\pi$ and $0 \le \beta \le \pi$. Since $SO(3)$ acts on the sphere, we can define a convolution. Let $L^1(S^2, d\mu)$ denote the space of integrable functions on S^2 with the rotation invariant measure on the sphere $d\mu(x) = \sin\theta d\theta d\varphi$. We define a convolution on S^2, by a section

$$[\cdot] : S^2 \to SO(3)$$

$$x = x(\theta, \varphi) \mapsto [x] = \rho(\varphi, \theta, 0) = R_{x_3}(\varphi)R_{x_2}(\theta).$$

Definition. Given $f, g \in L^1(S^2, d\mu)$ and define their convolution by

$$(f * g)(x) = \int_{S^2} f([y]^{-1}x)g(y)d\mu(y), \quad x \in S^2.$$

Lemma 4. *For any function f on S^2 and every $y \in S^2$, let f_y defined by $f_y(x) = f([y]^{-1}x)$, for all $x \in S^2$. If $1 \le p < \infty$ and if $f \in L^p(S^2, d\mu)$, then $y \mapsto f_y$ is a continuous mapping of S^2 into $L^p(S^2, d\mu)$.*

Proof. Similar to proof of Lemma 1. □

Theorem 5. *Given $f \in L^1(S^2, d\mu)$ and $\epsilon > 0$, there exists a neighborhood V of N in S^2 with the following property: if u is non-negative Borel function which vanishes outside V, and if $\int_{S^2} u(x)d\mu(x) \equiv 1$, then $\|f * u - f\|_1 < \epsilon$.*

Proof. By Lemma 4, We can choose V so that $\|f - f_y\|_1 < \epsilon$, for all $y \in V$. If u satisfies the hypotheses, we have

$$(f * u)(x) - f(x) = \int_{S^2} [(f[y]^{-1}x) - f(x)]u(y)d\mu(y).$$

So that

$$\|f * u - f\|_1 \le \int_{S^2} |u(y)|d\mu(y) \int_{S^2} |f([y]^{-1}x) - f(x)|d\mu(x)$$

$$= \int_{S^2} \|f - f_y\|_1 u(y)d\mu(y) < \epsilon. \qquad \square$$

Let ψ be a function on S^2. For $a > 0$ define

$$\psi_a(x) = \lambda(a, \theta)\psi(x_{\frac{1}{a}}),$$

where the Radon-Nikodym derivative $\lambda(a,\theta)$ is given by

$$\lambda(a,\theta)=\frac{d\mu(x_{\frac{1}{a}})}{d\mu(x)}=\frac{4a^2}{[(a^2-1)\cos\theta+(a^2+1)]^2}.$$

Theorem 6. *Suppose $\psi\in L^1(S^2,d\mu)$ with $\int_{S^2}\psi(x)dx=1$. If $g\in L^\infty(S^2,d\mu)$ and g is continuous at $x\in S^2$, then $\lim_{a\to 0}(g*\psi_a)(x)=g(x)$.*

Proof. Similar proof of Theorem 2. □

Theorem 7. *Suppose $\psi\in L^1(S^2,d\mu)$ with $\int_{S^2}\psi(x)dx=1$. If $1\le p<\infty$ and $f\in L^p(S^2,d\mu)$, then $\lim_{a\to 0}\|f*\psi_a-f\|_p=0$.*

Proof. Similar proof of Theorem 3. □

Example. (i) $\psi_a(\theta,\varphi)=\dfrac{a^2}{\pi[(a^2-1)\cos\theta-(a^2+1)]^2}$,

(ii) $\psi_a(\theta,\varphi)=\lambda(a,\theta)\dfrac{(1+a^2\tan^2(\frac{\theta}{2}))^2e^{-a^2\tan^2(\frac{\theta}{2})}}{4\pi^{\frac{3}{2}}a\tan(\frac{\theta}{2})}$,

are examples of sequence $(\psi_a)_a$ that satisfies in Theorem 7.

The respective action of R_{x_3} and R_{x_2} on S^2 are as follows:

$$R_{x_3}(A)x(\theta,\varphi)=x(\theta,\varphi+A), \tag{1}$$

$$R_{x_2}(A)x(\theta,0)=x(\theta+A,0). \tag{2}$$

Also we define

$$R_{x_2}(A)x(\theta,\varphi)=R_{x_3}(\varphi)R_{x_2}(A)R_{x_3}(-\varphi)x(\theta,\phi)=x(\theta+A,\varphi).$$

Then convolution on the sphere is commutative and associative and $L^1(S^2,d\mu)$ is a Banach algebra with approximate identity.

7. Conclusions

In this paper a constructive theory for the approximate identities on the one-she hyperboloid and sphere has been developed. After introducing the notation of convolution on these manifolds, we showed that the space of square integrable functions on these manifolds has approximate identities.

References

[1] A. Askari Hemmat and Z. Yazdani Fard, Continuous wavelet transform on the one-sheeted hyperboloid, *Communications in Mathematics and Applications*, **2** (2-3) (2011), 131–143.

[2] G. Bachman, L. Narici and E. Beckenstein, *Fourier and Wavelet Analysis*, Springer-Verlag, New York, 2000.

[3] I. Bogdanova, P. Vandergheynst and J.-P. Gazeau, Continuous Wavelet transform on the hyperboloid, *Applied Comput. Harmon. Anal* **23**(2007), 285–306.

[4] L. Debnath, *Wavelet Transforms and Their Applications*, Birkhauser, 2002.

[5] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, John Wiley and Sons Inc, 1974.

[6] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1986.

[7] W. Rudin, *Fourier Analysis on Groups*, Interscience Publishers, Inc., New York, 1962.

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