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Research Article

Uniform Convergence of Multigrid Methods for Elliptic Quasi-Variational Inequalities and Its Implementation

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Abstract. In this paper, algebraic multigrid methods on adaptive finite element discretisation are applied for solving elliptic quasi-variational inequalities. The uniform convergence of the multigrid scheme has been established which proves that the multigrid methods have a contraction number with respect to the maximum norm. Numerical results which demonstrate the high efficiency of these methods are given for a quasi-variational inequality arising from impulse control problem on a domain with nonpolygonal boundaries.

Keywords. Quasi-variational inequality, Finite element method, HJB equation, Multigrid method

Mathematics Subject Classification (2020). 65M55, 65N30, 49N05, 65K15, 35J86

1. Introduction

We treat multigrid methods for solving one-sided obstacle problems based on restating the *Quasi-Variational Inequality* (QVI) as a *Hamilton-Jacobi-Bellman* (HJB)-equation. A great deal of work for the analysis of elliptic quasi-variational inequalities related to the HJB-equation is given in [5, 6, 12], to get more details about the regularity of the continuous problem and the discretization error bound we refer to [1, 3–7, 18, 19]. In multigrid methods, we try to reduce the amount of geometric information (i.e., make these methods more algebraic), for a thorough treatment of these methods we refer to the monograph [13].

For the discretization, we use the finite element method to get a large sparse linear system of equations and then use an iterative procedure established by Hoppe in [20] to solve this system. However, In contrast to Hoppe [20] we apply a multigrid method directly to solve the algebraic systems obtained by the HJB reformulation (4.1). The convergence results proposed by Arnold for elliptic PDEs [24] yield the multigrid L^∞-convergence result described in this work. The proof of these results is based on the approximation and smoothing properties introduced by Hackbusch [13]. The analysis of the smoothing property presented here is the same as for the elliptic PDEs in [24], and as is shown in [15], the main points in the proof of the approximation property are the finite element L^∞-error estimate given in Theorem 4 and the results of Lemma 2, which do not depend on the bilinear form, but only on the triangulation.

We briefly outline the remainder of this paper. In Section 2, we introduce a continuous problem and we present some regularity results. In Section 3 we apply linear finite element discretizations to derive a system of equations, and in Section 4 we describe a multigrid method for the solution of the algebraic systems obtained by the HJB reformulation. In Section 5 we present uniformed convergence results of these multigrid methods. The results of numerical experiments are given in Section 6 for a QVI leading to a stochastic inventory problem with impulse control.

2. Continuous Problem

2.1 Notations and Assumptions

Let Ω be an open in $\mathbb{R}^N$, with sufficiently smooth boundary $\partial\Omega$ for $u,v \in V$ $(V = H_0^1(\Omega))$, consider

$$a(u,v) = \int_\Omega \left[\sum_{1\le i,j\le N} a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{1\le i\le N} a_i(x)\frac{\partial u}{\partial x_i} + a_0(x) u\cdot v \right] dx, \tag{2.1}$$

a bilinear form related to the linear operator A defined by

$$A = \sum_{1\le i,j\le N} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{1\le i\le N} a_i \frac{\partial}{\partial x_i} + a_0,$$

where $a_{ij}(x)$, $a_i(x)$, $a_0(x)$, $x \in \overline{\Omega}$, $1 \le i,\ j \le n$ are sufficiently regular and satisfy the following conditions:

$$\sum_{1\le i,j\le n} a_{ij}\xi_i\xi_j \ge \varrho|\xi|^2, \quad \xi \in \mathbb{R}^N, \varrho > 0,$$

$$a_0(x) \ge \beta > 0, \quad \beta \text{ is a constant.}$$

We defined the operator $M : V \cap L^\infty(\Omega) \to V \cap L^\infty(\Omega)$ by:

$$Mu = k + \inf_{\varepsilon\ge 0, x+\varepsilon\in\overline{\Omega}} u(x+\varepsilon), \quad k \text{ is a positive constant,} \tag{2.2}$$

$$M \in W^{2,p}(\Omega), \quad Mu \ge 0, \text{ on } \partial\Omega : 0 \le g \le Mu,$$

where g is a regular function defined on $\partial\Omega$. Let $K_g(u)$ an implicit convex and non empty set given by

$$K_g(u) = \{v \in V,\ v = g \text{ on } \partial\Omega,\ v \le Mu,\ \text{in } \Omega\}.$$

Consider the following problem: Find $u \in K_g(u)$ solution of

$$\begin{cases} a(u,v-u) \geq \langle f, v-u \rangle, & v \in K_g(u), \\ u \leq Mu, & Mu \geq 0, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

We present a theorem which ensure the existence and the uniqueness of the solution of the problem (2.3).

Theorem 2.1 ([17])**.** *Under the previous conditions the problem* (2.3) *has a unique solution* $u \in K_g(u)$. *Moreover, we have*

$$u \in W^{2,p}(\Omega), \quad 2 \leq p \leq \infty.$$

3. Discrete Problem

To apply a multigrid cycle, we propose a decreasing sequence (mesh size parameter)

$$\{h_k\}_{k=0}^{l}, \quad h_{k+1} < h_k, 0 \leq k \leq m-1.$$

We introduce a quasi-uniform family of nested triangulations $\{T_k, k \in N\}$ of $\Omega_k = \bigcup_{T \in T_k} T$. For all T_k we have

$$\Omega_k \subset \Omega_{k+1} \subset \Omega,$$
$$dist(\partial\Omega_k, \partial\Omega) \leq c_0 h_k^2,$$
$$h_k h_{k+1} \leq c_1.$$

On each level k, we choose a piecewise linear finite element space

$$V_k = \{v_k \in C(\Omega) \cap H^1(\Omega) |\ v_{k|\Omega_k} \in P_1\}, \tag{3.1}$$

and we associate to each h_k an analogous discretization of the problem (2.3) by a *finite element method* (FEM). To simplify the notation, we put

$$\Omega_k = \Omega_{h_k}, \;\; V_k = V_{h_k}, \;\; A_k = A_{h_k}.$$

The standard basis functions φ_k^i, $i \in (1, \dots, m(h_k))$ is defined by

$$\varphi_k^i(x_k^j) = \delta_{ij},$$

where x_k^i is a vertex of the triangulation T_k. Let $U_k = \mathbb{R}^{m_k}$, the usual finite element restriction operator from U_k into V_k is bijection defined by

$$r_k v(x) = \sum_{i=1}^{m(h_k)} v(x_k^i) \varphi_k^i(x). \tag{3.2}$$

On U_k we use a scaled Euclidean scalar product

$$\langle u, v \rangle_k = h_k^2 \sum_{i=1}^{m_k} u_i v_i, \quad \text{and a corresponding norm } \|u\|_k = \langle u, u \rangle_k^{1/2}.$$

Moreover, the adjoint operator $r_k^* : V_k \to U_k$ satisfies

$$\langle r_k u, v \rangle_{L^2} = \langle u, r_k^* v \rangle, \quad \forall\ u \in U_k,\ v \in V_k.$$

The maximum norm $\|\cdot\|_\infty$ (on U_k) and the norm $\|\cdot\|_{L^\infty}$ (on V_k) are equivalent, which are denoted by $\|\cdot\|_\infty$.

Lemma 3.1 ([24]). *For the restriction operator r_k defined by* (3.2)*, there exist constants C_1 and C_2 independent of k such that*

$$\|r_k(u)\|_{L^\infty} = \|u\|_\infty,\ \forall\ u \in U_k,$$
$$C_1\|v\|_{L^\infty} \le \|r_k^*(v)\|_\infty \le C_2\|v\|_{L^\infty}, \quad \forall\ v \in V_k.$$

We naturally introduce the discretization matrices A_k by the generic coefficient matrices

$$a(\varphi_k^l(x), \varphi_k^s(x)), \quad s \in (1,\dots,m(h_k)).$$

The numerical approximation of the QVI (2.3) by finite elements leads to the solution of the following discrete QVI in finite dimension. Find $u_k \in K_{g,k}$ such that

$$\begin{cases} \langle A_k u_k, v_k - u_k\rangle \ge \langle f, v_k - u_k\rangle, & \forall\ v_k \in K_{g,k}, \\ u_k \le M_k u_k, & v_k \le M_k u_k, \end{cases} \tag{3.3}$$

where

$$f \in L^\infty(\Omega),$$
$$M_k u_k = K + \inf_{\epsilon \ge 0, (x+\epsilon) \in \bar{\Omega}} u_k(x+\epsilon),\ k \text{ is a positive constant},$$
$$K_{g,k} = \{v \in V_k : v = \pi_k g \text{ on } \partial\Omega, v \le M_k u_k \text{ in } \Omega\}.$$

Denoted by π_k the interpolation operator on $\partial\Omega$.

Theorem 3.1 ([10]). *Assume that the discrete maximum principle holds* (*i.e., the matrices resulting from the discretization of the problems* (2.3) *are M-matrices* [9]). *Then, under the previous conditions the problem* (3.3) *has a unique solution.*

Theorem 3.2 ([11]). *Let u and u_k be the solutions of problems* (2.3) *and* (3.3) *respectively, then there exists a constant C independent of h_k such that:*

$$\|u - u_k\|_{L^\infty(\Omega)} \le Ch^2 |\log h_k|^2. \tag{3.4}$$

4. Description of Multigrid Methods for QVIs

4.1 The Well Defined HJB-formulation of the Discrete Problem

Formally, we can write the QVI (3.3) as the following HJB equation. Let u_k^ν be the unique solution of the discrete HJB equation

$$\max_{1 \le i \le N}(A_{k,i} u_{k,i}^\nu - f_{k,i},\ u_{k,i}^\nu - M_k u_{k,i}^{\nu-1}) = 0. \tag{4.1}$$

Choose an initial vector $u_k^0 \in U_k$. Starting from the iterate $u_k^\nu \in U_k$, $\nu \ge 0$, we may split the set

$$J_k = \{1,2,\dots,m_k\} \text{ by } \quad J_k = \bigcup_{p=1}^{3} J_k^p(u_k^\nu)$$

as

$$\left.\begin{aligned} J_k^1(u_k^\nu) &= \{i \in J_k \mid (A_k u_k^\nu - f_k)_i > u_{k,i}^\nu - M_k u_{k,i}^{\nu-1}\}, \\ J_k^2(u_k^\nu) &= \{i \in J_k \mid (A_k u_k^\nu - f_k)_i < u_{k,i}^\nu - M_k u_{k,i}^{\nu-1}\}, \\ J_k^3(u_k^\nu) &= \{i \in J_k \mid (A_k u_k^\nu - f_k)_i = u_{k,i}^\nu - M_k u_{k,i}^{\nu-1}\} \end{aligned}\right\} \tag{4.2}$$

and compute $u_k^{\nu+1} \in U_k$ as the solution of the equation

$$A_k^\nu u_k^{\nu+1} = f_k^\nu, \tag{4.3}$$

where

$$A_k^\nu = \begin{cases} A_{k,i}, & \text{if } i \in J_k^1(u_k^\nu), \\ I_{k,i}, & \text{if } i \in J_k^2(u_k^\nu) \cup J_k^3(u_k^\nu), \end{cases} \tag{4.4}$$

$$f_k^\nu = \begin{cases} f_{k,i}, & \text{if } i \in J_k^1(u_k^\nu), \\ M_k u_{k,i}^{\nu-1}, & \text{if } i \in J_k^2(u_k^\nu) \cup J_k^3(u_k^\nu), \end{cases} \tag{4.5}$$

with $A_{k,i}$ (resp., $f_{k,i}$) is the ith row of the discretization matrix A_k (resp., the ith component of the right-hand side f_k) of our discrete problem, and $I_{k,i}$ is the ith row of the identity matrix I_k.

We can prove the monotone convergence of the iterates if we assume A_k to be continuously differentiable.

Theorem 4.1 ([20])**.** *Let (u_k^ν) be the iterate obtained by the previous iterative scheme so it satisfies the H.J.B equation above, moreover we suppose that A_k to be continuously differentiable, then the sequence $(u_k^\nu)_{\nu\ge 0}$ converges monotone decreasingly towards the unique solution u_k^* of* (3.3)*.*

4.2 Multigrid Algorithm

In the multigrid method, choose an iterate u_k^ν, $\nu > 0$, we get $\bar{u}_k^\nu$ by α applications of an iterative method for the solution of the system (4.3), denoted by

$$\bar{u}_k^\nu = S_k^\alpha(u_k^\nu). \tag{4.6}$$

S_k is the iteration matrix of smoothing method, and α is the number of iterations performed.

We denote by u_k^* the solution of (4.3). Setting the error $e_k^\nu = \bar{u}_k^\nu - u_k^*$, and the residual $d_k^{(\nu)} = f_k^\nu - A_k^\nu \bar{u}_k^\nu$, we can write the equation (4.3) as

$$A_k^\nu(\bar{u}_k^\nu + e_k^\nu) = f_k^\nu.$$

Which results in the residual equation

$$A_k^\nu e_k^\nu = f_k^\nu - A_k^\nu \bar{u}_k^\nu = d_k^{(\nu)}.$$

On the fine grid, after the relaxation on $A_k^\nu \bar{u}_k^\nu = f_k^\nu$ the error will be smooth, while on the coarse grid this error seems to be more oscillatory, and the relaxation will be so efficient. So to determine e_k^ν completely, we need to calculate e_{k-1}^ν at level $(k-1)$ as the solution of the coarse grid system

$$A_{k-1}^\nu e_{k-1}^\nu = d_{k-1}^{(\nu)}. \tag{4.7}$$

We can interpret e_{k-1}^ν (resp., A_{k-1}^ν, $d_{k-1}^{(\nu)}$) as an approximation at the level $k-1$ of e_k^ν (resp., A_k^ν, $d_k^{(\nu)}$).

$$e_{k-1}^\nu = R_k e_k^\nu, \quad A_{k-1}^\nu = R_k A_k^\nu P_k, \quad d_{k-1}^{(\nu)} = R_k d_k^{(\nu)}.$$

Consequently, we determine an improved iterate at the level k by

$$u_k^{\nu+1} = \bar{u}_k^\nu + P_k(e_{k-1}^\nu). \tag{4.8}$$

Due to the nestedness we use the well-defined identity operator

$$\Pi : V_{k-1} \to V_k$$
$$\Pi v = v$$

to define the prolongation and the restriction operators, i.e.,

$$P_k = r_k^{-1} r_{k-1}, \quad R_k = P_k^t. \tag{4.9}$$

Remark 4.1. The previous algorithm describes one cycle of a two-grid iteration to solve (4.3) for two hierarchy of grids Ω_k and Ω_{k-1}. It is clear that the coarse grid system (4.7) has the same form as the system (4.3) Thus we can solve a system (4.7) approximately by applying the two-grid iteration recursively to all hierarchy of grids $\{\Omega_k, k = 0, \dots, m_k\}$.

The multigrid iteration may be described as the Algorithm 1.

Algorithm 1 Muligrid methods

$u \leftarrow MGM(A_k^v, f_k^v, u_0)$
$u_0 := smoother(A_k^v, b, u_0, \alpha_1)$; %$\alpha_1$ is a number of iterations performed (presmoothing)
$d_k := f_k^v - A_k^v u_0$; %residual computation
$R_k; P_k$; %define the prolongation and the restriction matrices
$A_{k-1}^v := R_k A_k^v P_k$; %restriction of A_k
$d_{k-1} := R_k d_k$; %restriction of d_k
$e_{k-1} := d_{k-1} \cdot 0$; %start value for coarse grid iteration
if $size(A_k^v) \le \mu$ % coarsest grid Ω_μ **then**
  $e_{k-1} := (A_k^v)^{-1} d_{k-1}$; %direct solve on the coarse grid
else
  $e_{k-1} := MGM(A_{k-1}^v, d_{k-1}, e_{k-1})$; %solve coarse problem
end if
$e_k := P e_{k-1}$; %prolongation of e_{k-1}
$u_k := u_0 + e_k$; %add correction to the solution
$u_k := smoother(A_k^v, f_k^v, u_k, \alpha_2)$; %$\alpha_2$ is a number of iterations performed (postsmoothing)
return u_k

5. The Multigrid Convergence on the Maximum Norm

In this section, we present the uniform convergence analysis for the multigrid algorithm described in the previous section.

5.1 The Matrix of Multigrid HJB Algorithm

The iteration matrix of the two-grid method with α_1 presmoothing and α_2 postsmoothing iterations on level k is given by

$$TG_k(\alpha_1, \alpha_2) = S_k^{\alpha_2} \left(\left(A_k^v\right)^{-1} - P_k \left(A_{k-1}^v\right)^{-1} R_k \right) \left(A_k^v\right) S_k^{\alpha_1}. \tag{5.1}$$

It is easy to verify that the multigrid method is a linear iterative method.

Theorem 5.1 ([23]). *The multigrid method is a linear iterative method with the iteration matrix MG_k given by*

$$MG_0 = 0, \tag{5.2a}$$

$$MG_k = S_k^{\alpha_2}\left(I_k - P_k(I_k - MG_{k-1})\left(A_{k-1}^{\nu}\right)^{-1}R_k\right)\left(A_k^{\nu}\right)S_k^{\alpha_1}, \tag{5.2b}$$

$$= TG_k + S_k^{\alpha_2}P_kMG_{k-1}\left(A_{k-1}^{\nu}\right)^{-1}R_k\left(A_k^{\nu}\right)S_k^{\alpha_1}, \quad k = 1,2,\dots. \tag{5.2c}$$

5.2 Approximation Property

The proof of the approximation property introduced by Arnold in [24] yields the approximation property presented here, the following approximation property is based on Theorem 3.2 and Lemma 3.1.

Theorem 5.2 ([15]). *Under the previous assumptions, the matrix*

$$\chi = \left[\left(A_k^{\nu}\right)^{-1} - P_k\left(A_{k-1}^{\nu}\right)^{-1}R_k\right],$$

satisfies the following approximation property:

$$\|\chi\|_\infty \le Ch_k^2|\log h_k|^2. \tag{5.3}$$

Proof. According to Theorem 3.2 we have

$$\|u-u_k\|_{L^\infty} \le Ch_k^2|\log h_k|^2\|f_k\|_\infty, \quad u \in W^{2,p}.$$

Then

$$\|u-u_k^*\|_{L^\infty} \le Ch_k^2|\log h_k|^2\|f_k\|_{L^\infty}.$$

Thus, we get

$$\begin{aligned}\|u_k^*-u_{k-1}^*\|_{L^\infty} &\le \|u_k^*-u\|_{L^\infty} + \|u_{k-1}^*-u\|_{L^\infty}\\ &\le C_1h_k^2|\log h_k|^2\|f_k\|_{L^\infty} + C_2h_k^2|\log h_k|^2\|f_k\|_{L^\infty}\\ &\le Ch_k^2|\log h_k|^2\|f_k\|_{L^\infty}.\end{aligned} \tag{5.4}$$

Then there exists a right-hand side $g \in U_k$ such that

$$a(r_k(A_k^{\nu})^{-1}g, v_k) = \langle (r_k^*)^{-1}g, v_k\rangle_{L^2}, \quad \forall\, v \in K_{g,k},$$

$$a(r_{k-1}^{-1}(A_{k-1}^{\nu})^{-1}R_kg, v_{k-1}) = \langle (r_k^*)^{-1}g, v_{k-1}\rangle_{L^2}, \quad \forall\, v \in K_{g,k-1}.$$

Using (5.4) and Lemma 3.1 we get

$$\|r_k^{-1}(A_k^{\nu})^{-1}g - r_{k-1}^{-1}(A_{k-1}^{\nu})^{-1}R_kg\|_\infty \le Ch_k^2|\log h_k|^2\|g\|_\infty.$$

Then

$$\|(A_k^{\nu})^{-1} - r_k^{-1}r_{k-1}(A_{k-1}^{\nu})^{-1}R_k\|_\infty \le Ch_k^2|\log h_k|^2.$$

This completes the proof

$$\|(A_k^{\nu})^{-1} - P_k(A_{k-1}^{\nu})^{-1}R_k\|_\infty \le Ch_k^2|\log h_k|^2. \qquad \square$$

5.3 Smoothing Property

In order to prove a smoothing property we decompose $A_k^\nu = E_k - N_k$, and use the following assumptions:

$$E_k \text{ is regular and } \|E_k^{-1}N_k\|_\infty \le 1, \quad \text{for all } k, \tag{5.5}$$

$$\|E_k\|_\infty \le Ch_k^{-2}, \quad \text{for all } k, \text{ with } C \text{ independent of } k. \tag{5.6}$$

As a smoother we use a relaxation method with iteration matrix

$$S_k = I_k - \omega E_k^{-1}N_k, \quad \omega \in (0,1).$$

Theorem 5.3 ([24])**.** *Assume that the previous assumptions and notations are satisfied, then there is a constant C independent of k and α such that the following smoothing property holds:*

$$\|(A_k^\nu)S_k^\alpha\|_\infty \le C\frac{1}{\sqrt{\alpha}}h_k^{-2}. \tag{5.7}$$

5.4 Convergence Result of Multigrid Methods

In this section, we prove the uniform convergence of a multigrid algorithm. Besides the approximation and smoothing property, we have to prove the following stability bound:

$$\exists\, C_s : \|S_k^\alpha\|_\infty \le C_s, \quad \text{for all } k \text{ and } \alpha. \tag{5.8}$$

The convergence analysis is based on the following splitting of the two-grid iteration matrix, with $\alpha_2 = 0$:

$$\begin{aligned}\|TG_k(\alpha_1,0)\|_\infty &= \|((A_k^\nu)^{-1} - P_k(A_{k-1}^\nu)^{-1}R_k)(A_k^\nu)S_k^{\alpha_1}\|_\infty, \\ &\le \|(A_k^\nu)^{-1} - P_k(A_{k-1}^\nu)^{-1}R_k\|_\infty \|(A_k^\nu)S_k^{\alpha_1}\|_\infty.\end{aligned}$$

The following theorem represents the main result of our work.

Theorem 5.4. *Under the previous assumptions, there is a constant C independent of k and α, such that the iterate u_k^ν, $\nu \ge 0$ for two grids k and $k-1$ satisfies:*

$$\|u_k^{\nu+1} - u_k^*\|_\infty \le \left(\frac{C}{\sqrt{\alpha}}|\log h_k|^2\right)\|u_k^\nu - u_k^*\|_\infty. \tag{5.9}$$

Proof. We have:

$$\begin{aligned}\|u_k^{\nu+1} - u_k^*\|_\infty &= \|((I_k - P_k(I_k - MG_{k-1})(A_{k-1}^\nu)^{-1}R_k)(A_k^\nu)S_k^{\alpha_1})(u_k^\nu - u_k^*)\|_\infty \\ &\le \|I_k - P_k(I_k - MG_{k-1})(A_{k-1}^\nu)^{-1}R_k\|_\infty \|(A_k^\nu)S_k^{\alpha_1}\|_\infty \|(u_k^\nu - u_k^*)\|_\infty \\ &\le \left(C_2\frac{1}{\sqrt{\alpha}}h_k^{-2}\right)(C_1h_k^2|\log h_k|^2)\|u_k^\nu - u_k^*\|_\infty \\ &\le \frac{C_1C_2}{\sqrt{\alpha}}|\log h_k|^2\|u_k^\nu - u_k^*\|_\infty. \qquad \square\end{aligned}$$

Usually, we will choose a hierarchy of more than two grids. In this case the iteration matrix (5.2) can be defined by recurrence using the iteration matrix (5.1) for all levels k. Therefore, if we suppose that (5.8) holds, then the L^∞-convergence result can be easily deduced from the preceding result.

Theorem 5.5 ([23]). *Consider the multigrid method with iteration matrix given in* (5.2). *Then under the previous assumptions and for parameter values* $\alpha_2 = 0$, $\alpha_1 = \alpha > 0$, $\tau = 2$.
For any $\zeta \in (0,1)$ *there exists a* α^* *such that for all* $\alpha \geq \alpha^*$

$$\|MG_k\|_\infty \leq \zeta, \quad k = 0,1,\dots \tag{5.10}$$

holds.

Proof. Combining by the approximation and smoothing property with (5.8), then we can apply the same arguments as in [23, Theorem 7.20]. □

6. Numerical Experiments

In this section, we introduce a numerical example of a quasi-variational inequality arising from a stochastic inventory problem with impulse control [20].

In order to apply this model to our example, we suppose that the data of our problem to be sufficiently smooth and apply the Bellman's principle of dynamic programming, we find that the minimal cost function u which is the solution of impulse control problem related to the QVI (2.3). Then, we solve (2.3) as we discussed before with the following data:

$$\begin{cases} Au \leq f, & \text{in } \Omega = \{(x_1,x_2) | x_1^2 + x_2^2 \leq 1\}, \\ \langle Au - f, u^\nu - k - u^{\nu-1} \rangle = 0, & \\ u^\nu \leq k + u^{\nu-1}, & k \geq 0, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \tag{6.1}$$

where

$$Au = -\Delta u,$$
$$f(x) = x_1 + x_2,$$
$$k = 0.01.$$

We are confine ourselves to the FEM discretization with a uniform triangulation and P1 nested finite element function spaces. For the discretization of the Domain, we have used the PDE toolbox in MATLAB (R2018a) to generate the mesh and then the multigrid FEM can be used to efficiently solve (6.1) as mentioned above.

This numerical example is reported to verify the high efficiency of the multigrid method. As a pre/post-smoothing we have chosen Gauss-Seidel method in the multigrid code. Regarding the recursion in multigrid method, we stop the recursion of the multigrid algorithm when the DOFs (the number of interior grid points) smaller than 5.

Figure 1 illustrates the convergence behavior of the multigrid solver (red and black curves for the maximum norm of the residual of multigrid (V and W_cycle)) with respect to the number of iterations performed. For comparison, the convergence behavior of Gauss-Seidel and damped Gauss-Seidel solvers ($\omega = 1.5$) (green and blue curves) are included. The multigrid V-cycle is carried out on the finest grid with 1045 nodes and 4 nodes on the coarsest one, and then we have applied Matlab-Operator solver, Gauss-Seidel, and damped Gauss-Seidel on this finest grid to get the solutions in Figures 2, 3, 4 and 5.

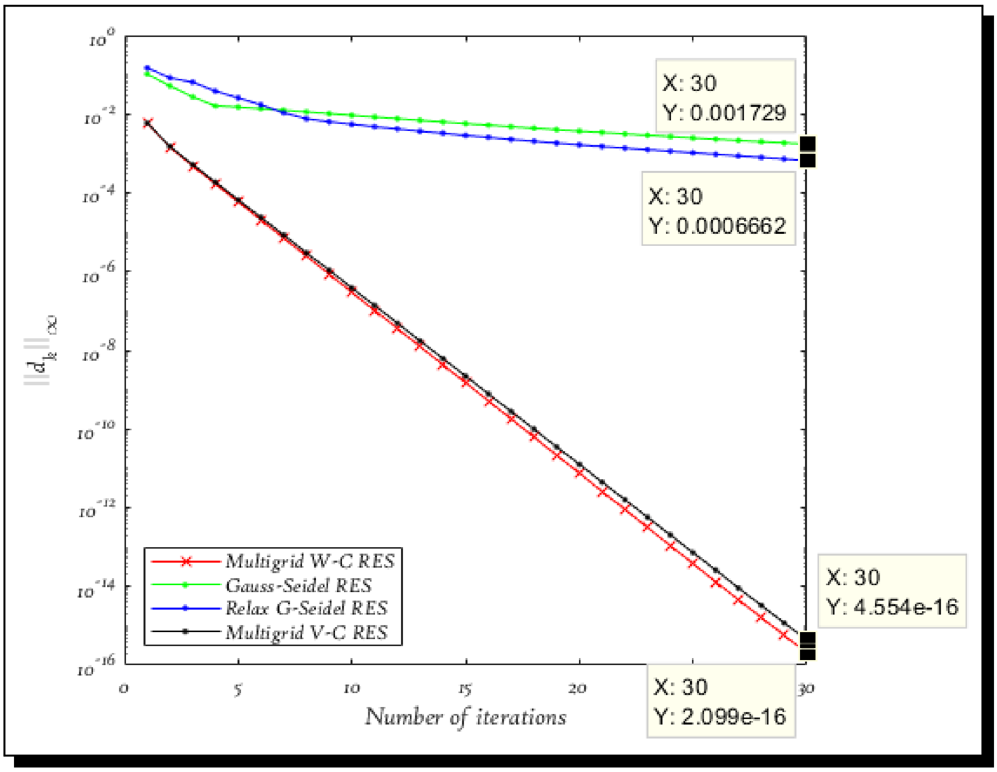

Figure 1. Comparison between the convergence behavior of Multigrid method, Gauss-Seidel and weighted Gauss-Seidel methods

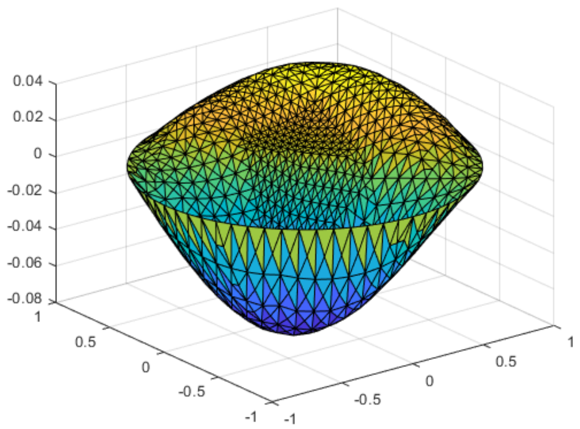

Figure 2. Solution of the problem (6.1) on a fine grid with 1045 DOFs after 30 iterations of multigrid method

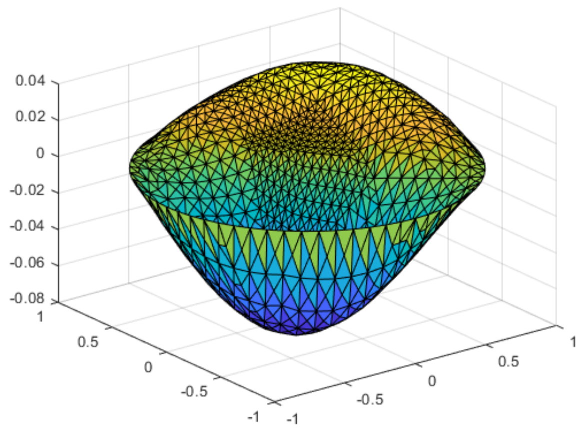

Figure 3. Solution of the problem (6.1) on a fine grid with 1045 DOFs using Matlab-Operator solver

6.1 Discussion

In this section, we have discussed the multigrid methods for solving the problem (6.1) in the finest grid with 225 interior grid points. We have chosen a start iterate $u_k^0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{225}$, given $u_k^\nu \in \mathbb{R}^{225}$ (we have chosen u_k^ν locally for which the maximum of the equation (4.1) is attained),

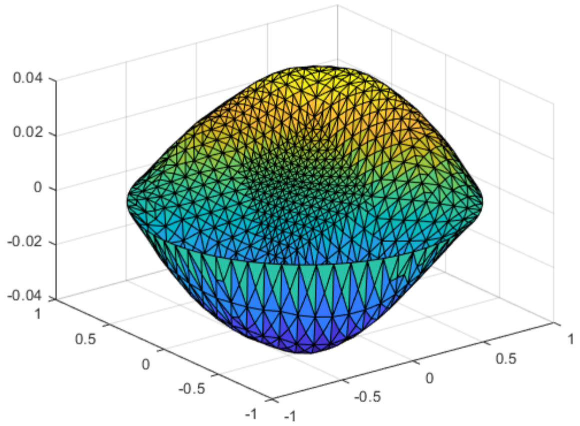

Figure 4. Solution of the problem (6.1) on a fine grid with 1045 DOFs after 30 iterations of a Gauss-Seidel method

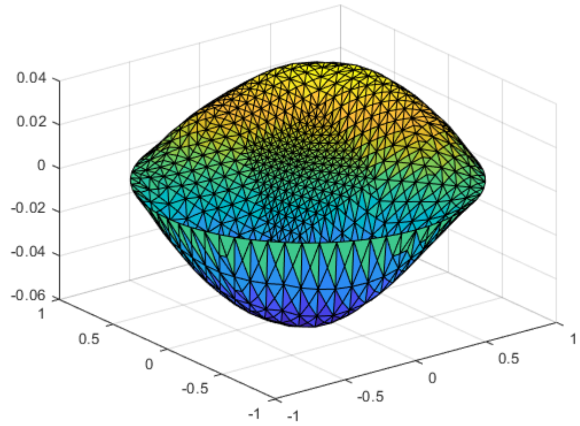

Figure 5. Solution of the problem (6.1) on a fine grid with 1045 DOFs after 30 iterations of a damped Gauss-Seidel method

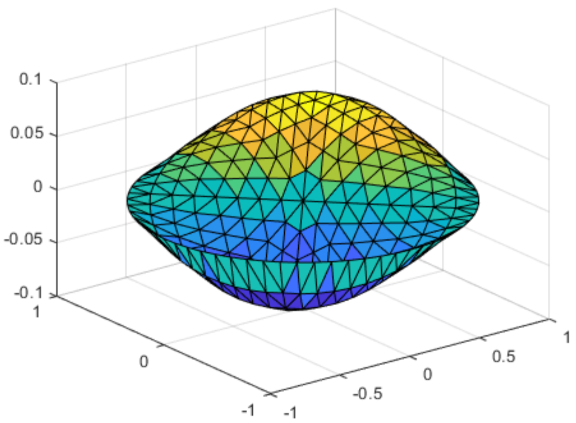

Figure 6. Solution of the equation (6.3) by 25 iterations of multigrid method with $k = 1$

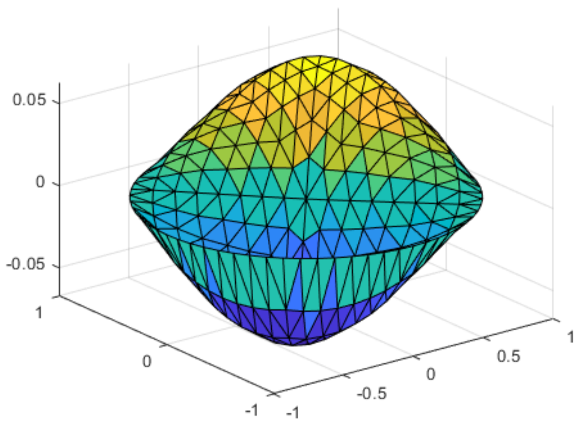

Figure 7. Solution of the equation (6.3) by 25 iterations of multigrid method with $k = 0.001$

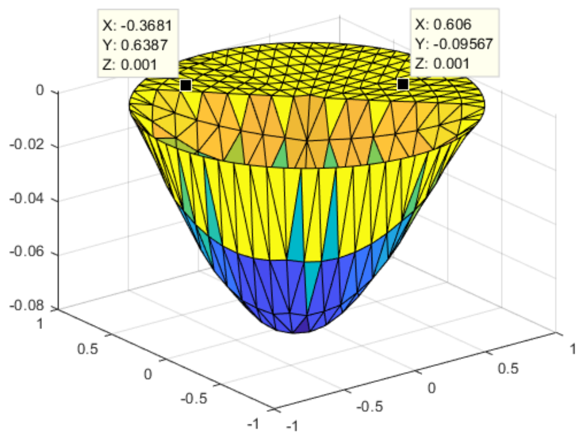

Figure 8. Solution of the VI (6.4) by 25 iterations of MGM with $k = 0.001$

compute $u_k^{\nu+1} \in \mathbb{R}^{225}$ as the solution of the equation

$$A_k^\nu u_k^{\nu+1} = f_k^\nu, \tag{6.2}$$

by multigrid method. Where A_k^ν and f_k^ν are defined in (4.4) and (4.5) with the data of the problem (6.1). The solution of (6.2) satisfies:

$$\max_{1\le i\le N}(A_{k,i}u_{k,i}^{\nu+1} - f_{k,i}, u_{k,i}^{\nu+1} - k - u_{k,i}^{\nu}) = \begin{cases} A_{k,i}u_{k,i}^{\nu+1} - f_{k,i}, & k \ge 0.02, \\ \max(A_{k,i}u_{k,i}^{\nu+1} - f_{k,i}, u_{k,i}^{\nu+1} - k - u_{k,i}^{\nu})_{0\le k\le 0.02}. \end{cases} \tag{6.3}$$

On the other hand, if we choose $u_k^{\nu-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{225}$, and $0 < k < 0.02$, the solution (8) of the problem (6.1) is the solution of the following variational Inequalities:

$$\begin{cases} Au \le f, & \text{in } \Omega = \{(x_1, x_2) \mid x_1^2 + x_2^2 \le 1\}, \\ \langle Au - f, u^\nu - k\rangle = 0, & \\ u^\nu \le k, & k = 0.001, \\ u = 0, & \text{in } \partial\Omega. \end{cases} \tag{6.4}$$

All codes were implemented in MATLAB and executed on a computer with Intel(R) Core(TM) i5-8265U CPU @ 1.60 GHz and 8 GB memory.

7. Conclusion

We have applied the algebraic multigrid methods for an efficient iterative solution of discretized elliptic quasi-variational inequalities. For the discretization, adaptive finite element approximation is used on a non-polygonal domain. After the discretization, we have explained a multigrid method for the solution of the discrete problem. We have established a uniform convergence of our problem and proved that the multigrid methods have a contraction number with respect to the maximum norm. In the numerical experiments, we have presented an example of a quasi-variational inequality arising from a stochastic inventory problem with impulse control. From these numerical results, we show that Gauss-Seidel and relaxed Gauss-Seidel are still not good even after a large number of iterations. Unlike the Multigrid method, which has a debugging feature (the high-frequency error is reduced by relaxation, while low-frequency error is mapped to coarse grids and reduced there), it converges in a few iterations.

Many extensions of the techniques given above are possible. An interesting future case is to apply parallel full multigrid methods for solving parabolic quasi-variational inequalities related to stochastic control problems and apply them to three-dimensional problems.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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