Communications in Mathematics and Applications
Vol. 14, No. 1, pp. 385–396, 2023
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v14i1.1970

http://www.rgnpublications.com

Research Article

Numerical Study of Layer Behaviour Differential-Difference Equations With Small Delay Arising in the Nerve Pulse Propagation

Kumar Ragula[1] and G.B.S.L. Soujanya*[2]

[1] *Department of Mathematics, Rajiv Gandhi University of Knowledge and Technologies, Basar, India*
[2] *Department of Mathematics, University College for Women, Kakatiya University, Warangal, India*
***Corresponding author:** gbslsoujanya@gmail.com

Received: February 26, 2022 **Accepted:** December 10, 2022

Abstract. In this study, we implement a numerical method to solve a singularly perturbed differential-difference equation with a small shift. Taylor series is used to deal with the small shift, and the given problem converted into a singularly perturbed boundary value problem. To solve this problem, a fourth order finite difference approach is used. The convergence of the method is investigated. The method is supported by the numerical results compared to the other method in the literature. Numerical experiments show how the small shift and perturbation parameter affects the boundary layer solution of the problem.

Keywords. Singularly perturbed differential-difference equation, Delay, Tridiagonal system, Truncation error

Mathematics Subject Classification (2020). 65L11, 65L12

1. Introduction

Delay differential equations with a miner dispersion parameter are common in engineering and environmental science applications for example in [10], the author deals with the model having layer behaviour with small shift that has bistable non-linearity and models electrical activity in neuron. These problems are well described in fluid flow at high Reynolds number

[13], [17], advection dominated heat and mass transfer, semiconductor device models [15], neuron variability [20], nerve pulse propagation [10] and in the study of travelling wave solutions [9]. Obtaining the solution of differential equation with a delay was a great task, but it was important because of the versatility of these equations in the mathematical modelling of processes in various application fields [2], [20]. Refer to [4], [6] for a complete theory of delay differential equations, also known as functional differential equations. In [3], [5], [7], [13], [14], [21], [17] the numerical solution to the problems of singular perturbations is very well described. In [1], Cryer derived a numerical scheme which uses finite differences to solve a second-order functional differential equations. Kadalbajoo and Sharma [8] introduced a numerical method for solving boundary layer problems having delay, which works well, when the delay argument is a larger one as well as a smaller one. The study by Swamy *et al.* [22] suggested a numerical integration for solving a delay differential equation with twin layers or oscillatory behavior. Lange and Miura [12] have addressed the problems that display layer behavior at one or both boundaries using Laplace transforms. In [11], Lange and Miura have investigated the problems having solutions which have turning point behavior. Phaneendra *et al.* [16] suggested a compact numerical higher order method for the solution of a boundary layer problem with a delay term. Trapezoidal integration rule was used by Swamy *et al.* [21] to solve a delay differential equation having dual layers or oscillatory structure. For the problems of delay differential equation with layer structure, Soujanya and Reddy [19] used Simpson's rule of integration to address the problems.

2. Description of the Problem

We consider the Dirichlet boundary value problem having a small delay of the form:

$$\varepsilon\omega''(\theta)+p(\theta)\omega'(\theta)+q(\theta)\omega(\theta-\delta)=f(\theta) \tag{1}$$

on $(0,1)$, under the boundary

$$\omega(\theta)=\varphi(\theta) \text{ on } \delta\le\theta\le 0,\ \omega(1)=\gamma \tag{2}$$

where the functions $p(\theta)$, $q(\theta)$ and $\varphi(\theta)$ are smooth, ε $(0<\varepsilon\ll 1)$ is a small perturbation parameter and a delay parameter δ $(0<\delta\ll 1)$ is of $o(\varepsilon)$ satisfying the condition $q(\theta)\le 0$, $\forall$ $\theta\in(0,1)$.

Since the solution $\omega(\theta)$ of the problem (1) with (2) is sufficiently differentiable, expanding the term $\omega(\theta-\delta)$ by Taylor series, we obtain

$$\omega(\theta-\delta)\approx\omega(\theta)-\delta\omega'(\theta)+O(\delta^2). \tag{3}$$

Using (3) in (1), we get an equivalent second order singular perturbation problem

$$\varepsilon\omega''(\theta)+a(\theta)\omega'(\theta)+q(\theta)\omega(\theta)=f(\theta). \tag{4}$$

Here,

$$a(\theta)=p(\theta)-\delta q(\theta). \tag{5}$$

3. Numerical Method

We divide the interval $[0,1]$ into N equal subintervals of mesh size $h = 1/N$ so that the mesh points are $\theta_i = ih$ for $i = 0,1,2,\dots,N$.

At $\theta = \theta_i$, (4) becomes

$$\varepsilon\omega_i'' + a_i\omega_i'' + q_i\omega_i = f_i. \tag{6}$$

We now rewrite the central difference formula for ω_i' and ω_i'' in new form as given below:

$$\omega_i'' \cong D^+D^-\omega_i - \frac{h^2}{12}\omega_i^{(4)} + R_1, \tag{7}$$

$$\omega_i'' = D^{\pm}\omega_i - \frac{h^2}{6}\omega_i''' + R_2, \tag{8}$$

where

$$D^+D^-\omega_i = \frac{\omega_{i-1} - 2\omega_i + \omega_{i+1}}{h^2}, \quad D^{\pm}\omega_i = \frac{\omega_{i+1} - \omega_{i-1}}{2h}, \quad R_1 = -\frac{2h^4\omega^{(6)}(\xi)}{6!},$$

$$R_2 = -\frac{h^4\omega^{(5)}(\eta)}{5!} \quad \text{for } \xi,\eta \in [\theta_{i-1},\theta_{i+1}].$$

From the equation (6), we obtain ω_i''', $\omega_i^{(4)}$ as

$$\omega_i''' = \left[-\frac{a_i}{\varepsilon}\omega_i'' - \frac{(a_i' + q_i)}{\varepsilon}\omega_i'' - \frac{q_i'}{\varepsilon}\omega_i + \frac{f'}{\varepsilon}\right],$$

$$\omega_i^{(4)} = \left[\frac{a_i{}^2}{\varepsilon^2} - \frac{(2a_i' + q_i)}{\varepsilon}\right]\omega_i'' + \left[\frac{a_i(a_i' + q_i)}{\varepsilon^2} - \frac{(a_i'' + 2q_i')}{\varepsilon}\right]\omega_i'' + \left[\frac{aq_i'}{\varepsilon^2} - \frac{q_i''}{\varepsilon}\right]\omega_i + \frac{1}{\varepsilon}f_i''.$$

Using these derivatives and substituting (7), (8) in (6), we get

$$\begin{aligned}
&\varepsilon\left\{\left[1 - \frac{h^2a_i^2}{12\varepsilon^2} + \frac{h^2(2a_i' + q_i)}{12\varepsilon}\right]\left(\frac{\omega_{i-1} - 2\omega_i + \omega_{i+1}}{h^2}\right) + \left[\frac{h^2(a_i'' + 2q_i')}{12\varepsilon} - \frac{h^2a_i(a_i' + q_i)}{12\varepsilon}\right]\right.\\
&\qquad\left.\cdot\frac{(\omega_{i+1} - \omega_{i-1})}{2h} - \left[\frac{h^2q_i''}{12\varepsilon} - \frac{a_iq_i'h^2}{12\varepsilon^2}\right]\omega_i - \frac{h^2}{12\varepsilon}f_i''\right\}\\
&\quad + a_i\left[\frac{a_ih^2}{6\varepsilon}\left(\frac{\omega_{i-1} - 2\omega_i + \omega_{i+1}}{h^2}\right) + \left(1 + \frac{h^2}{6\varepsilon}(a_i' + q_i)\right)\frac{(\omega_{i+1} - \omega_{i-1})}{2h} + \frac{h^2}{6\varepsilon}q_i'\omega_i - \frac{h^2f_i'}{6\varepsilon}\right] + q_i\omega_i\\
&= f_i.
\end{aligned}$$

Now introducing a fitting factor σ in the above finite difference scheme, we get

$$\begin{aligned}
&\sigma\varepsilon\left\{\left[1 - \frac{h^2a_i^2}{12\varepsilon^2} + \frac{h^2\left(2a_i' + q_i\right)}{12\varepsilon}\right]\left(\frac{\omega_{i-1} - 2\omega_i + \omega_{i+1}}{h^2}\right) + \left[\frac{h^2(a_i'' + 2q_i')}{12\varepsilon} - \frac{h^2a_i(a_i' + q_i)}{12\varepsilon}\right]\right.\\
&\qquad\left.\cdot\frac{(\omega_{i+1} - \omega_{i-1})}{2h} - \left[\frac{h^2q_i''}{12\varepsilon} - \frac{a_iq_i'h^2}{12\varepsilon^2}\right]\omega_i - \frac{h^2}{12\varepsilon}f_i''\right\}\\
&\quad + a_i\left[\frac{a_ih^2}{6\varepsilon}\left(\frac{\omega_{i-1} - 2\omega_i + \omega_{i+1}}{h^2}\right) + \left(1 + \frac{h^2}{6\varepsilon}(a_i' + q_i)\right)\frac{(\omega_{i+1} - \omega_{i-1})}{2h} + \frac{h^2}{6\varepsilon}q_i'\omega_i - \frac{h^2f_i'}{6\varepsilon}\right] + q_i\omega_i\\
&= f_i.
\end{aligned} \tag{9}$$

To find the value of the fitting factor, we have used the asymptotic solution of (4) (O'Malley [14]) given by

$$\omega(\theta)\approx\omega_n(\theta)+\frac{a(0)}{a(\theta)}(\varphi(0)-\omega_0(0)\exp\left\{-\int_0^\theta\left(\frac{a(\theta)}{\varepsilon}\right)d\theta\right\}.$$

Hence, at the mesh points we have

$$\omega(\theta_i)\approx\omega_0(\theta)+(\varphi(0)-\omega_0(0))\exp\left\{-\left(\frac{a(0)}{\varepsilon}\right)\theta_i\right\},\quad i=0,1,2,\dots,N$$

i.e.,

$$\omega(ih)\approx\omega_0(ih)+(\varphi(0)-\omega_0(0))\exp\left\{-\left(\frac{a(0)}{\varepsilon}\right)ih\right\}.$$

Therefore,

$$\lim_{h\to 0}\omega(ih)\approx\omega_0(0)+(\varphi(0)-\omega_0(0)\exp-a(0)i\rho. \tag{10}$$

Multiplying (9) by h, taking the limit $h\to 0$ and using the procedure given in [3], we get the fitting factor as

$$\sigma=\frac{a(0)}{2}\left(\frac{\coth\left(\frac{a(0)\rho}{2}\right)-\frac{\rho a^2(0)}{3}}{\left(\frac{1}{\rho}-\frac{\rho a^2(0)}{12}\right)}\right).$$

We rewrite the system of equations (9) in tridiagonal system of equations as

$$E_i\omega_{i-1}-F_i\omega_i+G_i\omega_{i+1}=H_i,\quad i=1,2,\dots,N-1, \tag{11}$$

where

$$E_i=\frac{\sigma\varepsilon}{h^2}-\frac{\sigma {a_i}^2}{12\varepsilon}+\frac{\sigma(2a_i'+q_i)}{12}+\frac{\sigma {a_i}^2}{6\varepsilon}-\frac{\sigma h}{24}(a_i''+2q_i')+\frac{\sigma h a_i(a_i'+q_i)}{24\varepsilon}-\frac{a_i}{2h}\left(1+\frac{h^2}{6\varepsilon}(a_i'+q_i)\right),$$

$$F_i=\frac{2\sigma {a_i}^2}{12\varepsilon}-\frac{2\sigma\varepsilon}{h^2}-\frac{2\sigma(2a_i'+q_i)}{12}-\frac{2\sigma {a_i}^2}{6}+\frac{\sigma h^2 q_i''}{12}-\frac{\sigma h^2 a_i q_i'}{12\varepsilon}+\frac{h^2 {a_i}^2 q_i'}{6\varepsilon}+q_i,$$

$$G_i=\frac{\sigma\varepsilon}{h^2}-\frac{\sigma {a_i}^2}{12\varepsilon}+\frac{\sigma(2a_i'+q_i)}{12}+\frac{{a_i}^2}{6\varepsilon}+\frac{\sigma h}{24}(a_i''+2q_i')-\frac{\sigma h a_i(a_i'+q_i)}{24\varepsilon}+\frac{a_i}{2h}\left(1+\frac{h^2}{6\varepsilon}(a_i'+q_i)\right),$$

$$H_i=\frac{\sigma\varepsilon h^2}{12\varepsilon}f_i''+\frac{a_i h^2}{6\varepsilon}f_i'+f_i.$$

We solve the tridiagonal system (11) by using the Thomas algorithm.

4. Convergence Analysis

Writing the tridiagonal system (11) in matrix-vector form, we get

$$AW=C \tag{12}$$

in which $A=(m_{ij})$, $1\le i,\, j\le N-1$ is a tridiagonal matrix of order $N-1$ with

$$m_{i\ i+1}=\frac{\sigma\varepsilon}{h^2}-\frac{\sigma {a_i}^2}{12\varepsilon}+\frac{\sigma(2a_i'+q_i)}{12}+\frac{{a_i}^2}{6\varepsilon}+\frac{\sigma h}{24}(a_i''+2q_i')-\frac{\sigma h a_i(a_i'+q_i)}{24\varepsilon}+\frac{a_i}{2h}\left(1+\frac{h^2}{6\varepsilon}(a_i'+q_i)\right),$$

$$m_{i\ i}=\frac{2\sigma {a_i}^2}{12\varepsilon}-\frac{2\sigma\varepsilon}{h^2}-\frac{2\sigma(2a_i'+q_i)}{12}-\frac{2\sigma {a_i}^2}{6}+\frac{\sigma h^2 q_i''}{12}-\frac{\sigma h^2 a_i q_i'}{12\varepsilon}+\frac{h^2 {a_i}^2 q_i'}{6\varepsilon}+q_i,$$

$$m_{i\ i-1} = \frac{\sigma\varepsilon}{h^2} - \frac{\sigma a_i{}^2}{12\varepsilon} + \frac{\sigma(2a_i' + q_i)}{12} + \frac{\sigma a_i{}^2}{6\varepsilon} - \frac{\sigma h}{24}(a_i'' + 2q_i') + \frac{\sigma h a_i(a_i' + q_i)}{24\varepsilon} - \frac{a_i}{2h}\left(1 + \frac{h^2}{6\varepsilon}(a_i' + q_i)\right),$$

and $C = (d_i)$ is a column vector with $d_i = \frac{\sigma\varepsilon h^2}{12\varepsilon} f_i'' + \frac{a_i h^2}{6\varepsilon} f_i' + f_i$ where $i = 1,2,\dots,N-1$ with local truncation error

$$|\tau_i(h)| \le \max_{\theta_{i-1}\le\theta\le\theta_{i+1}} \left\{ \frac{h^4 a(\theta_i)}{5!} |\omega^{(5)}(\theta_i)| \right\} + \max_{\theta_{i-1}\le\theta\le\theta_{i+1}} \left\{ \frac{2h^4\varepsilon}{6!} |\omega^{(6)}(\theta_i)| \right\},$$

i.e.,

$$|\tau_i(h)| \le o(h^4), \tag{13}$$

and $W = (\omega_0, \omega_1, \omega_2, \dots, \omega_N)^t$.

We also have

$$A\bar{W} - T_i(h) = C. \tag{14}$$

Let $\bar{W} = (\bar{\omega}_0, \bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_N)^t$ denotes the actual solution and the local truncation error be

$$T_i(h) = (T_0(h), T_1(h), \dots, T_N(h))^t.$$

From (12) and (14), we get

$$A(\bar{W} - W) = T_i(h). \tag{15}$$

Hence the equation of the error is

$$AE = T_i(h), \tag{16}$$

where $E = \bar{W} - W = (e_0, e_1, e_2, \dots, e_N)^t$.

Let the sum of the elements of ith row of the matrix be S_i, then we have

$$S_i = \sum_{j=1}^{N-1} m_{i\ j} = -\frac{\sigma\varepsilon}{h^2} + \frac{\sigma a_i{}^2}{12\varepsilon} - \frac{\sigma(2a_i' + q_i)}{12} - \frac{a_i{}^2}{6\varepsilon} + \frac{a_i h^2 q_i''}{12} - \frac{\sigma h^2 a_i q_i{}'}{12\varepsilon} + \frac{a_i{}^2 h^2 q_i{}'}{6\varepsilon} + q_i$$
$$+ \frac{\sigma h a_i''}{24\varepsilon} + \frac{\sigma h q_i'}{12} - \frac{\sigma h a_i a_i'}{24\varepsilon}\frac{\sigma h a_i q_i}{24\varepsilon} + \frac{a_i}{2h}\left(1 + \frac{h^2}{6\varepsilon}(a_i' + q_i)\right), \quad \text{for } i = 1\ ;$$

$$S_i = \sum_{j=1}^{N-1} m_{i\ j} = q_i - \frac{\sigma h^2 q_i''}{12} - \frac{\sigma a_i q_i' h^2}{12\varepsilon} + \frac{a_i h^2 q_i'}{6\varepsilon} = q_i + o\left(h^2\right) = B_{i_0}, \quad \text{for } i = 2,3,\dots,N-2\ ;$$

$$S_i = \sum_{j=1}^{N-1} m_{i\ j} = \frac{\sigma\varepsilon}{h^2} + \frac{\sigma a_i{}^2}{12\varepsilon} - \frac{\sigma(2a_i' + q_i)}{12} - \frac{a_i{}^2}{\varepsilon} - \frac{\sigma h(a_i'' + 2q_i')}{24} + \frac{\sigma h a_i(a_i' + q_i)}{24\varepsilon}$$
$$- \frac{a_i}{2h}\left(1 + \frac{h^2}{6\varepsilon}(a_i' + q_i)\right) + \frac{a_i h^2 q_i''}{12} - \frac{\sigma h^2 a_i q_i{}'}{12\varepsilon} + \frac{a_i{}^2 h^2 q_i{}'}{6\varepsilon} + q_i, \quad \text{for } i = N-1.$$

By choosing h sufficiently small and the matrix A is irreducible and monotone. Hence A^{-1} exists and it has non-negative elements.

Hence from (16), we get

$$E = A^{-1} T(h). \tag{17}$$

Also, from the *matrix theory* [23], we have

$$\sum_{i=1}^{N-1} \bar{m}_{k,i} S_i = 1, \quad k = 1,2,\dots,N-1, \tag{18}$$

where $\bar{m}_{k,i}$ is (k,i)th element of A^{-1} for some i_0 between 1 and $N-1$.

Therefore,

$$\sum_{i=1}^{N-1}\bar{m}_{k,i} \le \frac{1}{\min\limits_{1\le i\le N-1} S_i} = \frac{1}{B_{i_0}} \le \frac{1}{|B_{i_0}|}. \tag{19}$$

From (17), (19) and (13), we get

$$e_j = \sum_{i=1}^{N-1}\bar{m}_{k,i}T_i(h), \quad j = 1,2,\dots,N-1,$$

gives

$$e_j \le \frac{o(h^4)}{|B_{i_0}|}, \quad j = 1,2,\dots,N-1, \tag{20}$$

where $B_{i_0} = q_i$. Hence,

$$\|E\| = o(h^4),$$

i.e., our method reduces to a fourth order convergent on uniform mesh.

5. Numerical Examples

In this section, numerical examples in [18] are considered and solved by using the proposed method to illustrate the method with comparison.

Example 5.1. $\varepsilon\omega''(\theta)+(5-\delta)\omega'(\theta)+\omega(\theta) = 0$, under the conditions $\omega(\theta) = 0$, $-\delta \le \theta \le 0$, and $\omega(1) = 0$. The exact solution is

$$\omega(\theta) = \frac{-\exp(m_2)}{\exp(m_1)-\exp(m_2)}\exp(m_1\theta)+\frac{\exp(m_1)}{\exp(m_1)-\exp(m_2)}\exp(m_2\theta),$$

where

$$m_1 = \frac{-(5-\delta)+\sqrt{(5-\delta)^2-4\varepsilon}}{2\varepsilon}, \quad m_2 = \frac{-(5-\delta)-\sqrt{(5-\delta)^2-4\varepsilon}}{2\varepsilon}.$$

The numerical results are given in Tables 1-4 for different values of ε and the delay parameter δ. The effect of the small parameters on the boundary layer solution is shown in Figure 1.

Table 1. The pointwise errors in solution of Example 5.1 with $\varepsilon = 2^{-1}$ and $\delta = 0.1$

x	Pointwise errors by proposed scheme	Pointwise errors in [18]
0.1	2.34276674322409e-05	0.00789256273622
0.2	1.79370051162364e-05	0.00597781073993
0.3	1.02889531937433e-05	0.00339087970810
0.4	5.23403548477464e-06	0.00170461852578
0.5	2.48333510478697e-06	0.00079820716373
0.6	1.11781728659574e-06	0.00035378301643
0.7	4.34823856485531e-07	0.00014761503788
0.8	1.84166215681108e-07	0.00005571374275
0.9	5.54976747040042e-08	0.00001620803035

Table 2. The pointwise errors in solution of Example 5.1 with $\varepsilon = 2^{-2}$ and $\delta = 0.1$

x	Pointwise errors by proposed scheme	Pointwise errors in [18]
0.1	7.03861177466236e-05	0.00293539993187
0.2	2.02372077292082e-05	0.00083556718055
0.3	4.36389327890772e-06	0.00017838987890
0.4	8.36446313388960e-07	0.00003385416200
0.5	1.50291267231407e-07	0.00000602265390
0.6	2.59102855752184e-08	0.00000102792407
0.7	4.32935707189275e-09	0.00000016993948
0.8	6.95284489878321e-10	0.00000002693831
0.9	9.67151142240731e-11	0.00000000366954

Table 3. The pointwise errors in solution of Example 5.1 with $\varepsilon = 2^{-3}$ and $\delta = 0.1$

x	Pointwise errors by proposed scheme	Pointwise errors in [18]
0.1	7.92315428036830e-05	0.41125475328994E-3
0.2	3.20297215482517e-06	0.01648861952029E-3
0.3	9.71113592932933e-08	0.00049583154944E-3
0.4	2.61718809812781e-09	0.00001325399262E-3
0.5	6.61259112043285e-11	0.00000033215881E-3
0.6	1.60391026517306e-12	0.00000000799154E-3
0.7	3.78224253237220e-14	0.00000000018693E-3
0.8	8.73260896097546e-16	0.00000000000427E-3
0.9	1.94126488931012e-17	0.00000000000008E-3

Table 4. The pointwise errors in solution of Example 5.1 with $\varepsilon = 2^{-3}$ and $\delta = 0.01$

x	Pointwise errors by proposed scheme	Pointwise errors in [18]
0.1	2.48556607754247e-05	0.00760653663041
0.2	1.86857358309978e-05	0.00565825860890
0.3	1.05253819761078e-05	0.00315267951757
0.4	5.25880613667581e-06	0.00155710568811
0.5	2.45136706858613e-06	0.00071663012448
0.6	1.08469246605904e-06	0.00031237503064
0.7	4.54033257805356e-07	0.00012830756910
0.8	1.73248267172079e-07	0.00004773960348
0.9	5.15400928579728e-08	0.00001371682818

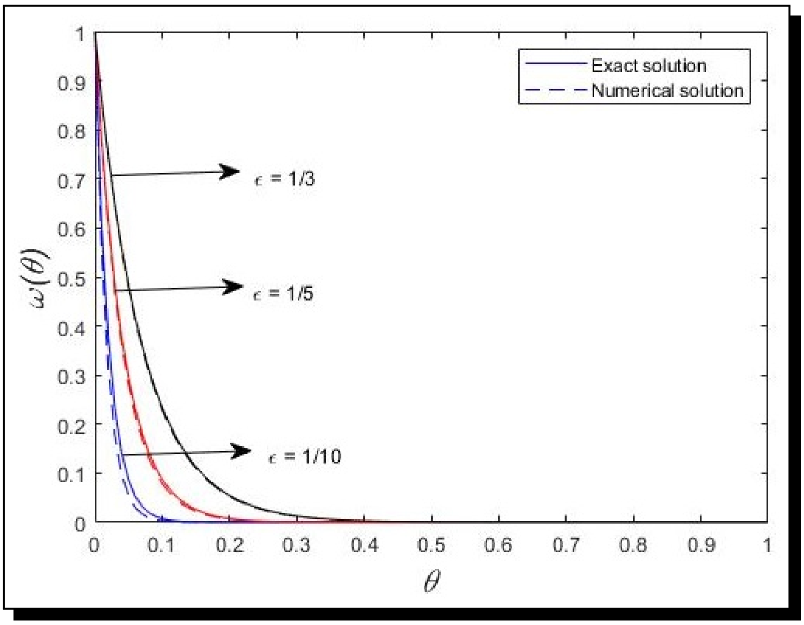

Figure 1. Layer behaviour in the solution of Example 5.1 when $\delta = 0.1$

Example 5.2. $\varepsilon\omega''(\theta) + (-5 - 2\delta)\omega'(\theta) + \omega(\theta) = 0$, under the conditions $\omega(\theta) = 0$, $-\delta \le \theta \le 0$, and $\omega(1) = 2$. The exact solution is

$$\omega(\theta) = \frac{2\exp(-m_1 - m_2)}{\exp(-m_2) - \exp(-m_1)} \exp(m_1\theta) - \frac{2\exp(-m_1 - m_2)}{\exp(-m_2) - \exp(-m_1)} \exp(m_2\theta),$$

where

$$m_1 = \frac{(5+2\delta) + \sqrt{(5+2\delta)^2 - 8\varepsilon}}{2\varepsilon}, \quad m_2 = \frac{(5+2\delta) - \sqrt{(5+2\delta)^2 - 8\varepsilon}}{2\varepsilon}.$$

Numerical results are shown in Tables 5-8 for different values of ε and different δ values. Figure 2 shows the effect of the small parameters on the solution of the boundary layer.

Table 5. The pointwise errors in solution of Example 5.2 with $\varepsilon = 0.1$ and $\delta = 0.1$

x	Pointwise errors by proposed scheme	Pointwise errors in [18]
0.1	1.86716421381654e-20	0.00000000000E-3
0.2	2.75764902651433e-18	0.00000000000E-3
0.3	3.98453860732086e-16	0.00000000000E-3
0.4	5.64385974019190e-14	0.00000000001E-3
0.5	7.77816329414833e-12	0.00000000218E-3
0.6	1.02988364365800e-09	0.00000031055E-3
0.7	1.27941374043098e-07	0.00004139239E-3
0.8	1.41391380242847e-05	0.00490467496E-3
0.9	0.00117283934264765	0.43592874661E-3

Table 6. The pointwise errors in solution of Example 5.2 with $\varepsilon = 2^{-2}$ and $\delta = 0.1$

x	Pointwise errors by proposed scheme	Pointwise errors in [18]
0.1	1.29572104025593e-09	0.00000001297418
0.2	1.02297192312132e-08	0.00000011682562
0.3	7.05347520713228e-08	0.00000094810121
0.4	4.68634930941959e-07	0.00000760202085
0.5	3.01964146555012e-06	0.00006086311380
0.6	1.86716397377545e-05	0.00048718987663
0.7	0.000108232252272788	0.00389970919703
0.8	0.000557669055780544	0.03121511401660
0.9	0.00215506026671314	0.24986042277630

Table 7. The pointwise errors in solution of Example 5.2 with $\varepsilon = 2^{-3}$ and $\delta = 0.1$

x	Pointwise errors by proposed scheme	Pointwise errors in [18]
0.1	1.12244319913553e-20	0.00000000000000
0.2	6.26442823218817e-19	0.00000000000000
0.3	3.37945663290366e-17	0.00000000000014
0.4	1.78529024568573e-15	0.00000000000758
0.5	9.16924267614167e-14	0.00000000039683
0.6	4.52094557399249e-12	0.00000001993824
0.7	2.08975977696740e-10	0.00000093927305
0.8	8.58639632431071e-09	0.00003933677906
0.9	1.07997080078803e-06	0.00123572413410

Table 8. The pointwise errors in solution of Example 5.2 with $\varepsilon = 2^{-3}$ and $\delta = 0.01$

x	Pointwise errors by proposed scheme	Pointwise errors in [18]
0.1	3.72302281149103e-20	0.00000000000000
0.2	1.80185182025770e-18	0.00000000000001
0.3	8.40582475489440e-17	0.00000000000040
0.4	3.83962299775995e-15	0.00000000001873
0.5	1.70513182628286e-13	0.00000000084819
0.6	7.26939837251670e-12	0.00000003687295
0.7	2.90542890166848e-10	0.00000150297543
0.8	1.03221344775620e-08	0.00005446304225
0.9	2.75036312731569e-07	0.00148037509233

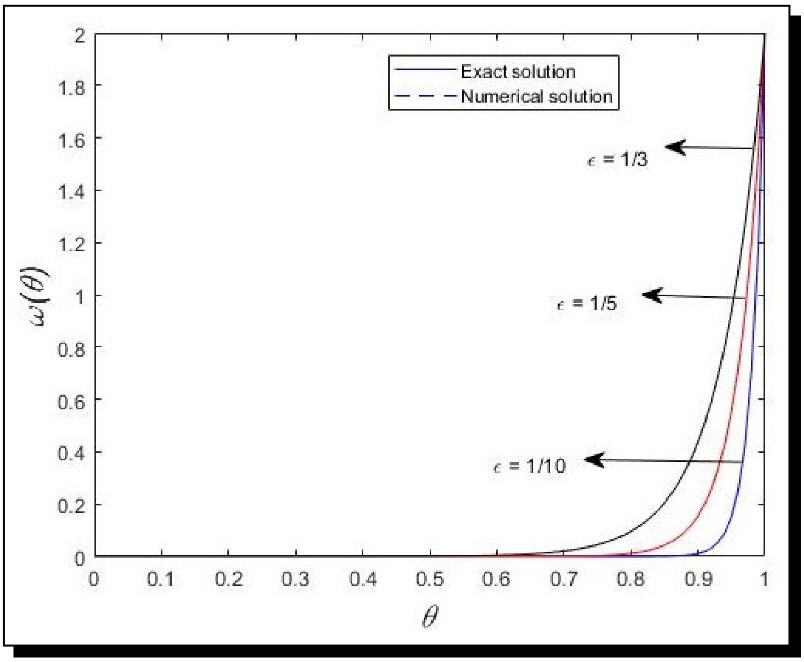

Figure 2. Layer behaviour in the solution of Example 5.2 when $\delta = 0.1$

6. Discussions and Conclusion

A layer behavior differential-difference equations with a delay parameter is solved by using a higher order finite difference method involving a fitting parameter. Two examples from [18] were chosen and solved for various values of the delay and perturbation parameter to demonstrate the applicability of the method. Even though the computational results are computed at all points of the mesh size, only a few results have been reported. The approximate solution have been compared with the exact solution and point wise errors are presented in Tables 1-8 with the comparison of the results given in [18]. The impact of the delay on the solution in the left and right boundary layer is negligible when the value of delay is increased. In Figure 1, the effect of the perturbation on the left boundary layer is shown. It is noticed that, as ε decreases, the width of the left layer also decreases. In Figure 2, we show the impact of the perturbation on the right boundary layer. It is observed that as ε decreases, the width of the right layer also decreases. Moreover, the proposed method is a simple and efficient technique for solving singularly perturbed boundary value problems.

Acknowledgement

The authors are thankful to the reviewers for their constructive comments/suggestions to improve the standards of the manuscript.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] C. W. Cryer, The numerical solution of boundary value problems for second order functional differential equations by finite differences, *Numerische Mathematik* **20** (1972), 288 – 289, DOI: 10.1007/BF01407371.

[2] M. W. Derstine, H. M. Gibbs, F. A. Hopf and D. L. Kaplan, Bifurcation gap in a hybrid optically bistable system, *Physical Review A* **26** (1982), 3720 – 3722, DOI: 10.1103/PhysRevA.26.3720.

[3] E. P. Doolan, J. J. H. Miller and W. H. A. Schilders, *Uniform Numerical Methods for Problems With Initial and Boundary Layer*, Boole Press, Dublin (1980), DOI: 10.1002/nme.1620180814.

[4] M. M. M. A. Feldstein, *Discretization Methods for Retarded Ordinary Differential Equations*, Doctoral thesis, University of California, Los Angeles (1964).

[5] E. C. Gartland, Uniform high-order difference schemes for a singularly perturbed two-point boundary value problem, *Mathematics of Computation* **48** (1987), 551 – 564, DOI: 10.1090/S0025-5718-1987-0878690-0.

[6] J. K. Hale, Functional differential equations, in: *Analytic Theory of Differential Equations*, P. F. Hsieh and A. W. J. Stoddart (editors), Lecture Notes in Mathematics, Vol. **183**, Springer, Berlin, Heidelberg (1971), DOI: 10.1007/BFb0060406.

[7] P. W. Hemker, *A Numerical Study to Stiff Two-Point Boundary Problems*, Mathematisch Centrum, Amsterdam, ix + 178 pages (1977).

[8] M. K. Kadalbajoo and K. K. Sharma, A numerical method on finite difference for boundary value problems for singularly perturbed delay differential equations, *Applied Mathematics and Computation* **197** (2008), 692 – 707, DOI: 10.1016/j.amc.2007.08.089.

[9] A. Kaushik, Nonstandard perturbation approximation and travelling wave solutions of nonlinear reaction diffusion equations, *Numerical Methods for Partial Differential Equations* **24**(1) (2007), 217 – 238, DOI: 10.1002/num.20244.

[10] A. Kaushik, Singular perturbation analysis of bistable differential equation arising in the nerve pulse propagation, *Nonlinear Analysis: Real World Applications* **9**(5) (2008), 2106 – 2127, DOI: 10.1016/j.nonrwa.2007.06.014.

[11] C. G. Lange and R. M. Miura, Singular perturbation analysis of boundary value problems for differential-difference equations. V. small shifts with layer behavior, *SIAM Journal of Applied Mathematics* **54**(1) (1994), 249 – 272, DOI: 10.1137/S0036139992228120.

[12] C. G. Lange and R. M. Miura, Singular perturbation analysis of boundary value problems for differential-difference equations, *SIAM Journal of Applied Mathematics* **42**(3) (1982), 502 – 531, DOI: 10.1137/0142036.

[13] J. J. H. Miller, E. O'Riordan and G. I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific Publishing Ltd. (2012), DOI: 10.1142/8410.

[14] R. E. O'Malley, *Introduction to Singular Perturbations*, 1st edition, Academic Press, New York, 214 pages (1974).

[15] R. O. O'Malley, *Singular Perturbation Methods for Ordinary Differential Equations*, Applied Mathematical Sciences series **89**, Springer-Verlag, viii + 225, Berlin (1991), DOI: 10.1002/zamm.19950750121.

[16] K. Phaneendra, Y. N. Reddy and G. B. S. L. Soujanya, A seventh order numerical method for singular perturbed differential-difference equations with negative shift, *Nonlinear Analysis: Modelling and Control* **16**(2) (2011), 206 – 219, DOI: 10.15388/NA.16.2.14106.

[17] H.-G. Roos, M. Stynes and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations: Convection-Diffusion and Flow Problems*, Springer Series in Computational Mathematics (SSCM, Vol. **24**), (1996), DOI: 10.1007/978-3-662-03206-0.

[18] M. Sharma, A. Kaushik and C. Li, Analytic approximation to delayed convection dominated systems through transforms, *Journal of Mathematical Chemistry* **52** (2014), 2459 – 2474, DOI: 10.1007/s10910-014-0394-1.

[19] G. B. S. L. Soujanya and Y. N. Reddy, Computational method for singularly perturbed delay differential equations with layer or oscillatory behaviour, *Applied Mathematics & Information Sciences* **10** (2016), 527 – 536, DOI: 10.18576/amis/100214.

[20] R. B. Stein, Some models of neuronal variability, *Biophysical Journal* **7**(1) (1967), 37 – 68, DOI: 10.1016/S0006-3495(67)86574-3.

[21] D. K. Swamy, K. Phaneendra and Y. N. Reddy, Accurate numerical method for singularly perturbed differential-difference equations with mixed shifts, *Khayyam Journal of Mathematics* **4**(2) (2018), 110 – 122, DOI: 10.22034/kjm.2018.57949.

[22] D. K. Swamy, K. Phaneendra, A. B. Babu and Y. N. Reddy, Computational method for singularly perturbed delay differential equations with twin layers or oscillatory behaviour, *Ain Shams Engineering Journal* **6**(1) (2015), 391 – 398, DOI: 10.1016/j.asej.2014.10.004.

[23] R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, NJ (1962), DOI: 10.1007/978-3-642-05156-2.