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Research Article

The Arrow Domination of Some Generalized Graphs

Dipshi* and Seema Mehra

Department of Mathematics, Maharshi Dayanand University, Rohtak 124001, Haryana, India
***Corresponding author:** dipshikhanagwal.333@gmail.com

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Abstract. The aim of this article is to apply the concept of arrow domination defined by Radhi *et al*. (The arrow domination in graphs, *International Journal of Nonlinear Analysis and Applications* **12**(1) (2021), 473 – 480) on some generalized graphs like Friendship graph or Fan graph, Gear graph, Helm graph, Flower graph, Sunflower graph, Triangular snake graph, Double triangular snake graph, Petersen graph, Dragon graph, Lollipop graph and Barbell graph.

Keywords. Domination, Arrow domination, Arrow domination number, Arrow dominating set

Mathematics Subject Classification (2020). 05C69

1. Introduction

Let G be an undirected graph which does not contain loops and parallel edges. Let $V(G)$ and $E(G)$ represent the set of the graph's vertices and edges, respectively. Let $D \subseteq V$ such that every vertex of $V-D$ is in open neighbourhood of at least one vertex of D then D is said to be a dominating set of G. Let D be a smallest minimal dominating set of a graph G then the cardinality of D is called the domination number of graph G, which is represented by $\gamma(G)$ [4]. Let $w \in V$ and $S \subseteq V$ then S is said to be dominated by vertex w if all vertices of S are in open neighbourhood of w. In 1958, Berge [1] was first to establish the concept of domination.

Let D be a dominating set of a graph G then D is called an arrow dominating set of G if $|N(d) \cap (V-D)| = r$ and $|N(d) \cap D| \geq s \ \forall \ d \in D$, where r and s are non-equal positive integers. Here, $N(d)$ represents the open neighbourhood of d. In 2021, Radhi *et al*. [10] was first to establish the notion of arrow domination in which arrow domination was evaluated for some standard graphs by applying bounds on r and s, i.e., only for $r = 1$ and $s = 2$. Let D be a smallest minimal arrow dominating set of G then $|D|$ is called arrow domination number of graph G, which is represented by $\gamma_{ar}(G)$ and D is called γ_{ar}-set.

In this article, we evaluate arrow domination number of some generalized graphs like Friendship graph, Gear graph, Helm graph, Flower graph, Sunflower graph, Triangular snake graph, Double triangular snake graph, Petersen graph, Dragon graph, Lollipop graph and Barbell graph. We also discuss the existence and non-existence of an arrow dominating set of some graphs in particular case, i.e., for $r = 1$ and $s = 2$.

2. Definitions and Preliminaries

Definition 2.1 ([10])**.** Let G be an undirected and simple graph, a set $D \subseteq V(G)$ is called an arrow dominating set, if $|N(d) \cap (V - D)| = r$ and $|N(d) \cap D| \geq s$ $\forall$ $d \in D$ where r and s are positive integers such that $r \neq s$.

Example 2.2. Let G_1 be a graph as:

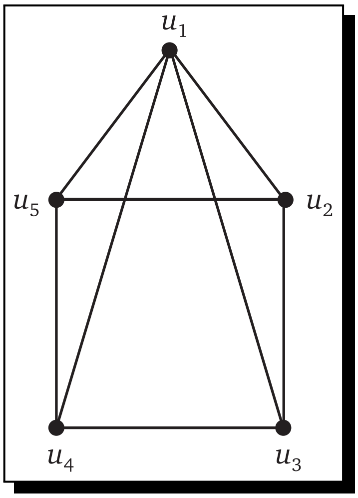

Figure 1. Graph G_1

If we take D as $\{u_1, u_2, u_4\} \subseteq V(G)$ then we obtain $|N(u) \cap (V - D)| = 2$ and $|N(u) \cap D| \geq 1$ $\forall$ $u \in D$. Thus, by using definition of arrow domination, D is an arrow dominating set of graph G_1.

Definition 2.3 ([10])**.** Let D be an arrow dominating set of graph G then D is called a γ_{ar}-set of G if D is smallest minimal arrow dominating set of graph G.

Definition 2.4 ([10])**.** The number of vertices of a smallest minimal arrow dominating set of graph G is said to be the arrow domination number of graph G and it is represented by $\gamma_{ar}(G)$.

Example 2.5. Let G be a graph as below:

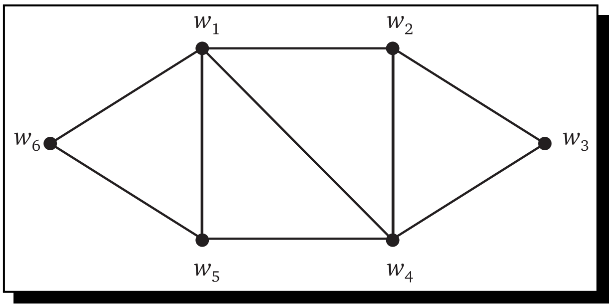

Figure 2. Graph G

Let us take $D = \{w_1, w_4\} \subseteq V(G)$ then D is an arrow dominating set of graph G where $r = 3$ and $s = 1$. Also, D is the smallest minimal arrow dominating set of graph G. Hence, D is γ_{ar}-set of G and $\gamma_{ar}(G) = 2$.

Observation 2.6. Let G be a graph and D be a γ_{ar}-set of G then $|D| \geq 2$.
Since according to the definition of arrow domination, r and s must be non-equal positive integers. If $|D| < 2$ then it implies either $|D| = 0$ or 1. But $|D| = 0$ implies $D = \phi$, which is not possible. Also, if $|D| = 1$ then D is a singleton set and no vertex of D has neighbouring vertices in D, i.e., $s = 0$, which leads us to a contradiction to the definition of arrow domination. This implies $|D| \geq 2$.

3. Main Results

Definition 3.1 ([3])**.** Let C_3 be a cycle graph. If we join n copies of cycle C_3 by a common vertex then the graph obtained is called Friendship graph or Fan graph. It is represented by F_n. The figure of Friendship graph for $n = 4$ is as below:

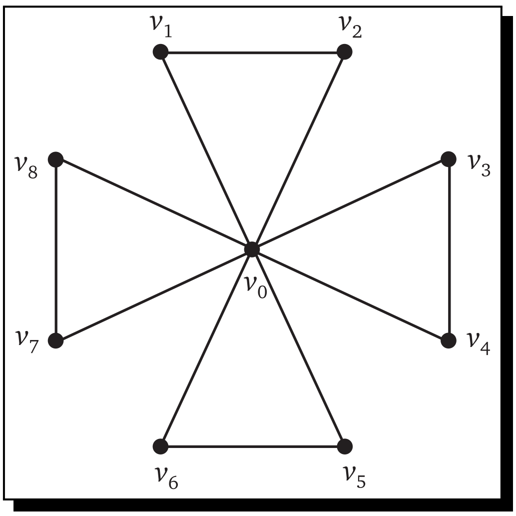

Figure 3. Friendship graph F_4

Theorem 3.2. *Friendship graph F_n has no γ_{ar}-set for any n.*

Proof. Since Friendship graph F_n contains $2n+1$ vertices in which $2n$ vertices have degree 2 and only one vertex has degree $2n$. For $n = 1$, all vertices of F_n have degree 2 and by [10, Remark 2.3] in arrow dominating set, we have $\deg(d) \geq 3\ \forall\ d \in D$. Therefore, $D = \phi$.
For $n > 1$, exactly one vertex of F_n, say v, is of degree > 2, i.e., $D = \{v\}$. In such a case, $|N(v) \cap (D)| = 0$, i.e., $s = 0$, which is a contradiction to the definition of arrow domination. Therefore, F_n has no γ_{ar}-set for any n. □

Remark 3.3. Since F_n has no any arrow dominating set for every natural number n, this implies F_n has no γ_{ar}-set for any value of r and s. Hence, for $r = 1$ and $s = 2$ also, arrow domination does not exist for F_n.

Example 3.4. Let $G = C_3$ be a 3-cycle graph then every vertex of G has degree 2. But degree of every vertex of arrow dominating set must be at least 3. This shows that arrow domination does not exist in case of C_3. If we take $r = 1$ and $s = 2$ then also condition remains same, i.e., arrow domination does not exist in particular case also.

Definition 3.5 ([2])**.** Let W_n represents the wheel graph. If we add a new vertex between every pair of vertices of outer cycle of W_n then the resulted graph is a Gear graph. It is denoted by G_n. The figure of Gear graph for $n = 5$ is as below:

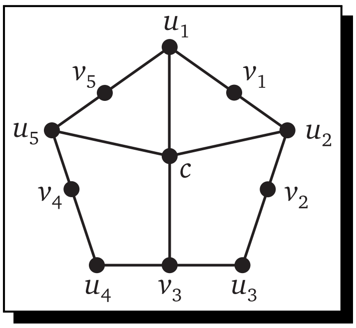

Figure 4. Gear graph G_5

Theorem 3.6. *Gear graph G_n has arrow dominating set except for $n = 3$ and $\gamma_{ar}(G_n) = n - 1$.*

Proof. As we know that Gear graph G_n has total $2n + 1$ vertices in which number of vertices of degree 2 are n, vertices of degree 3 are n and only one vertex has degree n. By [10, Remark 2.3(4)], we have degree of every vertex of D must be ≥ 3. Therefore, all n vertices of G_n of degree 2, say $\{v_1, v_2, \dots, v_n\}$, are in $V - D$. Let n vertices of G_n having degree 3 be $\{u_1, u_2, \dots, u_n\}$. Let $u_1 \in D$ then u_1 dominates two vertices in $V - D$. Take another vertex from $\{u_1, u_2, \dots, u_n\}$, say u_2 in D then u_2 is also adjacent to two vertices of $V - D$. Now, since central vertex, say c, is in neighbourhood of n vertices having degree 3. If $c \in V - D$ then all vertices of D dominate c and vertices of D are not adjacent to each other, i.e., $s = 0$, which is not possible.

Thus, c cannot be in $V-D$, i.e., $c \in D$. For this, there must be two vertices having degree 3, say u_j and u_s in $V-D$ which are in open neighbouhood of central vertex c and remaining $n-2$ vertices of degree 3 must be in D such that each vertex of $V-D$ is dominated by at least one vertex of D. From this, we obtain $|D| = n-2+1 = n-1$. We also observe that D is smallest minimal arrow dominating set. This implies D is γ_{ar}-set of G_n and cardinality of D is $n-1$, i.e., $\gamma_{ar}(G_n) = n-1$. But if $n = 3$ then three vertices, say u_1, u_2 and u_3 have degree 3 and three vertices, say v_1, v_2 and v_3 have degree 2 and one central vertex, say c, has degree 3. Therefore, $v_1, v_2, v_3 \in V-D$ and $c \in D$. If we take all vertices $u_1, u_2, u_3 \in D$ then no vertex of $V-D$ is in neighbourhood of c, which is not possible. If we take two vertices, say u_1 and u_2, in $V-D$ and u_3 in D then there is one vertex of $V-D$ such that it is not in neighbourhood of any vertex of D. In both cases, we obtain that G_n has no arrow dominating set only when $n = 3$. □

Corollary 3.7. *If $r = 1$ and $s = 2$ then G_n has no γ_{ar}-set for any n.*

Proof. Let $d \in G_n$ be a vertex having degree 3 then d dominates two vertices of degree 2 and also central vertex. These two vertices belongs to $V-D$, i.e., $|N(d) \cap (V-D)| = 2$. This holds for any $d \in D$, i.e., $r = 2$ and $s = 1$. Therefore, it is not possible for $r = 1$ and $s = 2$, G_n has any arrow dominating set for all n. □

Example 3.8. Let us consider Gear graph for $n = 4$, G_4 as:

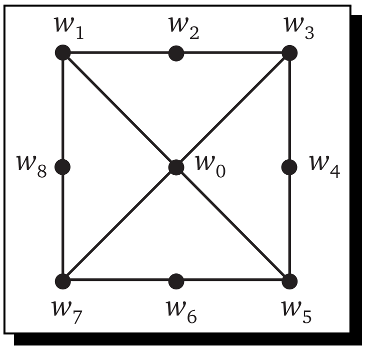

Figure 5. Gear graph G_4

Since w_2, w_4, w_6 and w_8 has degree 2. Thus, these must be in $V-D$. Now, let us take D as $\{w_0, w_1, w_5\}$ then we obtain $|N(d) \cap (V-D)| = 2$ and $|N(d) \cap D| \geq 1$ $\forall$ $d \in D$. Also, D is smallest minimal arrow dominating set of G_4. This implies $\gamma_{ar}(G_4) = 3$.

Definition 3.9 ([8])**.** If we add pendant or end vertex to every vertex of outer cycle of wheel graph W_n then the resulted graph is Helm graph. It is represented by H_n. The figure of Helm graph for $n = 5$ is given as below:

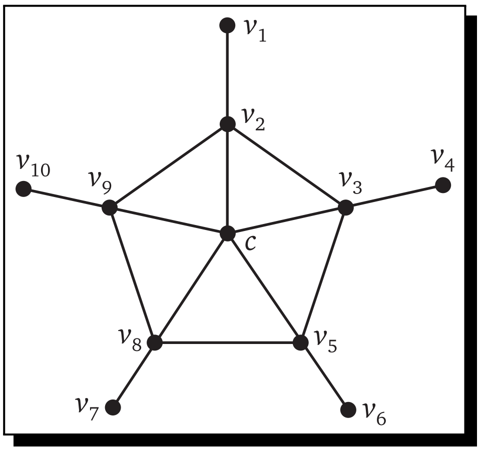

Figure 6. Helm graph H_5

Theorem 3.10. *Helm graph H_n has no γ_{ar}-set for any n.*

Proof. H_n has total $2n+1$ vertices in which number of vertices of degree 4 are n, n pendant vertices and one central vertex has degree n. All n pendant vertices belongs to $V-D$ because of degree 1 and every support vertex belongs to D. Therefore, n vertices having degree 4 must be in D. Now, if central vertex, say c, in D then $|N(c)\cap(V-D)|=0$, which is not possible. If $c\in V-D$ then $r=s=2$, which contradicts to the definition of arrow domination. In both cases, we cannot find any arrow dominating set of H_n. Thus, we observe that H_n has no γ_{ar}-set for any n. □

Definition 3.11 ([7])**.** If we join every pendant vertex of Helm graph H_n to the central vertex then the graph obtained is Flower graph. It is represented by fl_n. The figure of Flower graph for $n=3$ is given as below:

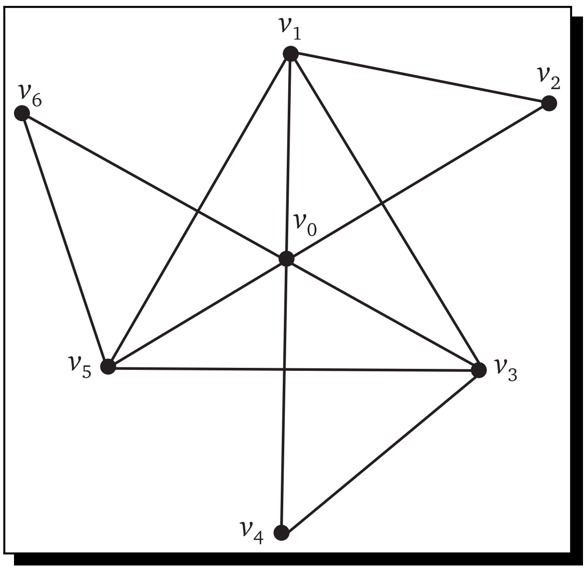

Figure 7. Flower graph fl_3

Theorem 3.12. *Flower graph fl_n has no γ_{ar}-set for any n.*

Proof. Flower graph fl_n contains total $2n+1$ vertices in which number of vertices of degree 2 are n, number of vertices of degree 4 are n and one central vertex has degree $2n$. Let D be an arrow dominating set of fl_n. Since all n vertices having degree 2 cannot be in D. Thus, these must be in $V-D$.

Case I. If all remaining $n+1$ vertices are in D then central vertex dominates all n vertices of $V-D$ but remaining n vertices of D dominates exactly one vertex or we can say that we cannot find any $r>0$ for which $|N(d)\cap(V-D)|=r\ \forall\ d\in D$ holds. Therefore, in this case, no arrow domination exists.

Case II. If remaining n vertices are in D and central vertex in $V-D$ then we obtain

$$|N(d)\cap(V-D)|=r=2 \quad \text{and} \quad |N(d)\cap D|\geq s=2,$$

i.e., $r=s=2$, which contradicts the definition of arrow domination. This shows that arrow domination in fl_n does not exists for all n. □

Definition 3.13 ([3]). If we join n pendant edges on central vertex of Flower graph then the graph obtained is called Sunflower graph. It is represented by sfl_n. The figure of Sunflower graph for $n=4$ is given as:

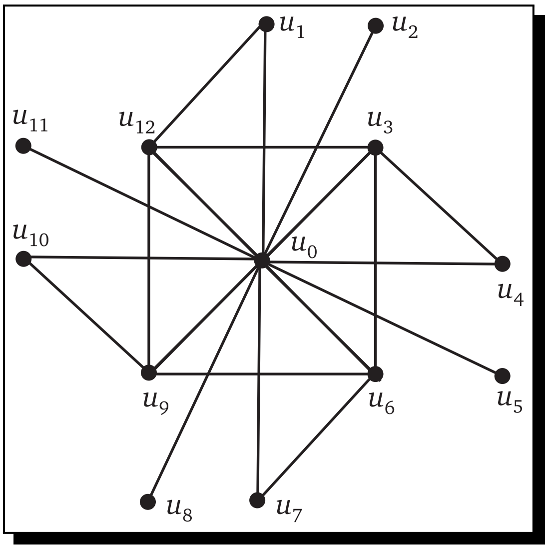

Figure 8. Sunflower graph sfl_4

Theorem 3.14. *Sunflower graph sfl_n has no γ_{ar}-set for any n.*

Proof. sfl_n has total $3n+1$ vertices in which number of vertices of degree 2 are n, n pendant vertices, n vertices are with degree 4 and one central vertex has degree $3n$. Let D be an arrow dominating set. Since n pendant vertices and n vertices having degree 2 belong to $V-D$ because of degree <3 and central vertex is only support vertex of all pendant vertices. Therefore, central vertex, say c, must be in D. But there is no vertex from remaining n vertices which is in neighbourhood of $2n$ vertices in $V-D$, i.e., $D=\{c\}$, i.e., $s=0$, which contradicts to the definition of arrow domination. Thus, we conclude that sfl_n has no γ_{ar}-set for any n. □

Definition 3.15 ([11])**.** Let P_n be a path of n vertices. If we add a new vertex, say u_j, between every pair of vertices, say v_j and v_{j+1} and join u_j to both v_j and v_{j+1}, for every $1 \le j \le (n-1)$ then the resulted graph is said to be the Triangular snake graph, which is represented by T_n. The figure of Triangular snake graph for $n=5$ is given as below:

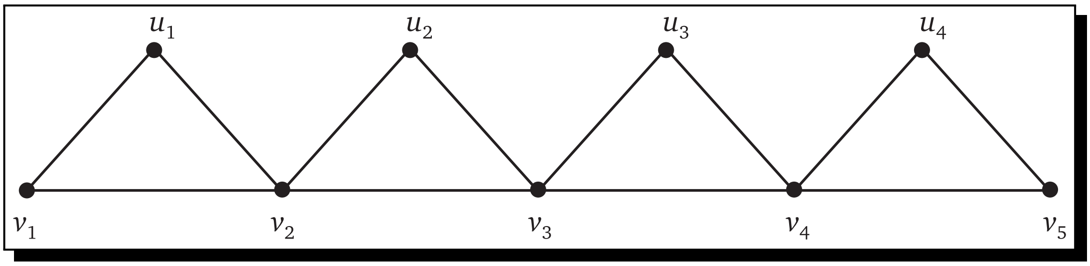

Figure 9. Triangular Snake graph T_5

Theorem 3.16. *Triangular snake graph T_n has an arrow dominating set only when $n=4$ and $\gamma_{ar}(T_4)=2$.*

Proof. In T_n graph, two end vertices of path graph and newly added vertices have degree 2, i.e., T_n graph has total $2n-1$ vertices in which $n-2$ vertices have degree 4 and $n+1$ vertices have degree 2. Also, these $n+1$ vertices belong to $V-D$ because of degree ≤ 2. If $n=2$ then T_2 becomes a cycle C_3 which has all vertices of degree 2. Therefore, T_2 has no arrow dominating set. If $n=3$ then T_3 has only one vertex having degree ≥ 3, i.e., D is a singleton set, i.e., $s=0$, which is a contradiction. If $n=4$ then T_4 has 2 vertices, say u and v, have degree 4 and remaining vertices have degree 2. Let $u,v \in D$ then we obtain

$$|N(u)\cap(V-D)| = |N(v)\cap(V-D)| = 3 = r$$

and also, u and v dominate each other, i.e., $s=1$. Thus, we obtain that T_4 has arrow dominating set with $\gamma_{ar}(T_4)=2$. Now, it remains to prove that for $n>4$, T_n has no γ_{ar}-set. For $n>4$, T_n has $n-2$ vertices i.e. more than 2 vertices of degree 4. If we take all these $n-2$ vertices in D then we obtain that we cannot find $r>0$ for which $|N(d)\cap(V-D)|=r \ \forall \ d \in D$ holds. If we take any one vertex from these $n-2$ vertices in $V-D$ then there is atleast one vertex in $V-D$ such that it is not in neighbourhood of any vertex of D. In all cases, we conclude that T_n has arrow dominating set only for $n=4$ with $\gamma_{ar}(T_4)=2$. □

Corollary 3.17. *If $r=1$ and $s=2$ then T_4 has no arrow dominating set.*

Proof. In T_4, there are only two vertices which have degree >2, i.e., $|D|\le 2$, i.e., $s\le 1$ and also each vertex of D has two neighbouring vertices in $V-D$, i.e., $r=2$. Therefore, T_4 has no arrow dominating set for $r=1$ and $s=2$. □

Definition 3.18 ([11])**.** If we join two vertices, say u_i and w_i, between each pair of vertices, say v_i and v_{i+1}, of path P_n and also join u_i and w_i to both v_i and v_{i+1} then the resulted graph we obtain is a Double triangular snake graph, which is represented by $D(T_n)$. The figure of Double triangular snake graph for $n = 5$ is given as below:

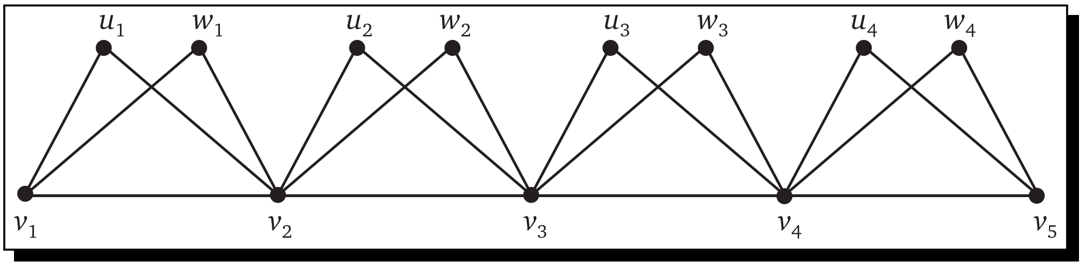

Figure 10. Double Triangular Snake graph $D(T_5)$

Theorem 3.19. *Double triangular snake graph has a γ_{ar}-set only when $n = 2$ and 4 and $\gamma_{ar}(D(T_n)) = 2$ for $n = 2,4$.*

Proof. If $n = 2$ then $D(T_n)$ has two vertices with degree 2 and two vertices, say u and v, having degree 3. Let $u,v \in D$ then u and v have exactly two neighbouring vertices in $V - D$, i.e., $r = 2$ and u, v both are adjacent to each other also, i.e., $s = 1$. Thus, D is γ_{ar}-set of $D(T_2)$ and $\gamma_{ar}(D(T_2)) = 2$. If $n = 4$ then $D(T_4)$ has six vertices having degree 2, two vertices having degree 3 and remaining two vertices, say v_1 and v_2, having degree 6. Let $v_1, v_2 \in D$ then both dominates all five vertices of $V - D$ and v_1, v_2 are adjacent to each other, i.e., we have $r = 5$ and $s = 1$. Therefore, D is γ_{ar}-set and $\gamma_{ar}(D(T_4)) = 2$. Now, we claim that $D(T_n)$ has no γ_{ar}-set for $n \neq 2,4$. For $n \neq 2,4$, $D(T_n)$ has total $3n - 2$ vertices in which $2(n-1)$ vertices have degree 2 and two vertices have degree 3 and remaining $n - 2$ vertices have degree 6. Since these $2(n-1)$ vertices belong to $V - D$ because of degree < 3. If we take any vertex $D(T_n)$ (other than these $2(n-1)$ vertices) in D then we observe that either vertices in D are not adjacent, i.e., $s = 0$ or all vertices of D do not dominate equal number of vertices in $V - D$. In both cases, we get a contradiction to the definition of arrow domination. Therefore, we conclude that no arrow domination exists in $D(T_n)$ for $n \neq 2,4$. □

Corollary 3.20. *If $r = 1$ and $s = 2$ then $D(T_n)$ has no γ_{ar}-set for any n.*

Proof. Since for $n \neq 2,4$, $D(T_n)$ has no arrow dominating set. That implies $D(T_n)$ has no γ_{ar}-set for $r = 1$ and $s = 2$ also. If $n = 2$ then $D(T_n)$ has only two vertices having degree 3, i.e., $s = 1$ and each vertex of D has two neighbouring vertices in $V - D$, i.e., $r = 2$. This shows that $D(T_2)$ has no any γ_{ar}-set for $r = 1$ and $s = 2$. Now, for $n = 4$, $D(T_n)$ has only one arrow dominating set which contains two vertices of degree 6 and these dominate all five vertices of $V - D$, i.e., $D(T_4)$ has γ_{ar}-set only when $r = 5$ and $s = 1$. This implies for every n, $D(T_n)$ has no arrow dominating set for $r = 1$ and $s = 2$. □

Remark. (i) If a graph G has γ_{ar}-set then it is not necessary that there is atleast one proper subgraph of G which has also a γ_{ar}-set.

Let $G = K_4 = \{a_1, a_2, a_3, a_4\}$ then $D = \{a_1, a_2, a_3\}$ is γ_{ar}-set of G but if we take any proper subgraph of G then it is either cycle C_3 or P_2 or singleton set which has no γ_{ar}-set.

(ii) If G has a proper subgraph having γ_{ar}-set then G may or may not have any γ_{ar}-set.

Example. If $S = K_4$, a proper subgraph of G where G is

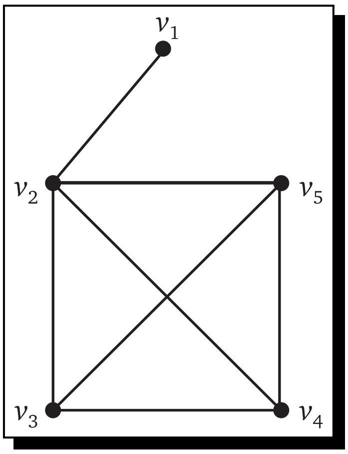

Figure 11. Graph G

Thus, we observe that S has γ_{ar}-set but G has no any γ_{ar}-set. But if $S = K_4$, a proper subgraph of G_1 and figure of G_1 is given as:

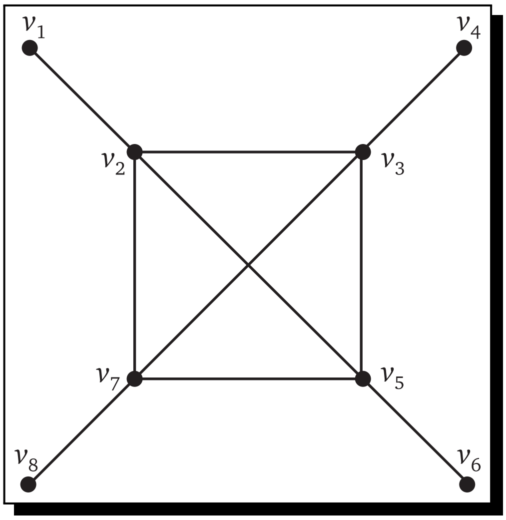

Figure 12. Graph G_1

Then S and G_1 both have arrow dominating set, i.e., if a proper subgraph of G has γ_{ar}-set then it is not necessary that G has also γ_{ar}-set.

Definition 3.21 ([6])**.** Let C_5 be a cycle graph. If there is a pentagram inside C_5 and we join corresponding vertices then the resulted graph is Petersen graph. It contains 10 vertices and 15 edges. The figure of Petersen graph is given as below:

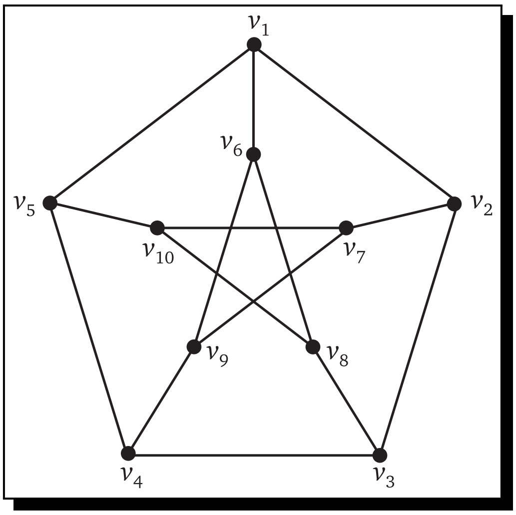

Figure 13. Petersen graph

Theorem 3.22. *Let G be a Petersen graph then G has γ_{ar}-set with $\gamma_{ar}(G) = 5$.*

Proof. Since we know Petersen graph has total 10 vertices in which every vertex has degree 3. Let these be $\{v_1, v_2, \dots, v_{10}\}$. Let D be an arrow dominating set of graph G. Take any vertex, say v_1, in D, since $d(v_1) = 3$ then $|N(v_1)| = 3$, and let us consider any two vertices from $N(v_1)$ are in D and remaining vertices of $N(v_1)$ in $V - D$. Similarly, take v_2 in D and repeat the same process. Continuing in this way, we obtain the arrow dominating set D for which cardinality of D is 5 and any vertex of D has exactly one neighbouring vertex in $V - D$ and each vertex of D has two neighbouring vertices in D, i.e., $r = 1$ and $s = 2$. We also observe that D is smallest minimal arrow dominating set, i.e., D is γ_{ar}-set of G with $\gamma_{ar}(G) = 5$. □

Definition 3.23 ([9])**.** A graph which is obtained by joining a cycle C_m with a path P_n by an edge between C_m and P_n is called Dragon graph $D(m,n)$. The figure of Dragon graph for $m = 5$ and $n = 4$ is given as below:

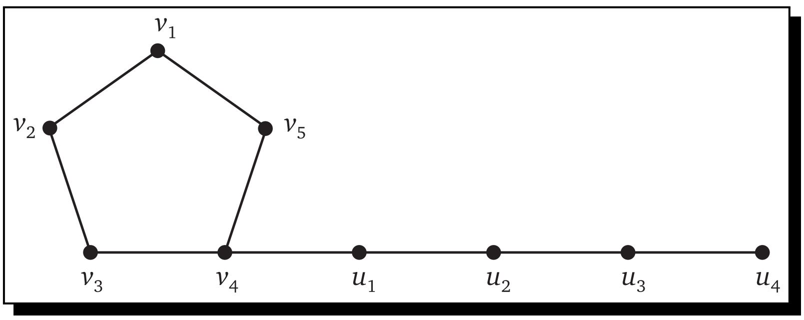

Figure 14. Dragon graph $D(5,4)$

Theorem 3.24. *Dragon graph $D(m,n)$ has no γ_{ar}-set for any m, n.*

Proof. In $D(m,n)$, there is join of an edge between a cycle graph C_m and a path graph P_n and there is one vertex of C_m, say u, such that $d(u) = 3$ and remaining vertices of C_m have degree 2 and all vertices of P_n have degree 2 except the end vertex which is pendant. This implies $D(m,n)$ has only one vertex of degree ≥ 3. If D is any arrow dominating set of $D(m,n)$ then D can contain at most one vertex which implies $s = 0$. This leads us to a contradiction of the definition of arrow domination. Thus, $D(m,n)$ has no γ_{ar}-set for any m, n. □

Definition 3.25 ([3])**.** If we join an edge between the complete graph K_n and the path graph P_m then resulted graph is said to be the Lollipop graph. It is represented by $L(m,n)$. The figure of Lollipop graph for $m = 5$ and $n = 2$ is given as below:

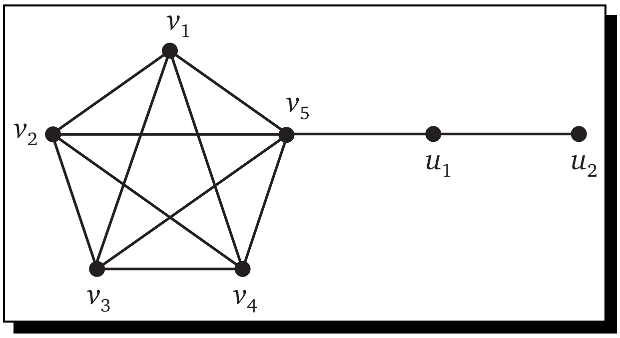

Figure 15. Lollipop graph $L(5,2)$

Theorem 3.26. *Lollipop graph $L(m,n)$ has no any γ_{ar}-set for all m, n.*

Proof. As we know that in lollipop graph $L(m,n)$, a complete graph K_m and a path P_n is joined by an edge and there is one vertex of K_m, say u, such that $d(u) = m$. For $n = 1$, Lollipop graph $L(m,n)$ has $m+1$ vertices in which $m-1$ vertices are of degree $m-1$ and exactly one vertex is of degree m and one end vertex is of degree one. Let v_1 be the support vertex of the end vertex then v_1 must be in D. If we take any vertex other than support vertex, say v_2, in D then we have

$$|N(v_1) \cap (V-D)| = |N(v_2) \cap (V-D)| + 1.$$

This holds for all v_2 in D other than v_1, i.e., we cannot find any $r > 0$ for which we obtain $|N(d) \cap (V-D)| = r \ \forall \ d \in D$. For $n > 1$, $L(m,n)$ has total $m+n$ vertices in which $n-1$ vertices are of degree 2 and one vertex is pendant. All these $n-1$ vertices cannot be in D, i.e., these are in $V-D$. If these $n-1$ vertices are in $V-D$ then these are not dominated by D. This conclude that $L(m,n)$ has no γ_{ar}-set for all m, n. □

Definition 3.27 ([5])**.** If we join an edge between two copies of complete graph, say K_m, then the graph obtained is Barbell graph. It is represented by B_n. The figure of Barbell graph for $n = 4$ is given as below:

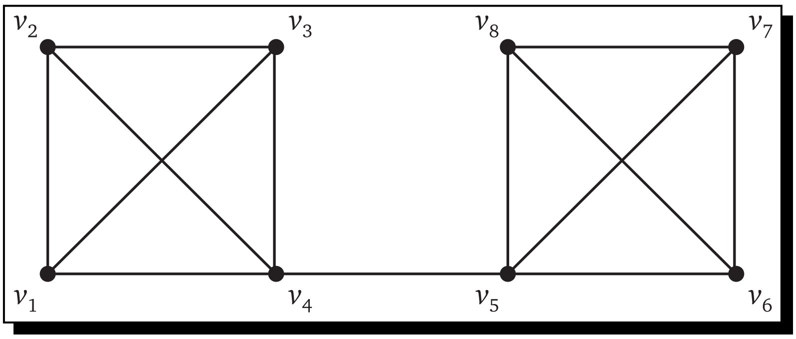

Figure 16. Barbell graph B_4

Theorem 3.28. *Barbell graph B_n has an arrow dominating set with $\gamma_{ar}(B_n) = 2$ for any n.*

Proof. As B_n has $2n$ vertices in which number of vertices of degree $n-1$ are $2(n-1)$ and two vertices, say u_1 and u_2, have degree n. Let D be an arrow dominating set of B_n. Let $u_1, u_2 \in D$ and all other vertices be in $V - D$ then u_1 and u_2 have exactly $(n-1)$ neighbouring vertices in $V - D$ and also, u and v are adjacent to each other. D is smallest minimal arrow dominating set also. Thus, we have D as γ_{ar}-set of B_n with $\gamma_{ar}(B_n) = 2$. □

Corollary 3.29. *If $r = 1$ and $s = 2$ then B_n has γ_{ar}-set only when $n = 4$ and $\gamma_{ar}(B_4) = 6$.*

Proof. If $n = 4$ then B_4 has six vertices having degree 3 and two vertices having degree 4. If we take all six vertices having degree 3 in D and remaining two vertices in $V - D$ then each vertex of D has exactly one neighbouring vertex in $V - D$ and has two neighbouring vertices in D, i.e., $r = 1$ and $s = 2$. D is smallest minimal arrow dominating set also. Therefore, $\gamma_{ar}(B_4) = 6$. For $n \neq 4$, D dominates more than two vertices for every arrow dominating set D, i.e., $s > 2$. Thus, we conclude that for $r = 1$ and $s = 2$, B_n except $n = 4$ has no any γ_{ar}-set. □

Example 3.30. Let us take barbell graph B_3, for $n = 3$ as:

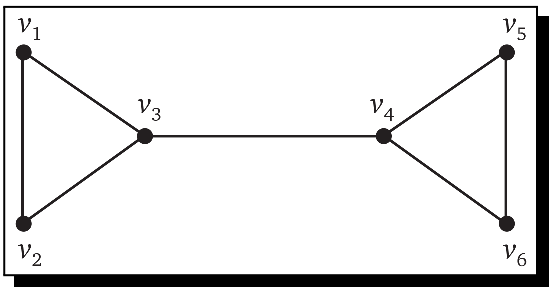

Figure 17. Barbell graph B_3

Here, all the vertices of B_3 has degree 2 except two vertices v_3 and v_4. If we consider D as $\{v_3, v_4\}$ then we have $|N(d) \cap (V - D)| = 2$ and $|N(d) \cap D| \geq 1 \ \forall \ d \in D$. Thus, by using definition of arrow domination, D is an arrow dominating set of B_3. Hence, D is γ_{ar}-set and $\gamma_{ar}(B_3) = 2$.

4. Conclusion

We evaluated the arrow domination number of some generalized graphs like friendship graph, gear graph, helm graph etc. and we also considered the particular case by applying bound on value of r and s by taking $r = 1$ and $s = 2$ to calculate arrow domination number of graphs. It is obvious that if a graph does not attain arrow dominating set then also in case of $r = 1$ and $s = 2$, arrow domination does not exist for that graph. We also concluded that existence of arrow dominating set and arrow domination number of a graph need not be hold in particular case also. Existence and non-existence of arrow dominating set of some generalized graphs is given in Table 1:

Table 1

Existence and Non-existence of γ_{ar}-set of some graphs		
Graphs	γ_{ar}-set	γ_{ar}-set for $r = 1$ and $s = 2$
Friendship graph (F_n)	No	No
Gear graph (G_n)	Yes but except $n = 3$	No
Helm graph (H_n)	No	No
Flower graph (fl_n)	No	No
Sunflower graph (sfl_n)	No	No
Triangular snake graph (T_n)	Yes, only for $n = 4$	No
Double triangular snake graph ($D(T_n)$)	Yes, only for $n = 2$ and 4	No
Petersen graph	Yes	Yes
Dragon graph ($D(m,n)$)	No	No
Lollipop graph ($L(m,n)$)	No	No
Barbell graph (B_n)	Yes	Yes, only for $n = 4$

Further, arrow domination number of graphs can also be derived under different interesting operations like corona, shadow, line, total, middle, join, cartesian of graphs etc. Also, arrow domination number can be calculated for some particular cases by applying different bounds on value of r and s under various above operations for further investigations.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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