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Research Article

Harmonic Centrality and Centralization of the Bow-Tie Product of Graphs

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Abstract. The bow-tie product is a newly named binary operation on graphs. In this article, we present some properties of the bow-tie product of graphs, as well as some results on both the harmonic centrality and the harmonic centralization of the bow-tie product of the path P_2 with any of the path P_m, cycle C_m, star S_m, fan F_m, and wheel W_m.

Keywords. Bow-tie product, Harmonic centrality, Harmonic centralization

Mathematics Subject Classification (2020). 05C12, 05C82, 91D30

1. Introduction

The four well-known standard products of graphs are the direct product, Cartesian product, strong product, and lexicographic product (Klavẑar [6]). According to Asmerom [1], there are a total of 256 distinct possible graph products depending on how the edge set is defined. One of these graph products is the bow-tie product, which is the focus of this article.

The main metrics tackled in this article are harmonic centrality and harmonic centralization. Note that centrality in graph theory and network analysis quantifies the importance of a vertex in a graph. It is used to identify which node takes up the critical position in a network (Zhang and Luo [14]) and is equated to either remarkable leadership, good popularity, or excellent reputation. Freeman [5] discussed the concepts behind the different measures of centrality.

Among the many measures of centrality is harmonic centrality. This was introduced in 2000 by Marchiori and Latora [8], and developed independently by Dekker [4], and Rochat [12]. For a related work on harmonic centrality in some graph families (Ortega and Eballe [9, 10]).

While centrality is on the node level of a graph, centralization quantifies the graph-level centrality score based on the various centrality scores of the nodes. Centralization may be used to compare how central graphs are. For a related work on harmonic centralization of some graph families (Ortega and Eballe [9, 11]).

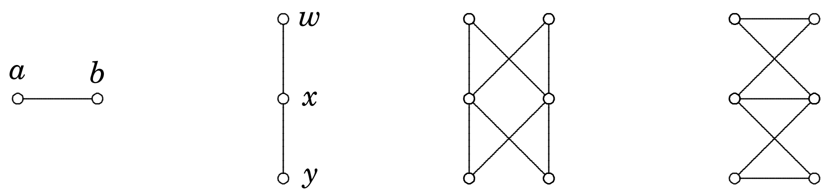
In this article, the harmonic centrality of the vertices and the harmonic centralization of the bow-tie products of path P_2 with any path P_m, cycle C_m, and fan F_m is derived. The expressions obtained could be of use when one determines the harmonic centrality and harmonic centralization of more complex graphs. For a study on graph products with betweenness centrality (Kumar and Balakrishnan [7]).

All graphs considered here are simple, undirected, and finite.

2. Preliminary Notes

For referencing, below are the definitions and some properties of the concepts discussed in this article.

Definition 2.1 (Bow-tie product of graphs)**.** Let G and H be graphs. The vertex set of the bow-tie product $G \bowtie H$ is the Cartesian product $V(G) \times V(H)$; where two vertices (u,v) and (u',v') are adjacent in $G \bowtie H$ if and only if either (i) $u = u'$ and v is adjacent to v' in H, or (ii) u is adjacent to u' in G and v is adjacent to v' in H.



(a) Path P_2 (b) Path P_3 (c) $P_2 \bowtie P_3$ (d) $P_3 \bowtie P_2$

Figure 1. (a) Path P_2, (b) Path P_3, (c) Bow-tie product $P_2 \bowtie P_3$, and (d) Bow-tie product $P_3 \bowtie P_2$

Figure 1 presents the paths P_2 and P_3, and the bow-tie products $P_2 \bowtie P_3$ and $P_3 \bowtie P_2$. The term “bow-tie" comes from the figure produced by the path P_2 multiplied to itself like those of the standard products of graphs. We can see from the figure that $P_2 \bowtie P_3$ has an order of 6 and a size of 8.

Further inspection of the graphs in Figure 1 reveals that $G \bowtie H \neq H \bowtie G$, where the bow-tie products have different sizes for $P_2 \bowtie P_3$ and $P_3 \bowtie P_2$. Therefore, the bow-tie product is not commutative.

Consider the graphs $A = P_2, B = P_2$, and $C = P_3$. We can see the graphs $A \bowtie (B \bowtie C)$ and $(A \bowtie B) \bowtie C$ in Figure 2 and Figure 3, respectively, are not the same since they have different sizes. Therefore, the bow-tie product is not associative.

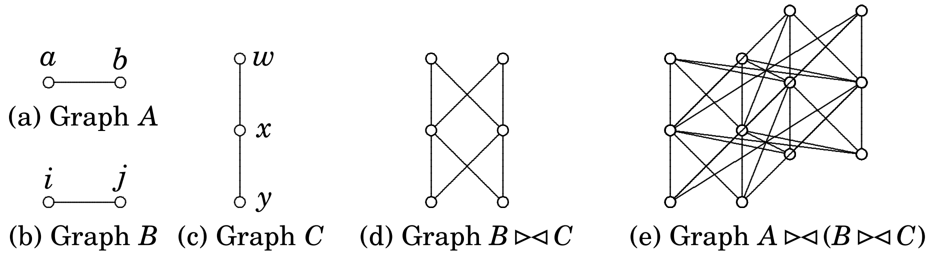


Figure 2. (a) Graph A, (b) Graph B, (c) Graph C, (d) Graph $B \bowtie C$, and (e) Graph $A \bowtie (B \bowtie C)$

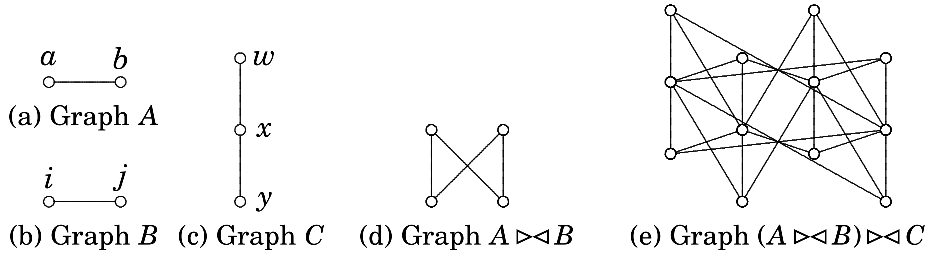


Figure 3. (a) Graph A, (b) Graph B, (c) Graph C, (d) Graph $B \bowtie C$, and (e) Graph $A \bowtie (B \bowtie C)$

Definition 2.2 ([3])**.** Let $G = (V(G), E(G))$ be a nontrivial graph of order m. If $u \in V(G)$, then the harmonic centrality of u is given by the expression

$$\mathcal{H}_G(u) = \frac{\mathcal{R}_G(u)}{m-1},$$

where $\mathcal{R}_G(u) = \sum\limits_{x \neq u} \frac{1}{d(u,x)}$ is the sum of the reciprocals of the shortest distance $d(u,x)$ in G between vertices u and x for each $x \in (V(G) \setminus u)$, with $\frac{1}{d(u,x)} = 0$ in case there is no path from u to x in G.

Definition 2.3 ([9])**.** The harmonic centralization of a graph G of order m is given by

$$C_{\mathcal{H}}(G) = \frac{\sum\limits_{i=1}^{m} (\mathcal{H}_{G\max}(u) - \mathcal{H}_G(u_i))}{\frac{m-2}{m}},$$

where $\mathcal{H}_{G\max}(u)$ is the largest harmonic centrality among all vertices in G.

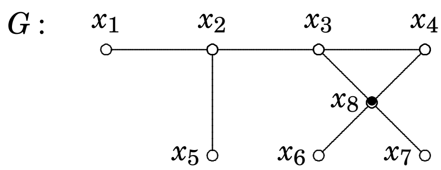


Figure 4. Graph G with $x_8 \in V(G)$, where $\mathcal{H}_G(x_8) = \frac{31}{42}$ and $C_{\mathcal{H}}(G) = \frac{4}{9}$

In the graph G given in Figure 4, we have

$$\mathcal{H}_G(x_1) = \frac{19}{42},\ \mathcal{H}_G(x_2) = \frac{2}{3},\ \mathcal{H}_G(x_3) = \frac{5}{7},\ \mathcal{H}_G(x_4) = \frac{25}{42},\ \mathcal{H}_G(x_5) = \frac{19}{42},\ \mathcal{H}_G(x_6) = \frac{10}{21},$$
$$\mathcal{H}_G(x_7) = \frac{10}{21}, \text{ and } \mathcal{H}_G(x_8) = \frac{31}{42}.$$

Clearly, $\mathcal{H}_{G\max}(x) = \frac{31}{42}$ from node x_8, thus,

$$C_{\mathcal{H}}(G) = \frac{\left(\frac{31}{42}-\frac{19}{42}\right)+\left(\frac{31}{42}-\frac{2}{3}\right)+\left(\frac{31}{42}-\frac{5}{7}\right)+\left(\frac{31}{42}-\frac{25}{42}\right)+\left(\frac{31}{42}-\frac{19}{42}\right)+\left(\frac{31}{42}-\frac{10}{21}\right)+\left(\frac{31}{42}-\frac{10}{21}\right)}{\frac{8-2}{2}} = \frac{\frac{4}{3}}{3} = \frac{4}{9}.$$

Definition 2.4 ([13]). The nth harmonic number H_n is the sum of the reciprocals of the first n natural numbers, that is $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

3. Main Results

The bow-tie products of special graph families considered in this article are again that of path P_2 with path P_m of order $m \geq 1$, path P_2 with cycle C_m of order $m \geq 3$, path P_2 with star S_m of order $m+1, m \geq 3$, path P_2 with fan F_m of order $m+1, m \geq 2$, and path P_2 with wheel W_m of order $m+1, m \geq 3$.

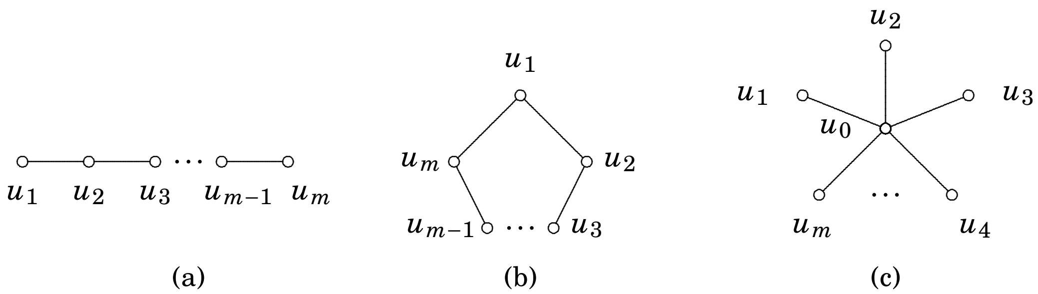


Figure 5. (a) Path P_m, (b) Cycle C_m, (c) Star S_m

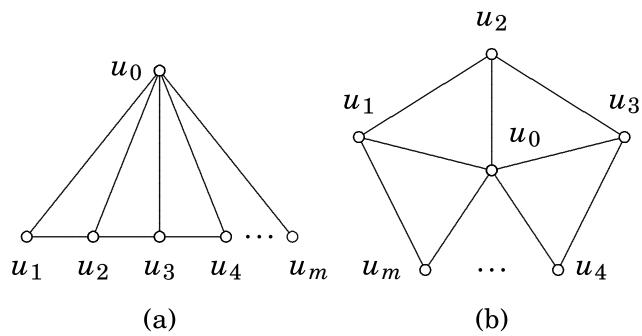


Figure 6. (a) Fan F_m, (b) Wheel W_m

Theorem 3.1. *The harmonic centrality of any vertex* (u_i, v_j) *for the bow-tie product of path* P_2 *of order* 2 *and a path graph* P_m *of order* m *is given by*

$$\mathcal{H}_{P_2 \bowtie P_m}(u_i, v_j) = \begin{cases} \frac{4H_{m-1}+1}{2(2m-1)}, & \text{for } i = 1,2 \text{ and } j = 1 \text{ or } m, \\ \frac{4H_{j-1}+4H_{m-j}+1}{2(2m-1)}, & \text{for } i = 1,2 \text{ and } 1 < j < m. \end{cases}$$

Proof. Let $P_2 = [u_1, u_2]$ and $P_m = [v_1, v_2, \ldots, v_m]$. Based on the structure of the graph product (see Figure 7), vertex (u_1, v_1) is of distance $1, 2, \ldots, m-1$ to each of the vertices $(u_1, v_2), (u_1, v_3), \ldots, (u_1, v_m)$, respectively. It is also of distance $1, 2, \ldots, m-1$ to each of the vertices $(u_2, v_2), (u_2, v_3), \ldots, (u_2, v_m)$, respectively, while it is of distance 2 to (u_2, v_1). Since vertices $(u_1, v_m), (u_2, v_1)$, and (u_2, v_m) have the same situation with vertex (u_1, v_1) as far as distances to all other distances are concerned, it follows that for $i = 1,2$ and $j = 1, m$. So that,

$$\mathcal{H}_{P_2 \bowtie P_m}(u_i, v_j) = \frac{2H_{m-1} + \frac{1}{2}}{2m-1} = \frac{4H_{m-1}+1}{2(2m-1)}.$$

Vertices (u_i, v_j) where $i = 1,2$ and $1 < j < m$, on the other hand, are connected to two sets of vertices for distances of $1, 2, \dots, j-1$ and another two sets of vertices with distances of up to $1, 2, \dots, m-j$ and also connected to one more vertex with a distance of 2. Thus, for $i = 1,2$ and $j \in \{2, 3, \dots, m-1\}$,

$$\mathcal{H}_{P_2 \bowtie P_m}(u_i, v_j) = \frac{2H_{j-1} + 2H_{m-j} + \frac{1}{2}}{2m-1} = \frac{4H_{j-1} + 4H_{m-j} + 1}{2(2m-1)}. \qquad \square$$

Figure 7 presents the skeletal graph of the bow-tie product of path P_2 with path P_m of order m.

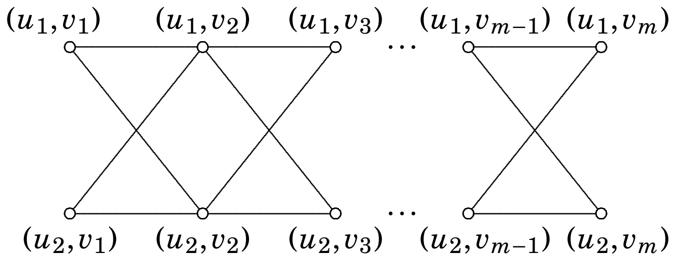


Figure 7. The graph of the bow-tie product $P_2 \bowtie P_m$

Theorem 3.2. *The harmonic centralization for the bow-tie product of path P_2 of order 2 and a Path graph P_m of order m is given by*

$$C_{\mathcal{H}}(P_2 \bowtie P_m) = \begin{cases} \frac{1}{(m-1)(2m-1)}\left(16H_{\frac{m-1}{2}} - 8H_{m-1} + (m-3)(8H_{\frac{m-1}{2}}+1) - 2\sum_{j=2}^{\frac{m-1}{2}}(4H_{j-1}+4H_{m-j}+1)\right), & \textit{if } m \textit{ is odd},\\ \frac{1}{(m-1)(2m-1)}\left((8m-16)H_{\frac{m-2}{2}} - 8H_{m-1} + m + 4 - \frac{16}{m} - 2\sum_{j=2}^{\frac{m-2}{2}}(4H_{j-1}+4H_{m-j}+1)\right), & \textit{if } m \textit{ is even}. \end{cases}$$

Proof. If m is odd, the maximum harmonic centrality of $P_2 \bowtie P_m$ is $H_{\frac{m-1}{2}}$ for vertices $(u_1, v_{\frac{m+1}{2}})$ and $(u_2, v_{\frac{m+1}{2}})$. From this harmonic centrality values the harmonic centrality of all other vertices are subtracted and normalized by dividing by $m-1$. Thus,

$$C_{\mathcal{H}}(P_2 \bowtie P_m) = \frac{1}{(m-1)(2m-1)}\left(16H_{\frac{m-1}{2}} - 8H_{m-1} + (m-3)(8H_{\frac{m-1}{2}}+1) - 2\sum_{j=2}^{\frac{m-1}{2}}(4H_{j-1}+4H_{m-j}+1)\right).$$

If m is even, however, the maximum harmonic centrality is $H_{\frac{m-2}{2}}$ for vertices $(u_1, v_{\frac{m}{2}})$, $(u_2, v_{\frac{m}{2}})$, $(u_1, v_{\frac{m+2}{2}})$ and $(u_2, v_{\frac{m+2}{2}})$. The harmonic centrality of all other vertices is subtracted from this value and normalized by dividing by $m-1$. Therefore,

$$C_{\mathcal{H}}(P_2 \bowtie P_m) = \frac{1}{(m-1)(2m-1)}\left((8m-16)H_{\frac{m-2}{2}} - 8H_{m-1} + m + 4 - \frac{16}{m} - 2\sum_{j=2}^{\frac{m-2}{2}}(4H_{j-1}+4H_{m-j}+1)\right). \qquad \square$$

Theorem 3.3. *The harmonic centrality of any vertex (u_i,v_j) for the bow-tie product of path P_2 of order 2 and a cycle graph C_m of order m is given by*

$$\mathcal{H}_{P_2 \bowtie C_m}(u_i,v_j) = \frac{2m+3}{2(2m-1)}.$$

Proof. Let $P_2 = [u_1,u_2]$ and $C_m = [u_1,u_2,\dots,u_m,u_1]$. Each vertex in $P_2 \bowtie C_m$ (see Figure 8) is adjacent to four vertices and have a distance of 2 to $2m-5$ other vertices, so that

$$\mathcal{H}_{P_2 \bowtie C_m}(u_i,v_j) = \frac{4(1)+\frac{2m-5}{2}}{2m-1} = \frac{2m+3}{2(2m-1)}.$$ □

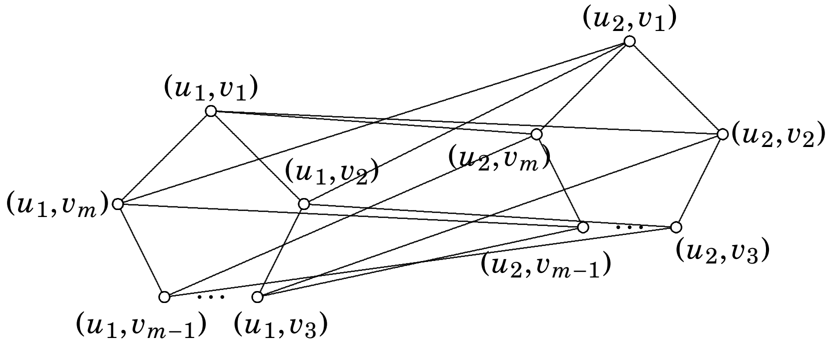


Figure 8. The graph of the bow-tie product $P_2 \bowtie C_m$

Theorem 3.4. *The harmonic centralization for the bow-tie product of path P_2 of order 2 and a cycle graph C_m of order m is zero.*

Proof. Since the harmonic centrality of the vertices in the bow-tie product of path P_2 and cycle graph C_m have the same values, the harmonic centralization of $P_2 \bowtie C_m$ therefore equates to zero. □

Theorem 3.5. *The harmonic centrality of any vertex (u_i,v_j) for the bow-tie product of path P_2 of order 2 and a star graph S_m of order $m+1$ is given by*

$$\mathcal{H}_{P_2 \bowtie S_m}(u_i,v_j) = \begin{cases} \frac{4m+1}{2(2m+1)}, & \text{for } i=1,2 \text{ and } j=0, \\ \frac{2m+3}{2(2m+1)}, & \text{for } i=1,2 \text{ and } 1 \le j \le m. \end{cases}$$

Proof. Let $P_2 = [u_1,u_2]$ and $S_m = [v_0,v_1,v_2,\dots,v_m]$. Based on the structure of the graph product $P_2 \bowtie S_m$ (see Figure 9), vertex (u_1,v_0) is adjacent to $2m$ vertices and have a distance of 2 to one other vertex. The same situation is exhibited by (u_2,v_0) in terms of distances to other vertices is concerned, it follows for $u=1,2$ and $j=0$

$$\mathcal{H}_{P_2 \bowtie S_m}(u_i,v_j) = \frac{2m(1)+1(\frac{1}{2})}{2m+1} = \frac{4m+1}{2(2m+1)}.$$

On the other hand, vertices (u_i,v_j), where $i=1,2$ and $j=1,2,\dots m$ are adjacent to two vertices and have a distance of 2 to $2m-1$ vertices, thus

$$\mathcal{H}_{P_2 \bowtie S_m}(u_i,v_j) = \frac{2(1)+(2m-1)\frac{1}{2}}{2m+1} = \frac{2m+3}{2(2m+1)}.$$ □

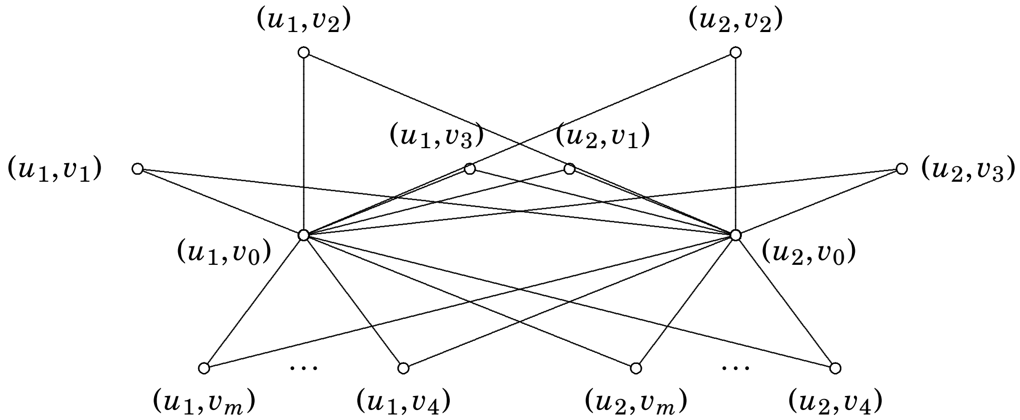


Figure 9. The graph of the bow-tie product $P_2 \bowtie S_m$

Theorem 3.6. *The harmonic centralization for the bow-tie product of path P_2 of order 2 and star graph S_m of order $m+1$ is given by*

$$C_{\mathcal{H}}(P_2 \bowtie S_m) = \frac{2m-2}{2m+1}.$$

Proof. For $P_2 \bowtie S_m$, vertices (u_1,v_0) and (u_2,v_0) have the maximum harmonic centrality of $\frac{4m+1}{2(2m+1)}$. From this value, $2m$ vertices with harmonic centralities of $\frac{2m+3}{2(2m+1)}$ is subtracted and normalized by dividing by m. Thus

$$C_{\mathcal{H}}(P_2 \bowtie S_m) = \frac{2m(4m+1-(2m+3))}{m(2(2m+1))} = \frac{2m-2}{2m+1}. \qquad \square$$

Theorem 3.7. *The harmonic centrality of any vertex (u_i,v_j) for the bow-tie product of path P_2 of order 2 and a fan graph F_m of order $m+1$ is given by*

$$\mathcal{H}_{P_2 \bowtie F_m}(u_i,v_j) = \begin{cases} \frac{4m+1}{2(2m+1)}, & \text{for } i=1,2 \text{ and } j=0, \\ \frac{m+2}{2m+1}, & \text{for } i=1,2 \text{ and } j=1,m, \\ \frac{2m+5}{2(2m+1)}, & \text{for } i=1,2 \text{ and } 1<j<m. \end{cases}$$

Proof. Let $P_2 = [u_1,u_2]$ and $F_m = [v_0,v_1,v_2,\dots,v_m]$. $P_2 \bowtie F_m$ (see Figure 10) has an order of $2m+2$ where vertex (u_1,j_0) is adjacent to $2m$ vertices and have a distance of 2 to one vertex. The same is true for vertex (u_2,j_0), thus, for $i=1,2$ and $j=0$,

$$\mathcal{H}_{P_2 \bowtie F_m}(u_i,v_j) = \frac{2m+\frac{1}{2}}{2m+1} = \frac{4m+1}{2(2m+1)}.$$

Vertex (u_1,j_1), on the other hand is adjacent to three vertices and have a distance of 2 to $2m-2$ other vertices. The same is true for vertices (u_2,j_1),(u_2,j_m) and (u_2,j_m), so that for $i=1,2$ and $j=1,m$,

$$\mathcal{H}_{P_2 \bowtie F_m}(u_i,v_j) = \frac{3+(2m-2)\frac{1}{2}}{2m+1} = \frac{m+2}{2m+1}.$$

As for all other vertices, they are adjacent to four vertices and have a distance of 2 to $2m-3$ other vertices, therefore, for $i=1,2$ and $1<j<m$,

$$\mathcal{H}_{P_2 \bowtie F_m}(u_i,v_j) = \frac{4+(2m-3)\frac{1}{2}}{2m+1} = \frac{2m+5}{2(2m+1)}. \qquad \square$$

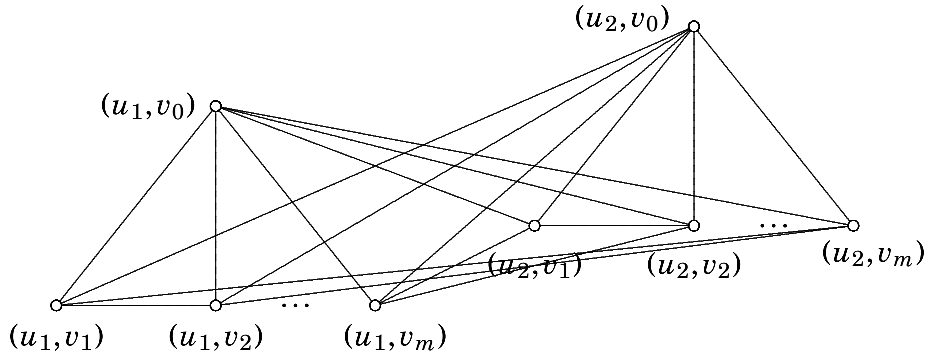


Figure 10. The graph of the bow-tie product $P_2 \bowtie F_m$

Theorem 3.8. *The harmonic centralization for the bow-tie product of path P_2 of order 2 and fan graph F_m of order $m+1$ is given by*

$$C_{\mathcal{H}}(P_2 \bowtie F_m) = \frac{2(m-1)^2}{m(2m+1)}.$$

Proof. The maximum harmonic centrality for $P_2 \bowtie F_m$ is $\frac{4m+1}{2(2m+1)}$ from which 4 vertices with harmonic centrality values of $\frac{m+2}{2m+1}$ and $2m-4$ vertices with harmonic centrality of $\frac{2m+5}{2(2m+1)}$ is deducted and normalized by dividing by m, therefore

$$C_{\mathcal{H}}(P_2 \bowtie F_m) = \frac{2m(4m+1)-8(m+2)-(2m-4)(2m+5)}{2m(2m+1)} = \frac{2(m-1)^2}{m(2m+1)}.$$ □

Theorem 3.9. *The harmonic centrality of any vertex (u_i, v_j) for the bow-tie product of path P_2 of order 2 and a wheel graph W_m of order $m+1$ is given by*

$$\mathcal{H}_{P_2 \bowtie W_m}(u_i, v_j) = \begin{cases} \frac{4m+1}{2(2m+1)}, & \text{for } i = 1,2 \text{ and } j = 0, \\ \frac{2m+7}{2(2m+1)}, & \text{for } i = 1,2 \text{ and } 1 \le j \le m. \end{cases}$$

Proof. Let $P_2 = [u_1, u_2]$ and $W_m = [v_0, v_1, v_2, \dots, v_m]$. The resulting graph product of $P_2 \bowtie W_m$ is of order $2m+2$ (see Figure 11).

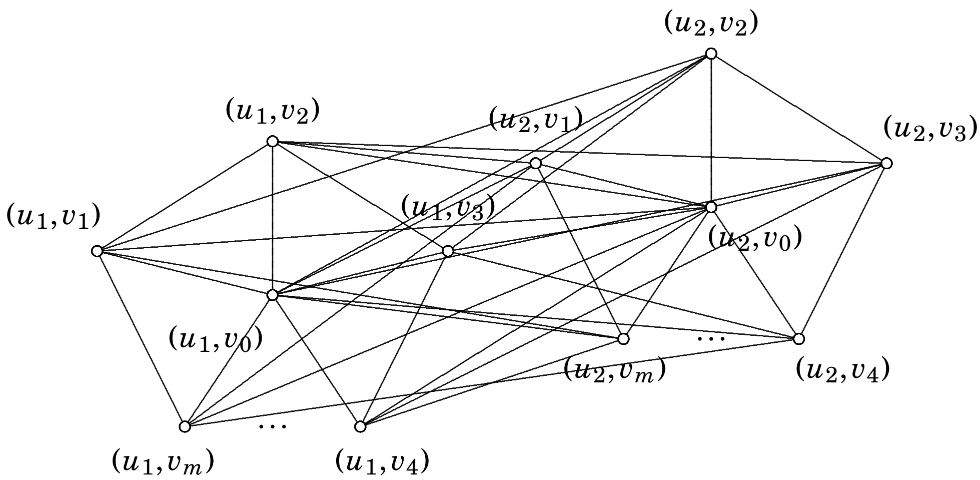


Figure 11. The graph of the bow-tie product $P_2 \bowtie W_m$

Based on its structure, vertex (u_1, j_0) is adjacent to $2m$ vertices and have a distance of 2 to one vertex. The same can be said about vertex (u_2, j_0), so for $i = 1,2$ and $j = 0$,

$$\mathcal{H}_{P_2 \bowtie W_m}(u_i, v_j) = \frac{2m + \frac{1}{2}}{2m+1} = \frac{4m+1}{2(2m+1)}.$$

As for vertex (u_1, v_1), it is adjacent to six vertices and have a distance of 2 to $2m-5$ vertices. The same is exhibited by other vertices (u_i, v_j) where $i = 1,2$ and $1 \le j \le m$, thus,

$$\mathcal{H}_{P_2 \bowtie W_m}(u_i, v_j) = \frac{6 + \frac{1}{2}(2m-5)}{2m+1} = \frac{2m+7}{2(2m+1)}.$$ □

Theorem 3.10. *The harmonic centralization for the bow-tie product of path P_2 of order 2 and wheel graph W_m of order $m+1$ is given by*

$$C_{\mathcal{H}}(P_2 \bowtie W_m) = \frac{2m-6}{2m+1}.$$

Proof. The maximum harmonic centrality of $P_2 \bowtie W_m$ is $\frac{4m+1}{2(2m+1)}$ from which $2m$ vertices with harmonic centrality of $\frac{2m+7}{2(2m+1)}$ is deducted and normalized by dividing by m, thus

$$C_{\mathcal{H}}(P_2 \bowtie W_m) = \frac{2m(4m+1-2m-7)}{2m(2m+1)} = \frac{2m-6}{2m+1}.$$ □

4. Conclusion

We introduced a newly named binary operation on graphs called the bow-tie product. Some properties as well as some results on both harmonic centrality and harmonic centralization of the bow-tie product of the path P_2 with any of the path P_m, cycle C_m, star S_m, fan F_m, and wheel W_m were also presented. For further studies, results can be derived for other families of graphs using the bow-tie product on graphs.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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