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Complex Fibonacci p-Numbers

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Abstract In the present paper, the complex Fibonacci p-numbers are defined by two-dimensional recurrence relation and some results are obtained.

1. Introduction

The complex Fibonacci numbers are considered by many authors. Harman [1] introduced complex Fibonacci numbers at Gaussian integers by two dimensional recurrence relation. In [1], for $n,m \in \mathbb{Z}$ and $(n,m) = n + im$, $G(n,m)$ numbers satisfy the following recurrence relations

$$G(n+2,m) = G(n+1,m) + G(n,m),$$
$$G(n,m+2) = G(n,m+1) + G(n,m),$$

with initial conditions

$$G(0,0) = 0, \quad G(1,0) = 1, \quad G(0,1) = i, \quad G(1,1) = 1+i\,.$$

In [1], Harman defined the complex Fibonacci numbers as

$$G(n,m) = F_{m+1}F_n + iF_{n+1}F_m.$$

Taking $m = 1$, $G(n,m)$ is the nth particular complex Fibonacci number

$$F_n^* = F_n + iF_{n+1}$$

given by Horadam [2].

In [3], Pethe defined the generalized Gaussian Fibonacci numbers. On the other hand, Berzsenyi [4] gave a different method by defining Gaussian Fibonacci

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numbers. Berzsenyi defined the Gaussian Fibonacci numbers F_{n+mi}, for n an integer and m a nonnegative integer by

$$F_{n+mi} = \sum_{k=0}^{m} \binom{m}{k} i^k F_{n-k}. \tag{1.1}$$

From (1.1), F_{n+mi} also satisfy

$$F_{n+mi} = F_{(n-1)+mi} + F_{(n-2)+mi}.$$

For a fixed m (or similarly a fixed n), in Harman's generalization [1], a second order recurrence relation is considered. In this paper for $p \geq 1$ and n, m be nonnegative integers, Harman's recurrence relation is generalized to a particular pth order recurrence relation, i.e.,

$$G_p(n+1,m) = G_p(n,m) + G_p(n-p,m), \quad n > p, \tag{1.2}$$

$$G_p(n,m+1) = G_p(n,m) + G_p(n,m-p), \quad m > p, \tag{1.3}$$

with initial conditions for all $r,s \in \{0,1,2,\ldots,p\}$

$$G_p(r,s) = F_{p,r} + iF_{p,s}, \tag{1.4}$$

where $F_{p,n}$ is the nth Fibonacci p-number.

Many authors deal with Fibonacci p-numbers and their applications ([5],[6],[7]). In [5], for any integer $p \geq 0$, $n \in \mathbb{Z}$ and $n > p$, the nth Fibonacci p-number is given by the recurrence relation

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \tag{1.5}$$

with the initial conditions

$$F_p(0) = 0,\ F_p(1) = F_p(2) = \cdots = F_p(p) = 1.$$

The Lucas p-numbers hold the same recurrence relation with the initial conditions

$$L_p(0) = p+1,\ L_p(1) = \cdots = L_p(p) = 1.$$

In the case $p = 1$ the classical Fibonacci numbers are obtained.

The well known relationship between $F_p(n)$ and $L_p(n)$ is given by

$$L_p(n) = F_p(n+1) + pF_p(n-p). \tag{1.6}$$

Throughout this paper the nth Fibonacci p-number and Lucas p-number will be denoted by $F_{p,n}$ and $L_{p,n}$, respectively.

2. Main Results

In this section some properties related to the recurrence relations in (1.2) and (1.3) are presented. It is obvious that

$$G_p(n,m) = \begin{cases} F_{p,n}, & m = 0 \\ iF_{p,m}, & n = 0. \end{cases} \tag{2.1}$$

Proposition 1.

$$G_p(n,1) = F_{p,n-p}G_p(0,1) + F_{p,n}G_p(1,1). \tag{2.2}$$

Proof. Use induction on n. Since $F_{p,-p} = 1$, $F_{p,0} = 0$ and from the initial conditions in (1.4), for $n = 0$ (2.2) is true. Suppose that (2.2) is true for $n-1$. From (1.2),

$$G_p(n,1) = G_p(n-1,1) + G_p(n-p-1,1).$$

By induction hypothesis,

$$\begin{aligned} G_p(n,1) = {} & F_{p,n-p-1}G_p(0,1) + F_{p,n-1}G_p(1,1) \\ & + F_{p,n-2p-1}G_p(0,1) + F_{p,n-p-1}G_p(1,1). \end{aligned}$$

Rearranging the RHS use of (1.5) shows that (2.2) is true for n. So, for all nonnegative integers (2.2) is true. □

In (2.2) by replacing the initial values in (1.4)

$$G_p(n,1) = F_{p,n} + iF_{p,n+1} \tag{2.3}$$

is obtained. By using (2.1), (2.3) can be written as

$$G_p(n,1) = G_p(n,0) + G_p(0,n+1).$$

Proposition 2.

$$G_p(n,m) = F_{p,m-p}G_p(n,0) + F_{p,m}G_p(n,1). \tag{2.4}$$

Proof. The proof is similar to Proposition 1. □

Proposition 3.

$$G_p(n,m) = F_{p,m+1}F_{p,n} + iF_{p,n+1}F_{p,m}\,. \tag{2.5}$$

Proof. Consider (2.4) with (2.1) and (2.3) gives

$$G_p(n,m) = F_{p,m-p}F_{p,n} + F_{p,m}(F_{p,n} + iF_{p,n+1}).$$

From (1.5),

$$G_p(n,m) = F_{p,m+1}F_{p,n} + iF_{p,n+1}F_{p,m}\,.$$

This completes the proof. □

Thus $G_p(n,m)$ can be written in terms of Fibonacci p-numbers. The complex Fibonacci p-numbers $G_p(n,m)$ can be defined by (2.5).

For $p = 1$, (2.5) gives the Harman's definition in [1].

In [7], Tuglu et al. give the sum of bivariate Fibonacci p-polynomials. Taking $x = y = 1$ in ([7], Prop. 6) gives the sum of Fibonacci p-numbers. So, from (2.1) and (2.3) the following sums are obvious.

$$\sum_{n=0}^{k} G_p(n,0) = F_{p,k+p+1} - 1,$$

$$\sum_{n=0}^{k} G_p(n,1) = (F_{p,k+p+1} - 1) + i(F_{p,k+p+2} - 1).$$

Taking $m = n$, it is obvious that

$$\sum_{n=1}^{k} G_p(n,n) = (1+i)(F_{p,1}F_{p,2} + F_{p,2}F_{p,3} + \cdots + F_{p,k}F_{p,k+1})$$
$$= (1+i)\sum_{n=1}^{k} F_{p,n}F_{p,n+1}.$$

The following proposition gives an analogy of (1.6).

Proposition 4.

$$G_p(n+1,m) + pG_p(n-p,m) = G_p(m+1,0)L_{p,n} + G_p(0,m)L_{p,n+1}.$$

Proof. From (2.5) and (1.6),

$$\begin{aligned} &G_p(n+1,m) + pG_p(n-p,m) \\ &\quad = F_{p,m+1}F_{p,n+1} + iF_{p,m}F_{p,n+2} + p(F_{p,m+1}F_{p,n-p} + iF_{p,m}F_{p,n-p+1}) \\ &\quad = (F_{p,n+1} + pF_{p,n-p})F_{p,m+1} + i(F_{p,n+2} + pF_{p,n-p+1})F_{p,m} \\ &\quad = L_{p,n}F_{p,m+1} + iL_{p,n+1}F_{p,m}. \end{aligned}$$

Use of (2.1) ends the proof. □

Thinking (1.2) and (1.3) jointly,

$$\begin{aligned} G_p(n+2,m+2) &= G_p(n+1,m+1) + G_p(n+1,m-p+1) \\ &\quad + G_p(n-p+1,m+1) + G_p(n-p+1,m-p+1) \end{aligned}$$

is obvious. This new recurrence relation denotes that each complex Fibonacci p-number $G_p(n,m)$ is sum of the four previous numbers at the vertices of a square on Gaussian lattice.

For $n = 5, m = 4$, it is clear that

$$\begin{aligned} G_3(7,6) &= 16 + 15i \\ &= G_3(6,5) + G_3(6,2) + G_3(3,5) + G_3(3,2). \end{aligned}$$

By choosing the initial conditions pursuant to Lucas p-sequence, consider the recurrence relations in (1.2) and (1.3) with initial conditions for all $r,s \in \{0,1,2,\ldots,p\}$

$$G_p(r,s) = L_{p,r} + iF_{p,s}, \tag{2.6}$$

where $F_{p,n}$ and $L_{p,n}$ is the nth Fibonacci and Lucas p-number, respectively.

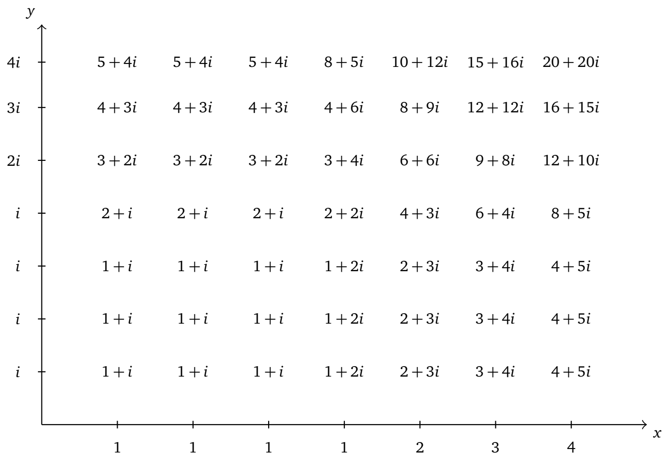

Figure 1. The first $G_p(n,m)$ numbers for $p = 3$.

By applying the same procedure, the propositions below can be given.

Proposition 5.

$$G_p(n,0) = L_{p,n}\,. \tag{2.7}$$

Proof. The proof is obvious. □

Proposition 6.

$$G_p(n,1) = F_{p,n-p}G_p(0,1) + F_{p,n}G_p(1,1)\,. \tag{2.8}$$

Proof. The proof can be seen by induction on n. □

Proposition 7.

$$G_p(n,1) = L_{p,n} + iF_{p,n+1}\,.$$

Proof. From (2.8) and (2.6),

$$\begin{aligned} G_p(n,1) &= F_{p,n-p}(L_{p,0} + iF_{p,1}) + F_{p,n}(L_{p,1} + iF_{p,1}) \\ &= F_{p,n-p}(p + 1 + i) + F_{p,n}(1 + i) \\ &= F_{p,n+1} + pF_{p,n-p} + iF_{p,n+1} \\ &= L_{p,n} + iF_{p,n+1}\,. \end{aligned}$$ □

Proposition 8.

$$G_p(n,m) = F_{p,m-p}G_p(n,0) + F_{p,m}G_p(n,1)\,. \tag{2.9}$$

Proof. The proof is obvious from induction on m. □

Replacing the values in (2.7) and (2.8) to (2.9), it is concluded that

$$G_p(n,m) = F_{p,m+1}L_{p,n} + iF_{p,n+1}F_{p,m}\,.$$

This paper outlines the concept of complex Fibonacci p-numbers and their applications and illustrates that this kind of generalization is possible for sequences in similar feature.

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