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Research Article

Two Regular Polygons with a Shared Vertex

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Abstract. For two non-congruent regular polygons of the same type, the method of finding a point in the plane at the equal distances to the vertices, is established. The existence of two points with this property is proved for two polygons with a shared vertex. For one of them, it is proved that it satisfies the Bottema theorem conditions and based on this, the generalized Bottema theorem for any two regular polygons is given.

Keywords. Bottema theorem, Cyclic averages, Regular polygon, Common vertex

Mathematics Subject Classification (2020). 51M15, 51M20, 51M35

1. Introduction

For a regular n-sided polygon $A_1A_2\cdots A_n$ and an arbitrary point M in the plane of the polygon, the distances from M to the vertices $A_1,A_2,\dots,A_n$ satisfy [2,3]:

$$\sum_{i=1}^{n} d_i^{2m} = n\left[(R^2+L^2)^m + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k}\binom{2k}{k} R^{2k}L^{2k}(R^2+L^2)^{m-2k}\right], \tag{1.1}$$

where $m=1,\dots,n-1$; R is the radius of the circumcircle Ω and L is the distance between M and the centroid O of the regular polygon.

Let us the second n-sided polygon $B_1B_2\cdots B_n$ is given in the plane. The distance from the point M to the vertices $B_1,B_2,\dots,B_n$ denote by $t_1,t_2,\dots,t_n$. Therefore, for given two n-sided regular polygons and the point M, we have two sets of distances:

$\{d_i\}$ and $\{t_i\}$

are there points in the plane, which have the same set of these distances? This problem is investigated in the present paper.

Denote by R_2 and L_2 the radius of the circumcircle Ω_2 and the distance between M and centroid O_2 of the regular polygon $B_1B_2\cdots B_n$. Equalize the right sides of (1.1), we get

$$(R_1^2+L_1^2)^m+\sum_{k=1}^{\lfloor \frac{m}{2}\rfloor}\binom{m}{2k}\binom{2k}{k}R_1^{2k}L_1^{2k}(R_1^2+L_1^2)^{m-2k}$$
$$=(R_2^2+L_2^2)^m+\sum_{k=1}^{\lfloor \frac{m}{2}\rfloor}\binom{m}{2k}\binom{2k}{k}R_2^{2k}L_2^{2k}(R_2^2+L_2^2)^{m-2k}.$$

For any n-gons, the first two relations ($m=1,2$) are:

$$R_1^2+L_1^2=R_2^2+L_2^2,$$
$$(R_1^2+L_1^2)^2+2R_1^2L_1^2=(R_2^2+L_2^2)^2+2R_2^2L_2^2.$$

So, we obtain two cases:

congruent case: $R_1=R_2$ and $L_1=L_2$,

non-congruent case: $R_1=L_2$ and $L_1=R_2$.

Equalize the left sides of (1.1), we get the $n-1$ relations for distances:

$$\left.\begin{aligned} d_1^2+d_2^2+\cdots+d_n^2&=t_1^2+t_2^2+\cdots+t_n^2,\\ d_1^4+d_2^4+\cdots+d_n^4&=t_1^4+t_2^4+\cdots+t_n^4,\\ &\vdots\\ d_1^{2(n-1)}+d_2^{2(n-1)}+\cdots+d_n^{2(n-1)}&=t_1^{2(n-1)}+t_2^{2(n-1)}+\cdots+t_n^{2(n-1)}.\end{aligned}\right\} \qquad (*)$$

Let's consider the cases separately.

2. Congruent Regular Polygons

Congruent n-gons case is divided into two subcases: $O_1=O_2$ and $O_1\neq O_2$.

If two polygons centroids coincide, we get two n-gons inscribed in the same circle and they differ only by rotation around the common centroid, see Figure 1 (for figures of any n-gons we use the squares in the Figures 1-6).

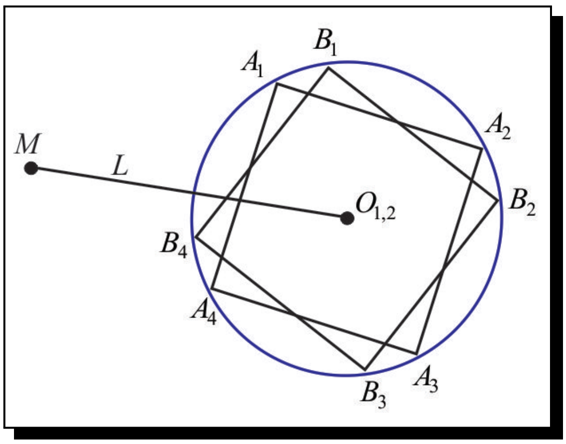

Figure 1

For M, we can take any point in the plane.

Proposition 2.1 (Rotational invariant). *If two congruent polygons are inscribed in the same circle, the distances from any point in the plane to the vertices of the polygons satisfy the system* $(*)$.

If $O_1 \neq O_2$, it is clear M lies on the perpendicular bisector of the line segment O_1O_2 (see Figure 2).

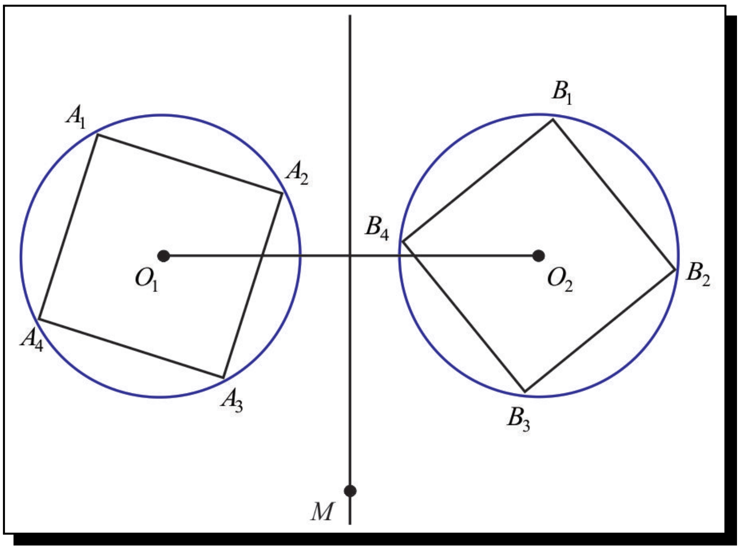

Figure 2

For M, we can take any point on the perpendicular bisector.

Proposition 2.2 (Reflection invariant). *If two congruent polygons are inscribed in symmetrical circles, the distances from any point on the axis of symmetry to the vertices of polygons satisfy the system* $(*)$.

3. Non-Congruent Regular Polygons

From the conditions of non-congruent case:

$$R_1 = L_2 \quad \text{and} \quad L_1 = R_2$$

follow — the point M is the intersection point of two circles:

$$\Omega_1(O_2, R_1) \quad \text{and} \quad \Omega_2(O_1, R_2).$$

It is clear, such point exists, if

$$|R_1 - R_2| \le O_1O_2 \le R_1 + R_2, \tag{3.1}$$

and if the circles intersect each other there are two points M_1 and M_2 of such property (see Figure 3).

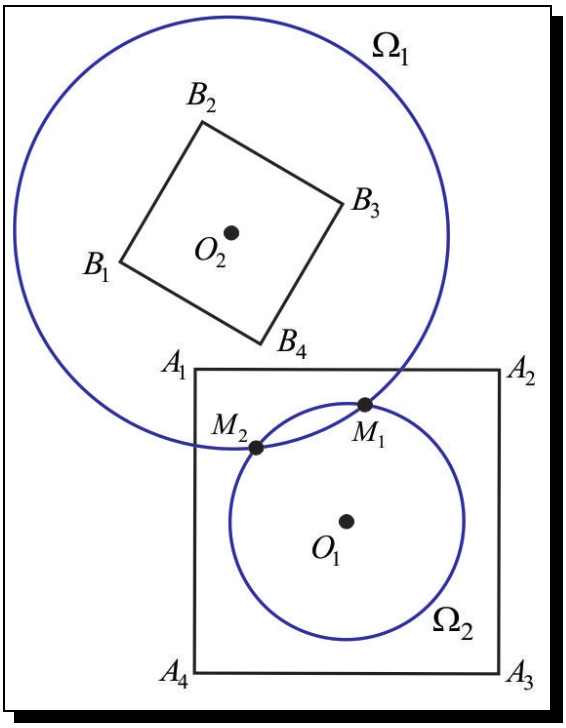

Figure 3

For M, we can take only points M_1 and M_2.

Proposition 3.1 (Necessary condition)**.** *For two non-congruent n-gons with centroids: O_1, O_2, circumradiii: R_1, R_2 and circumcircles Ω_1, Ω_2, the only points of the intersection*

$$\{M_1,M_2\} = \Omega_1(O_2,R_1) \cap \Omega_2(O_1,R_2)$$

satisfy the system $(*)$.

Until now we consider all cases for which the distances from the point to the vertices of two regular n-gons satisfy the system $(*)$, but for equality of the distances the system $(*)$ is only necessary condition. Let us establish the sufficient condition.

4. Equalization of Distances

Let us consider the distances from the given point M_1 to the vertices of the second regular polygon $B_1B_2\cdots B_n$ as variables:

$$t_1, t_2, \dots, t_n.$$

In the system $(*)$, we have $n-1$ equations and n variables in order to determine the variables uniquely we must eliminate one of them. It is possible by using Proposition 2.1. The rotation does not change the system $(*)$ thus we can rotate the second polygon, so that one pair of distances will be equal to each other.

Explain this procedure by using Figure 3. Construct circumscribe circles of the second polygon Ω_2, whose center is O_2. From the point M_1 as the center draw the auxiliary circle with radius M_1A_1.

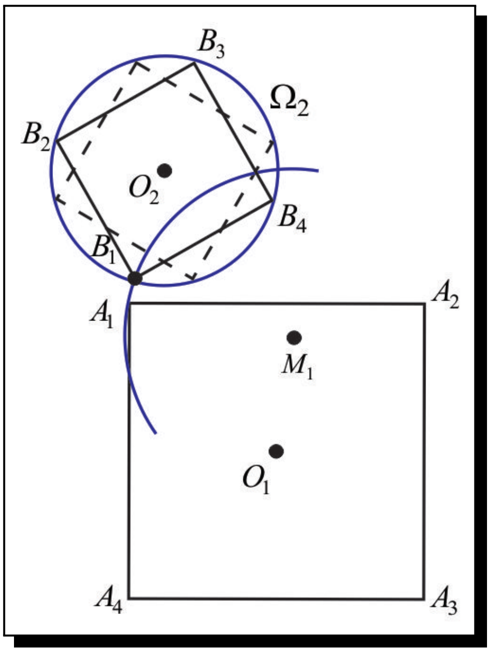

Figure 4

The intersection point of the auxiliary and Ω_2 circles is the new position of the vertex B_1 (see Figure 4). For the new position of the polygon $B_1B_2\cdots B_n$, we obtain

$$t_1 = d_1. \tag{$**$}$$

The condition $(**)$ and the system $(*)$, give us a new system:

$$\left.\begin{aligned} d_2^2 + \cdots + d_n^2 &= t_2^2 + \cdots + t_n^2, \\ d_2^4 + \cdots + d_n^4 &= t_2^4 + \cdots + t_n^4, \\ &\vdots \\ d_2^{2(n-1)} + \cdots + d_n^{2(n-1)} &= t_2^{2(n-1)} + \cdots + t_n^{2(n-1)}. \end{aligned}\right\} \tag{$***$}$$

Now the number of the variables equals to the number of the equations. By using Newton's Identities, elementary symmetric polynomials can be expressed in terms of power sums [4, 6]. Because of $(***)$, the power sums of $(d_2^2,\dots,d_n^2)$ equal to the power sums of $(t_2^2,\dots,t_n^2)$, so the corresponding elementary polynomials are the same, so

$$d_2^2,\dots,d_n^2 \quad \text{and} \quad t_2^2,\dots,t_n^2$$

are the roots of the same equation of degree $n-1$. Consequently, $t_2,\dots,t_n$ are the permutation of the $d_2,\dots,d_n$, therefore:

Proposition 4.1 (Sufficient condition)**.** *If in the system* $(***)$*, one pair of the distances is the same* $(t_1 = d_1)$*, from that it follows the equality of the sets*

$$\{t_2,\dots,t_n\} = \{d_2,\dots,d_n\}.$$

For congruent case, the rotation gives the coincidence (if $O_1 = O_2$, see Figure 1) and reflexion symmetry (if $O_1 \neq O_2$, see Figure 2).

For non-congruent polygons, we obtain:

Theorem 4.1. *If two non-congruent regular polygons are given in the plane, it is possible to fix both of them, so that there is the point M (in the plane) at equal distances from the vertices of the polygons. The point M satisfies:*

I. $M = \Omega_1(O_2,R_1) \cap \Omega_2(O_1,R_2)$; *where* Ω_1,Ω_2 *— are circumscribed circles,* R_1,R_2 *— circumradii and* O_1,O_2 *— centroids of the polygons.*

II. *One pair of distances must be equal, which is possible by rotation of one polygon over its centroid.*

By convenient enumeration of the vertices, we get

$$MA_k = MB_k, \quad \text{where } k = 1,\dots,n.$$

5. A Shared Vertex

If two n-gons have a shared vertex, the equality $(**)$ automatically holds for M_1 and M_2, so we do not need the rotation. For non-congruent case, we obtain two regular polygons and two points theorem.

Theorem 5.1. *If in the plane two regular non-congruent polygons of the same type have a shared vertex, there are two points in the plane separately having equal distances to the vertices of the polygons.*

The point M_1 and M_2 satisfy:

$$\{M_1,M_2\} = \Omega_1(O_2,R_1) \cap \Omega_2(O_1,R_2).$$

In the Figure 5 is given the case of two squares with shared vertex — A_1:

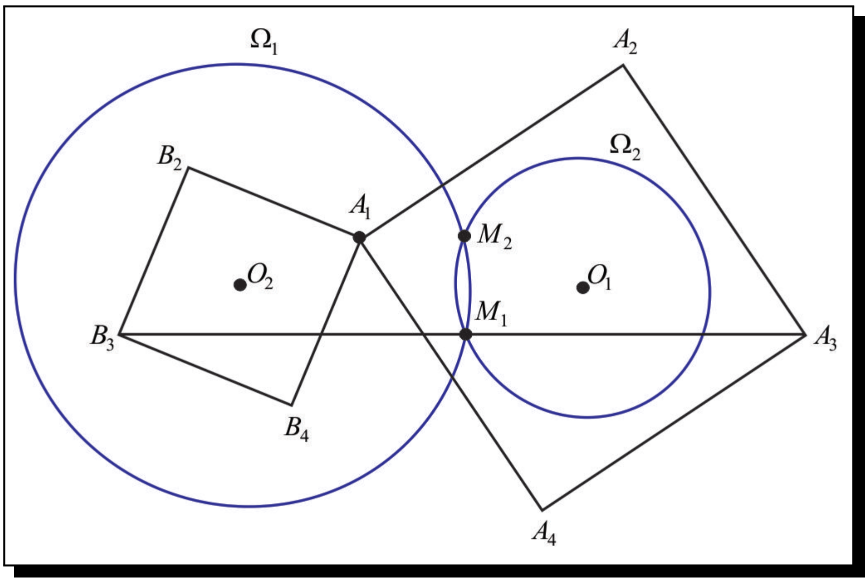

Figure 5

The same distances are:

$$M_1A_2 = M_1B_2, \quad M_1A_3 = M_1B_3, \quad M_1A_4 = M_1B_4 \tag{5.1}$$

and

$$M_2A_2 = M_2B_4, \quad M_2A_3 = M_2B_3, \quad M_2A_4 = M_2B_2. \tag{5.2}$$

In case of the shared vertex, from the triangle — $O_1A_1O_2$ the condition (3.1) is always held, so generally there are two points. If the shared vertex and the centroids are collinear, there is only one point having equal distances to the vertices (see Figure 6)

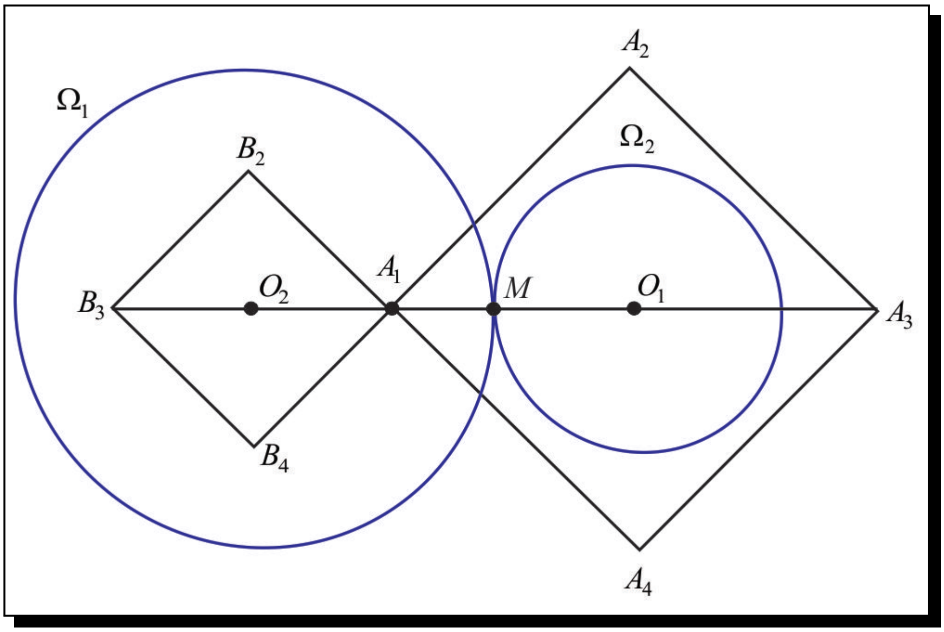

Figure 6

6. Corresponding Equal Distances

The equal distances (5.1) and (5.2), for points M_1 and M_2 are different in the order. If we take the vertices of the first polygon in clockwise direction:

$A_2, A_3, A_4,$

the vertices of the corresponding equal distances for the second polygon are:

- for M_2, in clockwise direction: B_4, B_3, B_2;
- for M_1, in anticlockwise direction: B_2, B_3, B_4;

i.e., the vertices of corresponding equal distances for points M_1 and M_2 are in opposite order. Is it true for any regular polygons of the same type? Let us consider two regular n-gons with the shared vertex A_1 (see Figure 7 and Figure 8).

For the point M_1 in Figure 7, we have

$$\triangle M_1A_1O_2 = \triangle M_1O_1A_1, \quad \text{so} \quad \angle A_1O_2M_1 = \angle M_1O_1A_1 \equiv \alpha.$$

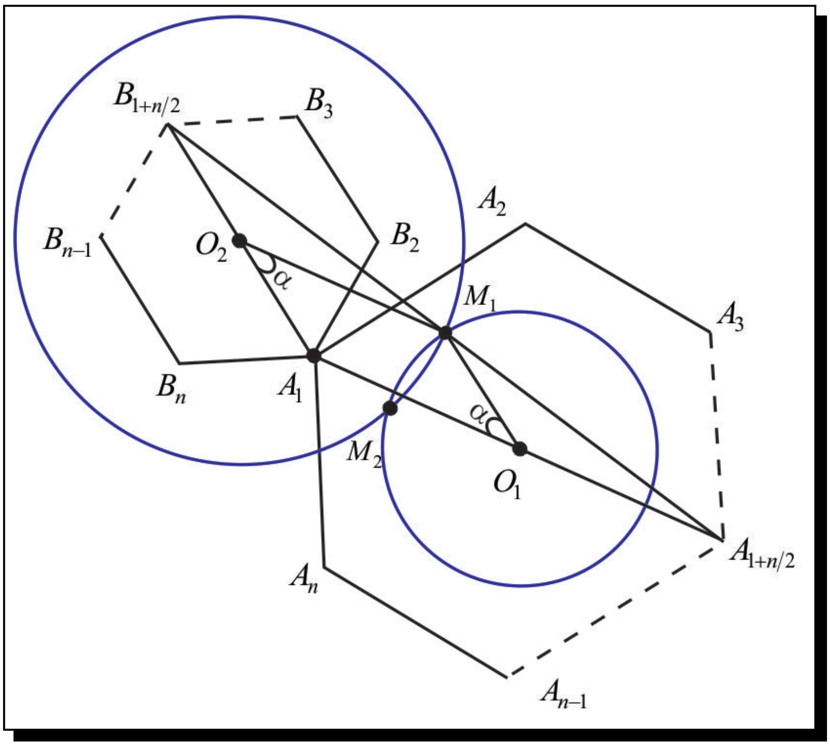

Figure 7

Then,

$$M_1A_2^2 = R_1^2 + R_2^2 - 2R_1R_2\cos\left(\frac{2\pi}{n} - \alpha\right),$$
$$M_1B_2^2 = R_1^2 + R_2^2 - 2R_1R_2\cos\left(\frac{2\pi}{n} - \alpha\right),$$
$$\vdots$$
$$M_1A_k^2 = R_1^2 + R_2^2 - 2R_1R_2\cos\left((k-1)\frac{2\pi}{n} - \alpha\right),$$
$$M_1B_k^2 = R_1^2 + R_2^2 - 2R_1R_2\cos\left((k-1)\frac{2\pi}{n} - \alpha\right).$$

Therefore,

$$M_1A_k = M_1B_k, \quad \text{where } k = 2,\dots,n,$$

i.e., the vertices of the corresponding equal distances are in anticlockwise direction.

In the same manner for the point M_2, we have (see Figure 8):

$$\triangle M_2O_1A_1 = \triangle M_2O_2A_1, \quad \angle M_2O_1A_1 = \angle M_2O_2A_1 \equiv \beta.$$

Then,

$$M_2A_2^2 = R_1^2 + R_2^2 - 2R_1R_2\cos\left(\frac{2\pi}{n} + \beta\right),$$
$$M_2B_n^2 = R_1^2 + R_2^2 - 2R_1R_2\cos\left(\frac{2\pi}{n} + \beta\right),$$
$$\vdots$$

$$M_2A_k^2 = R_1^2 + R_2^2 - 2R_1R_2\cos\left((k-1)\frac{2\pi}{n} + \beta\right),$$

$$M_2B_{n+2-k}^2 = R_1^2 + R_2^2 - 2R_1R_2\cos\left((k-1)\frac{2\pi}{n} + \beta\right).$$

Therefore,

$$M_2A_k = M_2B_{n+2-k}, \quad \text{where } k = 2, \dots, n,$$

i.e., the vertices of the corresponding equal distances are in clockwise direction.

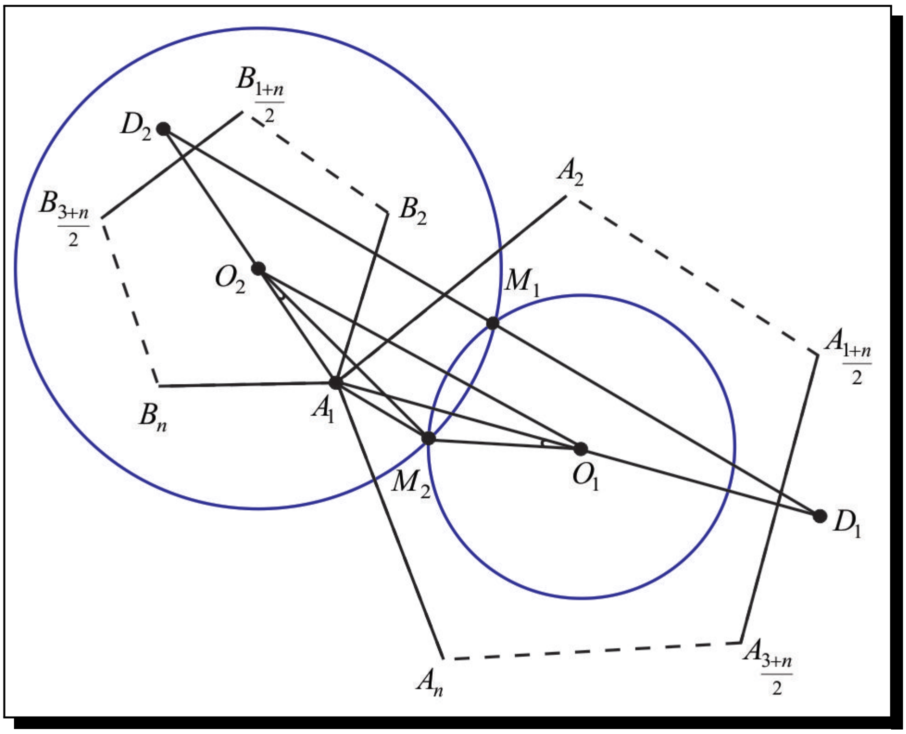

Figure 8

We obtain:

Theorem 6.1. *There are two points M_1 and M_2 in the plane having the equal distances to the vertices of two regular polygons $A_1A_2\cdots A_n$ and $A_1B_2\cdots B_n$ with a shared vertex A_1:*

$$M_1A_k = M_1B_k \quad \textit{and} \quad M_2A_k = M_2B_{n+2-k},$$

where $k = 2, \dots, n$.

7. Properties of M_1 and M_2

If the number n of the sides of the polygons is even, then diametrically opposed points of the shared vertex A_1 are the vertices

$$A_{1+\frac{n}{2}} \quad \text{and} \quad B_{1+\frac{n}{2}},$$

see Figure 7.

The quadrilateral $O_2M_1O_1M_2$ is parallelogram with sides: R_1, R_2, then

$$\angle B_{1+\frac{n}{2}}M_1A_{1+\frac{n}{2}} = \angle B_{1+\frac{n}{2}}M_1O_2 + \angle O_2M_1O_1 + \angle A_{1+\frac{n}{2}}M_1O_1 = \pi,$$

because of

$$\angle B_{1+\frac{n}{2}}M_1O_2 = \angle M_1A_{1+\frac{n}{2}}O_1 \quad \text{and} \quad \angle M_1A_{1+\frac{n}{2}}O_1 + \angle A_{1+\frac{n}{2}}M_1O_1 = \angle M_1O_1A_1.$$

Property 7.1. *If n is even, the point M_1 is the midpoint of the line segment $A_{1+\frac{n}{2}}B_{1+\frac{n}{2}}$.*

If the number n is odd, diametrically opposed points D_1, D_2 of the A_1 are the midpoints of the arcs

$$A_{\frac{1+n}{2}}A_{\frac{3+n}{2}} \quad \text{and} \quad B_{\frac{1+n}{2}}B_{\frac{3+n}{2}},$$

of the circles $\Omega_1(O_1,R_1)$, $\Omega_2(O_2,R_2)$, respectively (see Figure 8).

In odd case we can "double" the number of the vertices and then use the Property 7.1. For even case

$$D_1 = A_{1+\frac{n}{2}} \quad \text{and} \quad D_2 = B_{1+\frac{n}{2}},$$

so for both cases, we have:

Property 7.2. *The point M_1 is the midpoint of the line segment D_1D_2, where D_1, D_2 are diametrically opposed points of the shared vertex of the polygons.*

The triangle $M_2D_1D_2$ is isosceles (Theorem 6.1, $k = 1+\frac{n}{2}$), so

$$M_1M_2 \perp D_1D_2.$$

In the quadrelateral $A_1M_2O_1O_2$ (see Figure 8)

$$M_2O_1 = O_2A_1 \quad \text{and} \quad A_1O_1 = M_2O_2,$$

so $A_1M_2O_1O_2$ is the isosceles trapezoid and

$$A_1M_2 \parallel O_2O_1.$$

The line segment O_1O_2 is the midsegment of the triangle $D_1A_1D_2$, so we obtain:

Property 7.3. *The point M_2 lies on the perpendicular bisector of D_1D_2 and $A_1M_2 \parallel D_1D_2$.*

Property 7.4. *The line segment M_1M_2 is equal to the distance from the shared vertex to the line D_1D_2.*

8. Generalized Bottema Theorem

The Bottema theorem concerns two squares, which have a common vertex. The theorem can be stated as follows (see Figure 5) [1, 5]:

In any given triangle $A_4A_1B_4$, construct two squares on two sides A_1A_4 and A_1B_4. The midpoint of the line segment that connects the vertices of squares opposite the common vertex A_1, is independent of the location of A_1.

Let us change two squares by two regular polygons of the same number of sides in the Bottema theorem:

Theorem 8.1. *In any given triangle $A_nA_1B_n$, construct two regular n-gons $A_1A_2\cdots A_n$ and $A_1B_2\cdots B_n$ on two sides A_1A_n and A_1B_n. Take the points D_1, D_2 on the circumcircles of the* *polygons, which are diametrically opposed of the common vertex A_1. Then, the midpoint of the line segment D_1D_2 is independent of the location of A_1.*

Proof. In our notations (see Figure 7 and Figure 8) the midpoint is M_1, and from Theorem 6.1:

$$M_1A_k = M_1B_k, \quad \text{where } k = 2, \dots, n.$$

Draw the altitude M_1H of the triangle $A_nM_1B_n$ (see Figure 9). The triangle $A_nM_1B_n$ is isosceles $M_1A_n = M_1B_n$. Because

$$M_1O_1 = O_2B_n = R_2,$$

$$M_1O_2 = O_1A_n = R_1;$$

so

$$\triangle A_nO_1M_1 = \triangle M_1O_2B_n.$$

Because of $\angle M_1O_1A_1 \equiv \alpha$, then

$$\begin{aligned}\angle A_nA_1B_n &= 2\pi - \left(\angle O_1A_1O_2 + \angle O_1A_1A_n + \angle O_2A_1B_n\right)\\ &= \alpha + \frac{2\pi}{n} = \angle A_nO_1M_1\\ &= \angle B_nO_2M_1.\end{aligned}$$

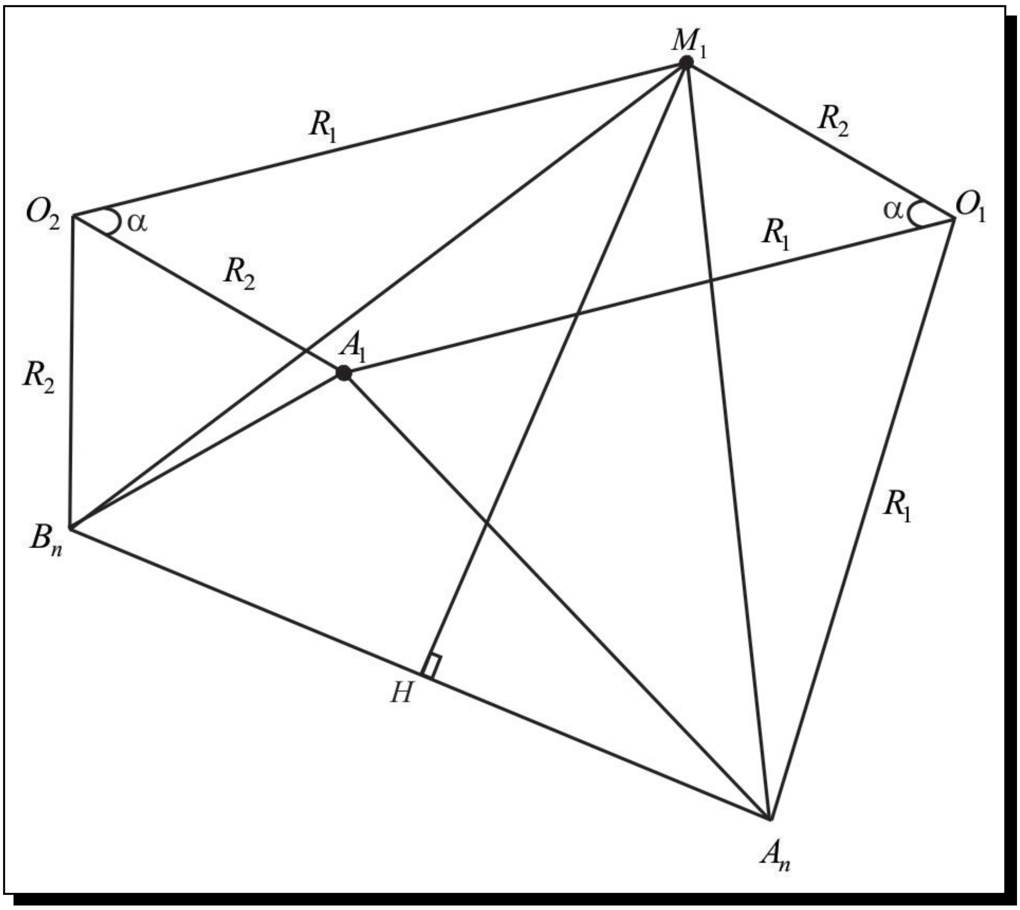

Figure 9

The sides A_1A_n, A_1B_n are proportional to R_1, R_2, respectively. Therefore,

$$\triangle A_nO_1M_1 \sim \triangle A_nA_1B_n,$$

so

$$\angle B_nA_nA_1 = \angle O_1A_nM_1.$$

Then,

$$\begin{aligned}\angle A_nM_1B_n &= 2\cdot\angle A_nM_1H \\ &= 2\left(\frac{\pi}{2} - \angle B_nA_nM_1\right) \\ &= \pi - 2(\angle B_nA_nA_1 + \angle M_1A_nA_1) \\ &= \pi - 2(\angle O_1A_nM_1 + \angle M_1A_nA_1) \\ &= \pi - 2\angle O_1A_nA_1 \\ &= \angle A_1O_1A_n \\ &= \frac{2\pi}{n}.\end{aligned}$$

For the position of the point M_1, we have:

$$HM_1 = \frac{1}{2}A_nB_n \cot\frac{\pi}{n}.$$

So the position of the M_1 depends only on the line segment A_nB_n and is independent of the location of A_1. Moreover, the obtained expression for HM_1 also shows that M_1 is the center of the regular n-gon formed on the side A_nB_n. □

For the triangle $\triangle A_2M_1B_2$ (see Figure 7 and Figure 8), analogically we get:

$$\angle A_2M_1B_2 = \frac{2\pi}{n},$$

i.e.,

$$\angle A_nM_1B_n = \angle A_2M_1B_2.$$

In the same manner, if we consider the triangles $A_{n-1}M_1B_{n-1}$ and $A_3M_1B_3$, we have

$$\angle A_{n-1}M_1B_{n-1} = \angle A_3M_1B_3 = \angle A_1O_1A_{n-1},$$

so:

Theorem 8.2. *In two regular n-gons $A_1A_2\cdots A_n$ and $A_1B_2\cdots B_n$ with the shared vertex A_1 and opposite orientation, take the points D_1, D_2 on the circumcircles of the polygons, which are diametrically opposed of the shared vertex. Then, for the midpoint M_1 of the line segment D_1D_2 holds:*

$$\angle A_kM_1B_k = \angle A_{n+2-k}M_1B_{n+2-k} = \frac{2\pi}{n}(k-1), \quad \textit{where } k = 2,\dots,n.$$

9. Conclusion

In the present paper general method is given — how to fix two regular polygons in the plane, so that there is a point M in the plane at equal distances to the vertices of the polygons? The result is trivial for the congruent n-gons, but for two non-congruent regular n-gons we get an unexpected result, which gives the new theorems in Euclidian geometry. For the existence of the point M two conditions must be satisfied:

I. $M = \Omega_1(O_2, R_1) \cap \Omega_2(O_1, R_2)$, where Ω_1, Ω_2: are circumscibed circles, R_1, R_2: circumradii and O_1, O_2: centroids of the regular polygons.

II. One pair of the distances from the point M must be equal, which is obtained by rotation of one of the polygons over its centroid.

In case of two n-gons with the shared vertex, the second condition is satisfied automatically, so we get two points M_1 and M_2. The properties of these points are investigated. We have established that M_1 is the point which satisfies the Bottema theorem ($n = 4$) conditions, from where we obtain generalized Bottema theorem for any two regular polygons.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] N.N. Giang, A new proof and some generalizations of the Bottema theorem, *International Journal of Computer Discovered Mathematics* **3** (2018), 49 – 54, URL: https://www.journal-1.eu/2018/Nguyen-Ngoc-Giang-Bottema-Theorem.pdf.

[2] M. Meskhishvili, Cyclic averages of regular polygons and platonic solids, *Communications in Mathematics and Applications* **11**(3) (2020), 335 – 355, DOI: 10.26713/cma.v11i3.1420.

[3] M. Meskhishvili, Cyclic averages of regular polygonal distances, *International Journal of Geometry* **10**(1) (2021), 58 – 65, URL: https://ijgeometry.com/wp-content/uploads/2020/12/4.-58-65.pdf.

[4] R. Seroul, Newton-Girard formulas, in: *Programming for Mathematicians*, pp. 278 – 279, Springer-Verlag (2000), URL: https://link.springer.com/book/9783540664222?cm_mmc=Google-_-Book%20Search-_-Springer-_-0.

[5] A. Shriki, Back to treasure island, *The Mathematics Teacher* **104**(9) (2011), 658 – 664, DOI: 10.5951/MT.104.9.0658.

[6] J.P. Tignol, *Galois' Theory of Algebraic Equations*, World Scientific, pages 348 (2001), DOI: 10.1142/4628.