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Research Article

Restrained Weakly Connected 2-Domination in the Vertex and Edge Coronas of Graphs

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Abstract. In this paper, we explore the concept of restrained weakly connected 2-domination in the vertex and edge coronas of graphs. In particular, we characterize the restrained weakly connected 2-dominating sets in the vertex and edge coronas of two graphs and obtain the corresponding restrained weakly connected 2-domination numbers of these graphs.

Keywords. Restrained weakly connected 2-domination, Vertex corona, Edge corona

Mathematics Subject Classification (2020). 05C69, 05C76

1. Introduction

Let $G = (V(G), E(G))$ be a simple, finite and undirected connected graph. A vertex u in G is called a *leaf* if $\deg_G(u) = 1$. The set of neighbors of a vertex $u \in V(G)$ is called the *open neighborhood* of u in G, denoted by $N_G(u)$, and the set $N_G[u] = N_G(u) \cup \{u\}$ is called the *closed neighborhood* of u in G. The *open neighborhood* and the *closed neighborhood* of a set $U \subseteq V(G)$ are $N_G(U) = \bigcup_{u\in U} N_G(u)$ and $N_G[U] = U \cup N_G(U)$, respectively. The *subgraph weakly induced* by a subset D of $V(G)$ is the subgraph $\langle D\rangle_w = (N_G[D], E_w)$, where E_w is the set of all edges in G incident with at least one vertex in D.

A set $S \subseteq V(G)$ is a *dominating set* in G if for every $u \in V(G)\setminus S$, there exists $v \in S$ such that $uv \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set in G. A dominating set $S \subseteq V(G)$ with $|S| = \gamma(G)$ is called a *γ-set* in G. Moreover, a dominating set $S \subseteq V(G)$ is a *restrained dominating set* if every vertex in $V(G)\backslash S$ is adjacent to another vertex in $V(G)\backslash S$. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set in G. A restrained dominating set $S \subseteq V(G)$ with $|S| = \gamma_r(G)$ is called a *γ_r-set* in G. The concept of restrained domination was introduced by Domke *et al.* [2]. A dominating set $S \subseteq V(G)$ is called *weakly connected dominating set* in G if the subgraph $\langle S\rangle_w$ weakly induced by S is connected. The *weakly connected domination number* of G, denoted by $\gamma_w(G)$, is the smallest cardinality of a weakly connected dominating set in G. A weakly connected dominating set $S \subseteq V(G)$ with $|S| = \gamma_w(G)$ is called a *γ_w-set* in G. The concept of weakly connected domination was studied by Dunbar *et al.* [3]. A set $D \subseteq V(G)$ is a *2-dominating set* in G if for every $u \in V(G)\backslash D$, $|D \cap N_G(u)| \geq 2$. The *2-domination number* of G, denoted by $\gamma_2(G)$, is the smallest cardinality of a 2-dominating set in G. A 2-dominating set $S \subseteq V(G)$ with $|S| = \gamma_2(G)$ is called a *γ_2-set* in G. This domination concept was studied by Fink and Jacobson [4]. A 2-dominating set $D \subseteq V(G)$ is called a *weakly connected 2-dominating* (WC2D) *set* if the subgraph $\langle D\rangle_w$ weakly induced by D is connected. The *weakly connected 2-domination number* of G, denoted by $\gamma_{2w}(G)$, is the smallest cardinality of a WC2D set in G. Any WC2D set $D \subseteq V(G)$ with $|D| = \gamma_{2w}(G)$ is called a *γ_{2w}-set* in G. This concept was studied by Militante and Eballe [5]. A *restrained weakly connected 2-dominating* (RWC2D) *set* in G is a subset D of $V(G)$ such that every vertex in $V(G)\backslash D$ is dominated by at least two vertices in D and is adjacent to a vertex in $V(G)\backslash D$, and the subgraph $\langle D\rangle_w$ weakly induced by D is connected. The *restrained weakly connected 2-domination number* of G, denoted by $\gamma_{r2w}(G)$, is the smallest cardinality of a RWC2D set in G. Any RWC2D set D with $|D| = \gamma_{r2w}(G)$ is called a *γ_{r2w}-set* in G. The concept of restrained weakly connected 2-domination in graphs was studied by Militante *et al.* [6].

In this paper, we investigate the restrained weakly connected 2-dominating sets in the vertex corona and edge corona of graphs.

Throughout this paper, every graph is considered in the context of being simple, finite, and connected. Readers may refer to Chartrand [1] for other graph theoretic terminologies which are not specifically given here.

Recall that the *union* of two graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, denoted by $G_1 + G_2$ where $V(G_1)$ and $V(G_2)$ are disjoint, is the graph $G_1 + G_2 = (V(G_1)\dot\cup V(G_2), E(G_1)\dot\cup E(G_2))$. The *join* of two graphs G and H, denoted by $G \vee H$, is the graph with vertex set $V(G \vee H) = V(G)\dot\cup V(H)$ and edge set

$$E(G \vee H) = E(G)\dot\cup E(H)\dot\cup \{uv : u \in V(G) \text{ and } v \in V(H)\}.$$

The symbol $\dot\cup$ denotes the disjoint union of sets.

To illustrate these two graph operations, consider the path P_2 and the cycle C_5 in Figure 1(a) and 1(b), respectively. The union $P_2 + C_5$ and the join $P_2 \vee C_5$ of P_2 and C_5 are shown in Figure 1(c) and 1(d), respectively.

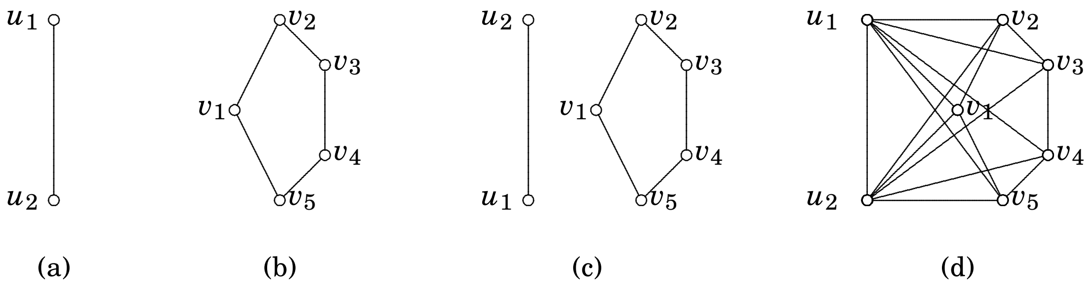


Figure 1. (a) The path P_2, (b) the cycle C_5, (c) the union $P_2 + C_5$, and (d) the join $P_2 \vee C_5$

2. Vertex Corona of Graphs

Let G and H be graphs. The *vertex corona* $G \circ H$ of G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Denote by $v \vee H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle \vee H^v$.

Figure 2(a) and 2(b) show the path P_3 and the cycle C_3, respectively. The vertex corona $P_3 \circ C_3$ of the path P_3 and the cycle C_3 is shown in Figure 2(c).

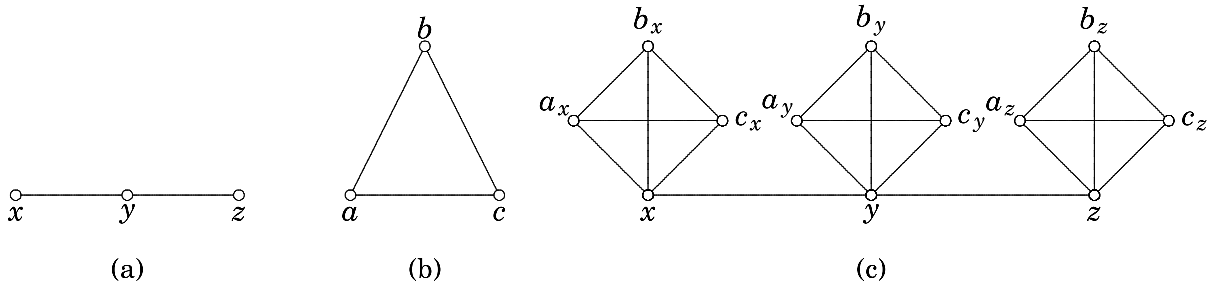


Figure 2. (a) The path P_3, (b) the cycle C_3, and (c) the vertex corona $P_3 \circ C_3$

We need the following published results in this section.

Theorem 2.1 ([5]). *Let G and H be nontrivial graphs with G connected. Then $C \subseteq V(G \circ H)$ is a* WC2D set in $G \circ H$ if and only if $C = D \cup \left(\bigcup_{v \in D} S^v \right) \cup \left(\bigcup_{a \in V(G) \setminus D} T^a \right)$, where D is a weakly connected dominating set in G, S^v is a dominating set in H^v for every $v \in D$, and T^a is a 2-dominating set in H^a for every $a \in V(G) \setminus D$. In particular, if $D = V(G)$, then $C = V(G) \cup \left(\bigcup_{v \in V(G)} S^v \right)$, where S^v is a dominating set in H^v for every $v \in V(G)$.

Theorem 2.2 ([6]). *Let $K_1 = \langle \{v\} \rangle$ and $H = H_1 + H_2 + \ldots + H_p + \left\langle \bigcup_{j=1}^{q} \{u_j\} \right\rangle$ where H_i is a component of H with $|V(H_i)| \geq 3$ for $1 \leq i \leq p$ and u_j is an isolated vertex for $1 \leq j \leq q$. Then $D \subseteq V(K_1 \vee H)$ is a* RWC2D *set in $K_1 \vee H$ if and only if one of the following holds:*

(i) $D = \{v\} \cup \left(\bigcup_{i=1}^{p} S_i \right) \cup \left(\bigcup_{j=1}^{q} \{u_j\} \right)$, *where S_i is a restrained dominating set in H_i for each i.*

(ii) $D = \left(\bigcup_{i=1}^{p} S_i'\right) \cup \left(\bigcup_{j=1}^{q} \{u_j\}\right)$, *where* S_i' *is a 2-dominating set in* H_i *for each* i *and* $S_i' \subsetneq V(H_i)$ *for some* i.

Remark 2.3. If G is connected and $\deg_G(x) = 2$, then x belongs to any RWC2D set in G.

Theorem 2.4. *Let* G *be a nontrivial connected graph. Let* $H = H_1 + H_2 + \ldots + H_k + \left\langle \bigcup_{j=1}^{q} \{u_j\} \right\rangle$ *where* H_i *is a component of* H *with* $|V(H_i)| \geq 3$ *for each* i, $1 \leq i \leq k$ *and* u_j *is an isolated vertex of* H *for each* j, $1 \leq j \leq q$. *Then* $C \subseteq V(G \circ H)$ *is a* RWC2D *set in* $G \circ H$ *if and only if*

$$C = D \cup \left(\bigcup_{v \in D} S^v\right) \cup \left(\bigcup_{a \in V(G)\setminus D} T^a\right),$$

where D *is a weakly connected dominating set in* G, S^v *is a restrained dominating set in* H^v *for every* $v \in D$, *and* T^a *is a 2-dominating set in* H^a *for every* $a \in V(G)\setminus D$, *where* $|T^a| \neq |V(H^a)|$ *whenever* $N_G(a) \cap (V(G)\setminus D) = \varnothing$. *In particular, if* $D = V(G)$, *then* $C = V(G) \cup \left(\bigcup_{v \in V(G)} S^v\right)$, *where* S^v *is a restrained dominating set in* H^v *for every* $v \in V(G)$.

Proof. Suppose $C \subseteq V(G \circ H)$ is a RWC2D set in $G \circ H$. Then C is a WC2D set in $G \circ H$. By Theorem 2.1, $C \cap V(G)$ is a weakly connected dominating set in G. Set $D = C \cap V(G)$.

Let $v \in D$. Since $v \vee H^v \cong K_1 \vee H$ and C is a RWC2D set in $G \circ H$, $C \cap V(v \vee H^v)$ is a RWC2D set in $v \vee H^v$. By Theorem 2.2(i), $C \cap V(H^v)$ is a restrained dominating set in H^v. Let $S^v = C \cap V(H^v)$ for every $v \in D$.

Let $a \in V(G)\setminus D$. Since C is a RWC2D set in $G \circ H$, every $a \in V(G \circ H)\setminus C$ is adjacent to a vertex either in $V(H^a)\setminus C$ or in $V(G)\setminus C$. Since $a \vee H^a \cong K_1 \vee H$, by Theorem 2.2(ii), $C \cap V(H^a)$ is a 2-dominating set in H^a where $C \cap V(H^a) \subsetneq V(H^a)$. If there exists $w \in V(G)\setminus D$ such that $aw \in E(G)$, then $C \cap V(H^a)$ is a 2-dominating set in H^a. This means that if there does not exist $w \in V(G)\setminus C$, then $C \cap V(H^a)$ is a proper subset of $V(H^a)$. Let $T^a = C \cap V(H^a)$ having the following properties: T^a is a 2-dominating set in H^a for every $a \in V(G)\setminus D$, where $|T^a| \neq |V(H^a)|$ whenever $N_G(a) \cap (V(G)\setminus D) = \varnothing$.

Conversely, suppose that $C = D \cup \left(\bigcup_{v \in D} S^v\right) \cup \left(\bigcup_{a \in V(G)\setminus D} T^a\right)$, where D is a weakly connected dominating set in G, S^v is a restrained dominating set in H^v for every $v \in D$ and T^a is a 2-dominating set in H^a for every $a \in V(G)\setminus D$ where $|T^a| \neq |V(H^a)|$ whenever $N_G(a) \cap (V(G)\setminus D) = \varnothing$. Then it is clear that S^v is a dominating set in H^v. By Theorem 2.1, $C = D \cup \left(\bigcup_{v \in D} S^v\right) \cup \left(\bigcup_{a \in V(G)\setminus D} T^a\right)$ is a WC2D set in $G \circ H$. To show that C is a restrained set in $G \circ H$, let $z \in V(G \circ H)\setminus C$. Consider the following cases:

Case 1. $z \in V(G)\setminus D$. By the properties of T^z mentioned in the hypothesis, we consider the following subcases:

Subcase 1.1: There exists $x \neq z$, that is $x \in V(G)\setminus D$ such that $zx \in E(G)$. Thus, $zx \in E(G \circ H)$.

Subcase 1.2: There does not exist $y \in V(G)\backslash D$ such that $zy \in E(G)$. Then $T^z \subsetneq V(H^z)$ by assumption. This means that for $y^* \in V(H^z)\backslash T^z$ it is clear that $zy^* \in E(G \circ H)$.

Case 2. $z \in V(H^v)\backslash S^v$ where $v \in D$. Since S^v is a restrained dominating set in H^v, there exists $z^* \in V(H^v)\backslash S^v$, $z^* \neq z$ such that $z^*z \in E(\langle\{v\}\rangle \vee H^v)$. Thus, $z^*z \in E(G \circ H)$.

Case 3. $z \in V(H^a)\backslash T^a$ where $a \in V(G)\backslash D$. Then $a \in V(G \circ H)\backslash C$. By definition of the corona of graphs, $az \in E(\langle\{a\}\rangle \vee H^a)$ implying that $az \in E(G \circ H)$.

Combining all the observations shown in cases 1, 2 and 3, C is a restrained set in $G \circ H$. Thus, C is a RWC2D set in $G \circ H$. In particular, suppose $D = V(G)$, then $\bigcup_{a \in V(G)\backslash D} T^a = \varnothing$. Thus, $C = V(G) \cup \left(\bigcup_{v \in V(G)} S^v \right)$, where S^v is a restrained dominating set in H^v for every $v \in V(G)$. □

Corollary 2.5. *Let G be a nontrivial connected graph of order m. Let $H = H_1 + H_2 + \ldots + H_k + \left\langle \bigcup_{j=1}^{q} \{u_j\} \right\rangle$ where H_i is a component of H with $|V(H_i)| \geq 3$ for each i, $1 \leq i \leq k$ and $1 \leq j \leq q$. Then*

$$\gamma_{r2w}(G \circ H) = \begin{cases} \gamma_w(G)(1+\gamma_r(H)-\gamma_2(H)) + m\gamma_2(H), & \text{if } 1+\gamma_r(H) > \gamma_2(H) \\ m(1+\gamma_r(H)), & \text{if } 1+\gamma_r(H) \leq \gamma_2(H). \end{cases}$$

That is,

$$\gamma_{r2w}(G \circ H) = \min\{\gamma_w(G)(1+\gamma_r(H)-\gamma_2(H)) + m\gamma_2(H), m(1+\gamma_r(H))\}.$$

Proof. Suppose that C is a γ_{r2w}-set in $G \circ H$. Then by Theorem 2.4,

$$C = D \cup \left(\bigcup_{v \in D} S^v \right) \cup \left(\bigcup_{a \in V(G)\backslash D} T^a \right),$$

where D is a weakly connected dominating set in G, S^v is a restrained dominating set in H^v for every $v \in D$ and T^a is a 2-dominating set in H^a for every $a \in V(G)\backslash D$ where $|T^a| \neq |V(H^a)|$ whenever $N_G(a) \cap (V(G)\backslash D) = \varnothing$. We see that the value $|C|$ depends on $\gamma_r(H)$ and $\gamma_2(H)$, thus the following three cases are considered:

Case 1. Suppose that $1+\gamma_r(H) > \gamma_2(H)$. Since C is a γ_{r2w}-set in $G \circ H$ and $D \neq \varnothing$, it is necessary that set D in the expression for C above must be γ_w-set in G so that

$$\begin{aligned} \gamma_{r2w}(G \circ H) &= |C| \\ &= |D| + |D|\gamma_r(H) + (m-|D|)\gamma_2(H) \\ &= |D|(1+\gamma_r(H)-\gamma_2(H)) + m\gamma_2(H) \\ &= \gamma_w(G)(1+\gamma_r(H)-\gamma_2(H)) + m\gamma_2(H). \end{aligned}$$

Case 2. Suppose that $1+\gamma_r(H) = \gamma_2(H)$. In this case, the set D in the expression for C can be just any weakly connected dominating set in G so that

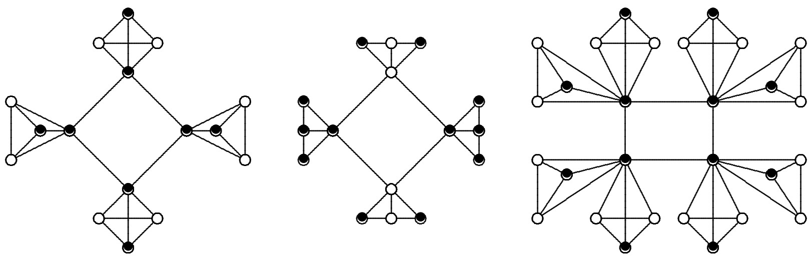
$$\gamma_{r2w}(G \circ H) = \gamma_w(G)(1+\gamma_r(H)-\gamma_2(H)) + m\gamma_2(H) = m(1+\gamma_r(H)).$$

Case 3. Suppose that $1+\gamma_r(H) < \gamma_2(H)$. Since C is a γ_{r2w}-set, $D = V(G)$. Thus, we have $C = V(G) \cup \left(\bigcup_{v \in V(G)} S^v \right)$. Hence, $\gamma_{r2w}(G \circ H) = |C| = |V(G)| + |V(G)|\gamma_r(H) = m(1+\gamma_r(H))$.

From all the three cases above, we have

$$\gamma_{r2w}(G \circ H) = \begin{cases} \gamma_w(G)(1+\gamma_r(H)-\gamma_2(H)) + m\gamma_2(H), & \text{if } 1+\gamma_r(H) > \gamma_2(H), \\ m(1+\gamma_r(H)), & \text{if } 1+\gamma_r(H) \le \gamma_2(H). \end{cases}$$ □

Example 2.6. Figure 3(a,b,c) show that $\gamma_{r2w}(C_4 \circ K_3) = 4(1+\gamma_r(K_3)) = 4(1+1) = 8$, $\gamma_{r2w}(C_4 \circ P_3) = \gamma_w(C_4)(1+\gamma_r(P_3)-\gamma_2(K_3)) + |V(C_4)|\gamma_2(P_3) = 12$ and also, $\gamma_{r2w}(C_4 \circ 2K_3) = |V(C_4)|(1+\gamma_r(2K_3)) = 12$, respectively.



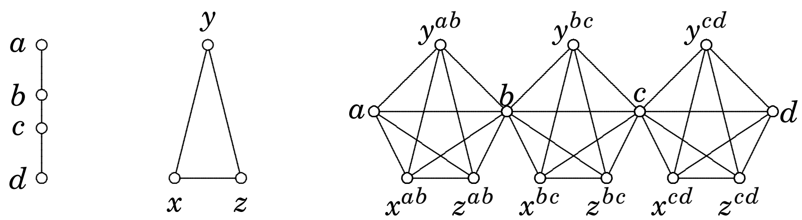
(a) (b) (c)

Figure 3. The graphs of (a) $C_4 \circ K_3$, (b) $C_4 \circ P_3$, (c) $C_4 \circ 2K_3$, with darkened vertices in some of their respective γ_{r2w}-sets

Remark 2.7. Let G be a connected graph of order $m \ge 2$ and $H = H_1 + H_2 + \ldots + H_k$ where H_i is a component of H with $|V(H_i)| \le 2$ for $1 \le i \le k$. Then C is a RWC2D set in $G \circ H$ if and only if $C = D \cup \left(\bigcup_{v \in V(G)} V(H^v) \right)$, where D is a RWCD set in G. Moreover, $\gamma_{r2w}(G \circ H) = \gamma_{rw}(G) + m|V(H)|$.

3. Edge Corona of Graphs

In this section, we obtain results in the edge corona of two graphs. Let G and H be two graphs on disjoint sets of n_1 and n_2 vertices, m_1 and m_2 edges, respectively. The *edge corona* $G \diamond H$ of G and H is the graph obtained by taking one copy of G and m_1 copies of H and then joining the two end-vertices of the ith edge of G to every vertex in the ith copy of H. If $ab \in E(G)$, then the copy H whose vertices are connected one by one to both a and b in $G \diamond H$ is called the ab-copy of H and is denoted by H^{ab}. If $V(H) = \{v_1, v_2, \ldots, v_n\}$, then the vertices of H^{ab} may be denoted by $v_1^{ab}, v_2^{ab}, \ldots, v_n^{ab}$. To illustrate this graph operation, consider the path P_4 and the cycle C_3 in Figure 4(a) and 4(b), respectively. The graph of the edge corona $P_4 \diamond C_3$ is shown in Figure 4(c).



(a) (b) (c)

Figure 4. The (a) path P_4; (b) the cycle C_3; and the edge corona (c) $P_4 \diamond C_3$

We note that $\deg_{G\diamond H}(x) = 2$ for every isolated vertex x in H. Hence, x belongs to any RWC2D set in the edge corona $G \diamond H$ as ascertained in Remark 2.3.

We need the following result in this section.

Theorem 3.1 ([5]). *Let G and H be nontrivial graphs with G connected. Then $T \subseteq V(G\diamond H)$ is a* WC2D *set in $G\diamond H$ if and only if*

$$T = S \cup \Bigg(\bigcup_{\substack{a,b\in V(G)\setminus S\\ ab\in E(G)}} T_0^{[ab]}\Bigg) \cup \Bigg(\bigcup_{\substack{a,b\in V(G), ab\in E(G)\\ |\{a,b\}\cap S|=1}} T_1^{[ab]}\Bigg) \cup \Bigg(\bigcup_{\substack{a,b\in S\\ ab\in E(G)}} T_2^{[ab]}\Bigg),$$

where $S \subseteq V(G)$, $T_0^{[ab]}$ is a 2-dominating set in H^{ab} for every pair a and b in $V(G)\setminus S$ with $ab \in E(G)$, $T_1^{[ab]}$ is a dominating set in H^{ab} for every pair a and b in $V(G)$ with $|\{a,b\}\cap S| = 1$ and $ab \in E(G)$, and $T_2^{[ab]} \subseteq V(H^{ab})$ for every pair a and b in S with $ab \in E(G)$.

Theorem 3.2. *Let G be a connected graph of order $m \geq 2$ and size n. Let H be any nonempty graph. Then the set $D = V(G)\cup S$ such that $S = \bigcup_{a,b\in V(G), ab\in E(G)} I^{[ab]}$ where $I^{[ab]}$ is the set of isolated vertices in H^{ab} for every pair a and b in $V(G)$ with $ab \in E(G)$, is a* RWC2D *set in $G\diamond H$. As a consequence, $\gamma_{r2w}(G\diamond H) \leq m + n|I^{[ab]}|$.*

Proof. From the definition of edge corona $G\diamond H$ it is clear that $V(G)\cup S$ such that $S = \bigcup_{a,b\in V(G), ab\in E(G)} I^{[ab]}$ where $I^{[ab]}$ is the set of all isolated vertices in H^{ab} is a 2-dominating set in $G\diamond H$. Moreover, the subgraph $\langle V(G)\cup S\rangle_w$ of $G\diamond H$ is exactly what remains from $G\diamond H$ after removing all the edges between any two adjacent vertices in each of the component in the copies of H. Since G is connected by assumption, $\langle V(G)\rangle_w$ is also connected. Thus, $\langle V(G)\cup S\rangle_w$ is connected. Also, $\langle V(G\diamond H)\setminus(V(G)\cup S)\rangle$ consist of $|E(G)|$ components of H each component has at least two vertices. Hence, $\langle V(G\diamond H)\setminus(V(G)\cup S)\rangle$ has no isolated vertex. This means that D is a restrained set in $G\diamond H$. Therefore, D is a RWC2D set in $G\diamond H$. Consequently, $\gamma_{r2w}(G\diamond H) \leq |D| = |V(G)\cup S| = |V(G)| + |S| = m + n|I^{[ab]}|$. □

Theorem 3.3. *Let G be a connected graph of order $m \geq 2$ and let H be any nonempty graph. Then $T \subseteq V(G\diamond H)$ is a* RWC2D *set in $G\diamond H$ if and only if*

$$T = S \cup \Bigg(\bigcup_{\substack{a,b\in V(G)\setminus S\\ ab\in E(G)}} T_0^{[ab]}\Bigg) \cup \Bigg(\bigcup_{\substack{a,b\in V(G), ab\in E(G)\\ |\{a,b\}\cap S|=1}} T_1^{[ab]}\Bigg) \cup \Bigg(\bigcup_{\substack{a,b\in S\\ ab\in E(G)}} T_2^{[ab]}\Bigg),$$

where $S \subseteq V(G)$, $T_0^{[ab]}$ is a 2-dominating set in H^{ab} for every pair a and b in $V(G)\setminus S$ with $ab \in E(G)$, $T_1^{[ab]}$ is a dominating set in H^{ab} where $T_1^{[ab]} \neq V(H^{ab})$ whenever $N_G(\{a,b\}\setminus(\{a,b\}\cap S))\cap(V(G)\setminus S) = \varnothing$ for every pair a and b in $V(G)$ with $ab \in E(G)$ and $|\{a,b\}\cap S| = 1$, and $T_2^{[ab]} \subseteq V(H^{ab})$ where $\langle V(H^{ab})\setminus T_2^{[ab]}\rangle$ has no isolated vertex for every pair a and b in S with $ab \in E(G)$.

Proof. Suppose that $T \subseteq V(G\diamond H)$ is a RWC2D set in $G\diamond H$. Then T is a WC2D set in $G\diamond H$. Let $S = T\cap V(G)$ and let $a,b \in V(G)$ such that $ab \in E(G)$. Since $V(H^{a_1b_1})\cap V(H^{a_2b_2}) = \varnothing$ for every $a_1b_1, a_2b_2 \in E(G)$, it follows that $T\cap V(\langle\{a,b\}\cup V(H^{ab})\rangle)$ is nonempty. Moreover, due to the adjacency of the vertices in the subgraph $\langle \{a,b\}\cup V(H^{ab})\rangle$, it can be seen that $T\cap V(\langle \{a,b\}\cup V(H^{ab})\rangle)$ is a RWC2D set in $\langle \{a,b\}\cup V(H^{ab})\rangle$. Let us consider the cases where $|\{a,b\}\cap S|=0,1,2$.

Case 1. Let $a,b\in V(G)\backslash S$. By Theorem 3.1, $T\cap V(H^{ab})=T_0^{[ab]}$ is a 2-dominating set in H^{ab} for every pair a and b in $V(G)\backslash S$ with $ab\in E(G)$.

Case 2. Let $|\{a,b\}\cap S|=1$, say $\{a,b\}\cap S=\{a\}$. By Theorem 3.1, $T\cap V(H^{ab})=T_1^{[ab]}$ is a dominating set in H^{ab}. Since T is a restrained set in $G\diamond H$, then either $T_1^{[ab]}$ is a dominating set in H^{ab} and $T\cap V(H^{ab})\subsetneq V(H^{ab})$, or $T_1^{[ab]}$ is a dominating set in H^{ab} and $bc\in E(G)$ for some $c\in V(G)\backslash S$ whenever $T\cap V(H^{ab})=V(H^{ab})$.

Case 3. Let $a,b\in S$. By Theorem 3.1, $T\cap V(H^{ab})=T_2^{[ab]}\subseteq V(H^{ab})$ for every pair a and b in $V(G)$ with $ab\in E(G)$. Since T is a restrained set in $G\diamond H$, $T\cap V(H^{ab})$ is a restrained set in H^{ab}, that is $\langle V(H^{ab})\backslash T_2^{[ab]}\rangle$ has no isolated vertex.

Combining all the aforementioned observations above, we have

$$T=S\cup\Bigg(\bigcup_{\substack{a,b\in V(G)\backslash S\\ ab\in E(G)}} T_0^{[ab]}\Bigg)\cup\Bigg(\bigcup_{\substack{a,b\in V(G),ab\in E(G)\\ |\{a,b\}\cap S|=1}} T_1^{[ab]}\Bigg)\cup\Bigg(\bigcup_{\substack{a,b\in S\\ ab\in E(G)}} T_2^{[ab]}\Bigg),$$

where $S\subseteq V(G)$, $T_0^{[ab]}$ is a 2-dominating set in H^{ab} for every pair a and b in $V(G)\backslash S$ with $ab\in E(G)$, $T_1^{[ab]}$ is a dominating set in H^{ab} where $T_1^{[ab]}\subsetneq V(H^{ab})$ whenever $N_G(\{a,b\}\backslash(\{a,b\}\cap S))\cap(V(G)\backslash S)=\varnothing$ for every pair a and b in $V(G)$ with $ab\in E(G)$ and $|\{a,b\}\cap S|=1$, and $T_2^{[ab]}\subseteq V(H^{ab})$ where $\langle V(H^{ab})\backslash T_2^{[ab]}\rangle$ has no isolated vertex for every pair a and b in S with $ab\in E(G)$.

For the converse, suppose that

$$T=S\cup\Bigg(\bigcup_{\substack{a,b\in V(G)\backslash S\\ ab\in E(G)}} T_0^{[ab]}\Bigg)\cup\Bigg(\bigcup_{\substack{a,b\in V(G),ab\in E(G)\\ |\{a,b\}\cap S|=1}} T_1^{[ab]}\Bigg)\cup\Bigg(\bigcup_{\substack{a,b\in S\\ ab\in E(G)}} T_2^{[ab]}\Bigg),$$

where S, $T_0^{[ab]}$, $T_1^{[ab]}$ and $T_2^{[ab]}$ satisfy the given properties stated in the theorem. By Theorem 3.1, T is a WC2D set in $G\diamond H$. We only need to show that T is a restrained set in $G\diamond H$ and to do this, let $S=T\cap V(G)$ and let $a,b\in V(G)$ such that $ab\in E(G)$ and consider three cases where $|\{a,b\}\cap S|=0,1,2$.

Case 1. $|\{a,b\}\cap S|=0$. Then $a,b\notin S$. Using the property of $T_0^{[ab]}$, it can be checked that $\langle(\{a,b\}\cup V(H^{ab}))\backslash T_0^{[ab]}\rangle\cong K_2\vee\langle V(H^{ab})\backslash T_0^{[ab]}\rangle$ which contains no isolated vertex.

Case 2. $|\{a,b\}\cap S|=1$, say $a\in S$ and $b\notin S$. Suppose that $V(H^{ab})\backslash T_1^{[ab]}\neq\varnothing$, then $\langle(\{a,b\}\cup V(H^{ab}))\backslash T_1^{[ab]}\rangle\cong K_1\vee\langle V(H^{ab})\backslash T_1^{[ab]}\rangle$ which contains no isolated vertex. On the other hand, if $V(H^{ab})\backslash T_1^{[ab]}=\varnothing$, by the property of $T_1^{[ab]}$, $N_G(b)\cap V(G)\backslash S\neq\varnothing$. It means that there exists $c\in V(G)\backslash S$ such that $bc\in E(G)$.

Case 3. $|\{a,b\}\cap S|=2$. Then $a,b\in S$. By the property of $T_2^{[ab]}$, $\langle V(H^{ab})\backslash T_2^{[ab]}\rangle$ contains no isolated vertex.

The three cases above show that T is a restrained set in $G\diamond H$. Therefore, T is a RWC2D set in $G\diamond H$. □

Theorem 3.4. *Let G be a connected graph of order $m \geq 2$ and let H be any nonempty graph. Then $\gamma_{r2w}(G \diamond H) \geq m + |E(G)||I^{[ab]}|$, where $I^{[ab]}$ is the set of all isolated vertices in H^{ab} for every pair a and b in $V(G)$ with $ab \in E(G)$.*

Proof. We are going to show that every RWC2D set in $G \diamond H$ has cardinality at least $m + |E(G)||I^{[ab]}|$. Let T be a RWC2D set in $G \diamond H$. By Theorem 3.3, T can be expressed as

$$T = S \cup \left(\bigcup_{\substack{a,b \in V(G)\setminus S \\ ab \in E(G)}} T_0^{[ab]} \right) \cup \left(\bigcup_{\substack{a,b \in V(G), ab \in E(G) \\ |\{a,b\} \cap S|=1}} T_1^{[ab]} \right) \cup \left(\bigcup_{\substack{a,b \in S \\ ab \in E(G)}} T_2^{[ab]} \right),$$

where $S \subseteq V(G)$, $T_0^{[ab]}$ is a 2-dominating set in H^{ab} for every pair a and b in $V(G)\setminus S$ with $ab \in E(G)$, $T_1^{[ab]}$ is a dominating set in H^{ab} where $T_1^{[ab]} \subsetneq V(H^{ab})$ whenever $N_G(\{a,b\}\setminus(\{a,b\} \cap S)) \cap (V(G)\setminus S) = \varnothing$ for every pair a and b in $V(G)$ with $ab \in E(G)$ and $|\{a,b\} \cap S| = 1$, and $T_2^{[ab]} \subseteq V(H^{ab})$ where $\langle V(H^{ab})\setminus T_2^{[ab]} \rangle$ has no isolated vertex for every pair a and b in S with $ab \in E(G)$. For convenience, set some subsets of T in the given expression above as follows:

$$\mathscr{A}_0 = \bigcup_{\substack{a,b \in V(G)\setminus S \\ ab \in E(G)}} T_0^{[ab]}, \quad \mathscr{A}_1 = \bigcup_{\substack{a,b \in V(G), ab \in E(G) \\ |\{a,b\} \cap S|=1}} T_1^{[ab]} \quad \text{and} \quad \mathscr{A}_2 = \bigcup_{\substack{a,b \in S \\ ab \in E(G)}} T_2^{[ab]}.$$

We then consider two simple cases for the set S.

Case 1. $S = V(G)$. Then $\mathscr{A}_0$ and $\mathscr{A}_1$ are both equal to the empty set. This implies that $T = V(G) \cup \mathscr{A}_2$. Since T is a RWC2D set in $G \diamond H$ and by Remark 2.3, all isolated vertices in H belong to T. Thus, we have $|T| \geq m + |E(G)||I^{[ab]}|$ where $a, b \in V(G)$ with $ab \in E(G)$.

Case 2. $S \subsetneq V(G)$. Let $x \in V(G)\setminus S$. Since G is connected and nontrivial, there exists $y \in V(G)$ such that $xy \in E(G)$. If $y \notin S$, then T has a subset $T_0^{[xy]}$ contained in $\mathscr{A}_0$ such that $T_0^{[xy]}$ is 2-dominating in H^{xy}. Since $|T_0^{[xy]}| \geq 2$, it follows that the absence of both $x, y \in V(G)$ from S, and hence from T, does not affect T cardinality-wise since T would also contain at least two elements from $V(H^{xy})$. On the other hand, if $y \in S$, then T has a subset $T_1^{[xy]}$ contained in $\mathscr{A}_1$ such that $T_1^{[xy]}$ is dominating in H^{xy}. Since $|T_1^{[xy]}| \geq 1$ in this subcase, it follows that relative to the edge $xy \in E(G)$, where $y \in S$, the absence of $x \in V(G)$ from S, and hence from T, is compensated by the fact that T contains at least one element from $V(H^{xy})$. Since T is a RWC2D set in $G \diamond H$ and by Remark 2.3, all isolated vertices in H belong to T. Since the vertex sets of the various copies of H as viewed in $G \diamond H$ are pairwise disjoint, it follows now from the two subcases, all for the case when $V(G)\setminus S \neq \varnothing$, that $|T| \geq |V(G)| + |E(G)||I^{[ab]}| = m + |E(G)||I^{[ab]}|$. In either case 1 or case 2, we get $\gamma_{2w}(G \diamond H) \geq m + |E(G)||I^{[ab]}|$ where $a, b \in V(G)$ with $ab \in E(G)$. □

The next result is an immediate consequence of Theorem 3.2 and Theorem 3.4.

Corollary 3.5. *If G and H are nontrivial graphs with G connected and H a nonempty graph, then $\gamma_{r2w}(G \diamond H) = |V(G)| + |E(G)||I^{[ab]}|$, where $I^{[ab]}$ is the set of isolated vertices of H^{ab} for every pair $a, b \in V(G)$ with $ab \in E(G)$.*

Example 3.6. The edge corona of two graphs C_4 and the union of complete graphs K_2 and K_1 is given in Figure 5 with $\gamma_{r2w}(C_4 \diamond (K_2 + K_1)) = 4 + 4(1) = 8$.

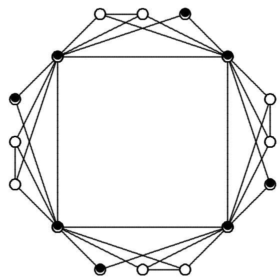


Figure 5. The edge corona $C_4 \diamond (K_2 + K_1)$ with darkened vertices in some γ_{r2w}-set

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

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